Finite Difference Techniques for One and Two Dimension Option Valuation Problems

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February 1990
Revised: November 1992

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Funding for this work was provided by the ESRC and Corporate Members of the Financial Options Research centre: Bankers Trust, London International Financial Futures and Options Exchange, London Clearing House, London FOX, Midland Montagu, Mitsubishi Finance, Nomura Bank, UBS Phillips and Drew.

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FORC Preprint: 90/10

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ABSTRACT

Finite difference methods represent an important numerical technique in the valuation of options for which analytical solutions cannot be obtained.

In this paper we review the finite difference methods which have appeared in the finance literature and we compare their advantages and disadvantages in terms of stability, convergence and efficiency.

The motivation for this paper is to present a logical development from the simplest one-dimensional method through to more complex one and two-dimensional methods. We deal with the problems of stability and convergence as they arise naturally within the logical development. We also attempt to show the theoretical relationship between the various methods and their relationship to discrete stochastic processes and iterative methods.

1 Notation and Transformations

We will use the following notation throughout the paper,

- $U, W, u \equiv$ option value
- $S \equiv$ stock price
- $\sigma \equiv$ annualised standard deviation of the stock returns
- $r \equiv$ continuously compounded riskless rate of interest
- $t \equiv$ real-time
- $\tau \equiv$ time to maturity

U will represent the theoretical option value, W will represent the value of the option under some transformation and u will represent the finite difference approximation to U.

The standard Black-Scholes partial differential equation (PDE) in real-time

$$\frac{\partial U}{\partial t} = rU - rS\frac{\partial U}{\partial S} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 U}{\partial S^2} \tag{1}$$

will be written in the standard partial differential short-hand

$$U_t = rU - rSU_S - \frac{1}{2}S^2\sigma^2 U_{SS} \tag{2}$$

where the subscripts mean the partial derivative with respect to that variable. Equation (2) can be written in terms of the time to maturity

$$U_\tau = \frac{1}{2}S^2\sigma^2 U_{SS} + rSU_S - rU \tag{3}$$

The Black-Scholes partial differential equation in this form can be simplified by applying transforms to the option value and the stock price. These reduce the number of terms and remove the stock price from the coefficients of the partial differentials.

The first transform, introduced by Courtadon (1982), removes the rU term.

Let

$$W = e^{r\tau}U \tag{4}$$

Therefore the partial differentials of W are

$$W_\tau = e^{r\tau}U_\tau + Ue^{r\tau}r \tag{5}$$

$$W_S = e^{r\tau}U_S \tag{6}$$

$$W_{SS} = e^{r\tau}U_{SS} \tag{7}$$

Substituting these into equation (3) gives

$$W_\tau = \frac{1}{2}S^2\sigma^2 W_{SS} + rSW_S \tag{8}$$

The second transform, introduced by Brennan and Schwartz (1978), removes the stock price from the coefficients, resulting in constant coefficients and so greater stability when finite difference methods are employed.

Let

$$x = \ln(S) \tag{9}$$

The partial differentials are

$$W_S = W_x \frac{1}{S} \tag{10}$$

$$W_{SS} = W_x(\frac{-1}{S^2}) + \frac{1}{S} W_{xx} \frac{1}{S} \tag{11}$$

Substituting these into equation (8) gives

$$W_\tau = \frac{1}{2}\sigma^2 W_{xx} + (r - \frac{1}{2}\sigma^2) W_x \tag{12}$$

Geske and Shastri (1985) noted that the Black-Scholes PDE can be reduced to the following form

$$U_t = \frac{1}{ak} U_{yy} \tag{13}$$

by using transformations introduced by Black and Scholes (1973) and Merton (1973).

We will define x, the natural logarithm of the stock price on the interval $[0, x_I]$ subdivided into I intervals such that $Ih = x_I$, $ih = x_i$, and we will define τ, the time to maturity, on the interval $[0, \tau]$ subdivided into M intervals such that $Mk = \tau_M$, $mk = \tau_m$. We will also define a second space variable when we consider two-dimensional problems on the interval $[0, b]$ subdivided into J intervals such that $Jh = b$, $jh = y_j$. We can then approximate $W(S, \tau) = W(e^x, \tau)$ by u_i^m (or $u_{i,j}^m$ in the two-dimensional case). We will call the space spanned by these discrete intervals the lattice.

2 Finite Difference Approximations to Derivatives

We will now briefly review the theoretical basis for the finite difference approach. If $u(x)$ and its derivatives are single-valued, finite and continuous functions of x then by Taylors Theorem (Smith (1975)) we can expand $u(x+h)$

$$u(x+h) = u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + ... \tag{14}$$

and $u(x-h)$

$$u(x-h) = u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + ... \tag{15}$$

Adding equations (14) and (15) we obtain

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + O(h^4) \tag{16}$$

Where $O(h^4)$ denotes terms containing fourth and higher powers of h so that the size of these terms will be of order h^4. Rearranging this we obtain an expression for $u''(x)$ in terms of the value of u at $x-h$, x and $x+h$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2) \tag{17}$$

Subtracting equations (14) and (15) and rearranging we obtain

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2) \tag{18}$$

These are called the central difference approximations because the range over which we are approximating $u(x)$ is centred on x.

From equation (14) neglecting second and higher powers of h we obtain the forward difference approximation with $O(h)$ error

$$u'(x) = \frac{u(x+h) - u(x)}{h} \tag{19}$$

From (15) neglecting second and higher powers of h we obtain the backward difference approximation with $O(h)$ error

$$u'(x) = \frac{u(x) - u(x-h)}{h} \tag{20}$$

3 One-Dimensional Methods

3.1 The Explicit Finite Difference Approximation

The explicit finite difference approximation replaces the space differentials by their central difference approximations but the time differential by the forward difference approximation (Smith (1975)). If we perform this operation on equation (12) we obtain the following finite difference equation

$$\frac{(u_i^{m+1} - u_i^m)}{k} = \frac{1}{2}\sigma^2 \frac{(u_{i+1}^m - 2u_i^m + u_{i-1}^m)}{h^2} + (r - \frac{1}{2}\sigma^2)\frac{(u_{i+1}^m - u_{i-1}^m)}{2h} \tag{21}$$

Equation (21) can be re-written as follows

$$u_i^{m+1} = p^- u_{i-1}^m + p u_i^m + p^+ u_{i+1}^m \tag{22}$$

$$p^- = k(\sigma^2/2h^2 - (r - \frac{1}{2}\sigma^2)/2h) \tag{23}$$

$$p = 1 - k\sigma^2/h^2 \tag{24}$$

$$p^+ = k(\sigma^2/2h^2 + (r - \frac{1}{2}\sigma^2)/2h) \tag{25}$$

We therefore have an expression for the value of u_i at the next time step explicitly in terms of the values of u_{i-1}, u_i and u_{i+1} at the current time step. Figure 1 shows this diagramatically. The coefficients p^-, p and p^+ being constant across the lattice.

In order to obtain the value of the option at some time to maturity τ we proceed backwards in time from maturity, at which we know the value of the option for all S, $(u_0^0, ..., u_I^0)$. Firstly we compute $(u_1^{m+1}, ..., u_{I-1}^{m+1})$ using equation (22), note that we cannot compute (u_0^{m+1}, u_I^{m+1}) since we require values either side at time m for equation (22). We must compute (u_0^{m+1}, u_I^{m+1}) using the boundary conditions, for example for a call

$$U_S = 1, S \to \infty \tag{26}$$

$$U_S = 0, S \to 0 \tag{27}$$

Which, in terms of our finite difference approximation, are

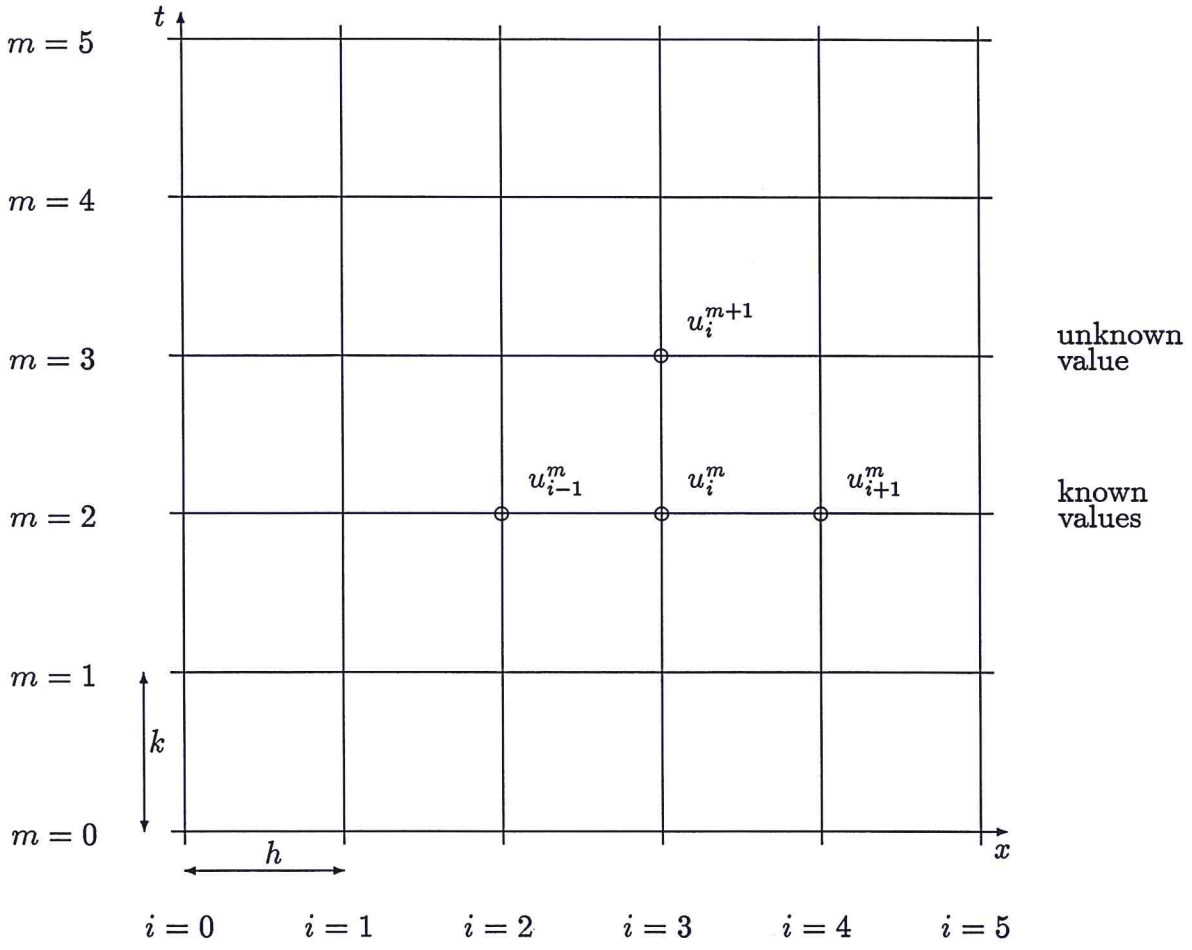
$$u_I^{m+1} - u_{I-1}^{m+1} = (\exp(x_I) - \exp(x_{I-1}))\exp(r\tau) \tag{28}$$

$$u_1^{m+1} - u_0^{m+1} = 0 \tag{29}$$

We can then repeat this process until we reach the time to maturity we require.

The accuracy of this method, determined by the accuracy of the finite difference approximations, is $O(h^2 + k)$.

Figure 1: The Explicit Finite Difference Method



It is stable and convergent only for a mesh ratio of $R = k/h^2 \leq \frac{1}{2}$. We will return to this point in more detail in section 5.

3.2 The Implicit Finite Difference Approximation

This method was introduced to the literature by Schwartz (1975,1977). The implicit finite difference approximation replaces the space differentials by their central difference approximations at the next time step rather than the current time step as in the explicit method. The time differential is again replaced by the forward difference approximation. If we perform this operation on equation (12) we obtain the following finite difference equation

$$\frac{(u_i^{m+1} - u_i^m)}{k} = \frac{1}{2}\sigma^2 \frac{(u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1})}{h^2} + (r - \frac{1}{2}\sigma^2)\frac{(u_{i+1}^{m+1} - u_{i-1}^{m+1})}{2h} \tag{30}$$

This can be re-written in the following form

$$p^- u_{i-1}^{m+1} + p u_i^{m+1} + p^+ u_{i+1}^{m+1} = u_i^m \tag{31}$$

$$p^- = -k(\sigma^2/2h^2 - (r - \frac{1}{2}\sigma^2/2h)) \tag{32}$$

$$p = 1 + k\sigma^2/h^2 \tag{33}$$

$$p^+ = -k(\sigma^2/2h^2 + (r - \frac{1}{2}\sigma^2/2h)) \tag{34}$$

Equation (31) is an implicit expression for the values of u at time step $m+1$ in terms of the values at time step m. Figure 2 shows this diagramatically. In order to obtain the values of u at $m+1$ we must solve a set of $I+1$ simultaneous equations,

$I-1$ of the form of equation (31) together with 2 boundary conditions. We will see in section 3.3 that these simultaneous equations form a tridiagonal matrix equation which can be solved without full inversion of the coefficients matrix.

The accuracy of this method is the same as for the explicit method, $O(h^2+k)$. However the implicit method is unconditionally stable and convergent, therefore we do not have to worry about the mesh ratio, we can simply consider the trade-off between the step sizes and the accuracy we require.

3.3 Solving a Tridiagonal Matrix Equation

A set of simultaneous equations in the form of equation (31) can be written in the following simplified form

...

$$a_i u_{i-1} + b_i u_i + c_i u_{i+1} = d_i \tag{35}$$

$$a_{i+1} u_i + b_{i+1} u_{i+1} + c_{i+1} u_{i+2} = d_{i+1} \tag{36}$$

...

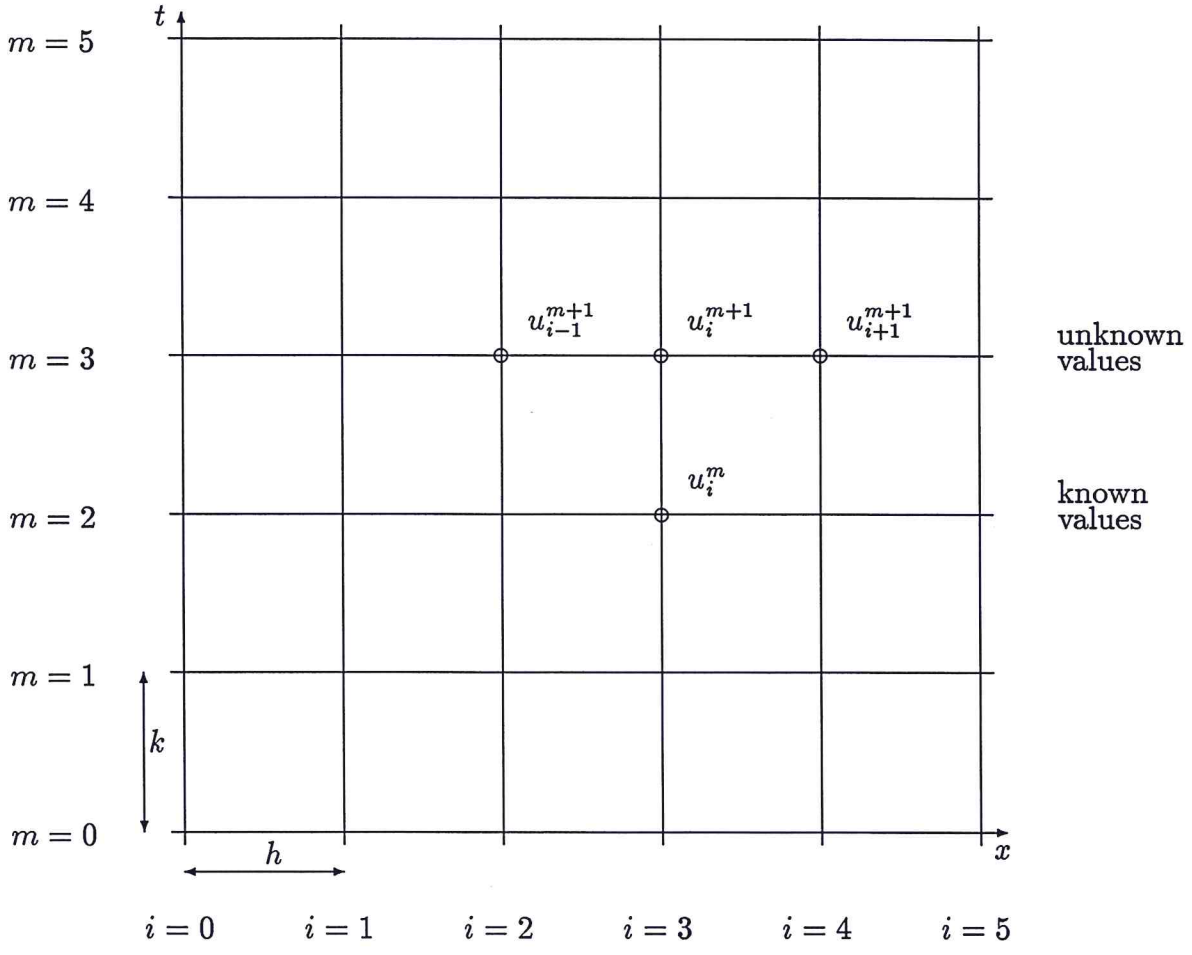
The important property to note is that two consecutive equations have two unknown u's in common. If we assume u_0 is known by a boundary condition then equation (35) at $i = 1$ becomes

$$b_i u_i + c_i u_{i+1} = d'_i \tag{37}$$

Therefore (36) becomes

$$b'_{i+1} u_{i+1} + c_{i+1} u_{i+2} = d'_{i+1} \tag{38}$$

Figure 2: The Implicit Finite Difference Method



Where

$$b'_{i+1} = b_{i+1} - \frac{a_{i+1}c_i}{b_i} \tag{39}$$

and

$$d'_{i+1} = d_{i+1} - \frac{a_{i+1}d'_i}{b_i} \tag{40}$$

This process is repeated for i from 1 up to $I-1$ so we obtain a set of simultaneous equations of the form of equation (38) with two unknowns.

Now if u_I is known by a boundary condition then we can substitute into (38) at $i+1 = I-1$ for u_I and rearrange to obtain u_{I-1}.

This process is repeated for i from $I-1$ down to 1 and we have solved for the complete set of unknowns $(u_0, ..., u_I)$.

3.4 The Crank-Nicholson Finite Difference Approximation

This method was developed by Crank and Nicholson (1947) and introduced to the finance literature by Courtadon (1982). Consider equation (12) which we repeat here

$$W_\tau = \frac{1}{2}\sigma^2 W_{xx} + (r - \frac{1}{2}\sigma^2)W_x \tag{41}$$

The Crank-Nicholson method replaces the space differentials by the mean of their central finite differences at time step m and $m+1$

$$(u_{xx})_i^{m+\frac{1}{2}} = \frac{1}{2h^2}[(u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}) + (u_{i+1}^m - 2u_i^m + u_{i-1}^m)] \tag{42}$$

$$(u_x)_i^{m+\frac{1}{2}} = \frac{1}{4h}[(u_{i+1}^{m+1} - u_{i-1}^{m+1}) + (u_{i+1}^m - u_{i-1}^m)] \tag{43}$$

The time differential has the same form as for the explicit and implicit methods but is now a central difference approximation

$$(u_\tau)_i^{m+\frac{1}{2}} = \frac{1}{k}[u_i^{m+1} - u_i^m] \tag{44}$$

All the finite differences are therefore centred at the same time point which leads to improved accuracy over the previous methods. Substituting these finite differences into equation (41) we obtain the following finite difference equation

$$p^- u_{i-1}^{m+1} + p u_i^{m+1} + p^+ u_{i+1}^{m+1} = \bar{p} \tag{45}$$

$$p^- = k(\frac{\sigma^2}{4h^2} - \frac{(r - \frac{1}{2}\sigma^2)}{4h}) \tag{46}$$

$$p = k(-\frac{\sigma^2}{2h^2}) - 1 \tag{47}$$

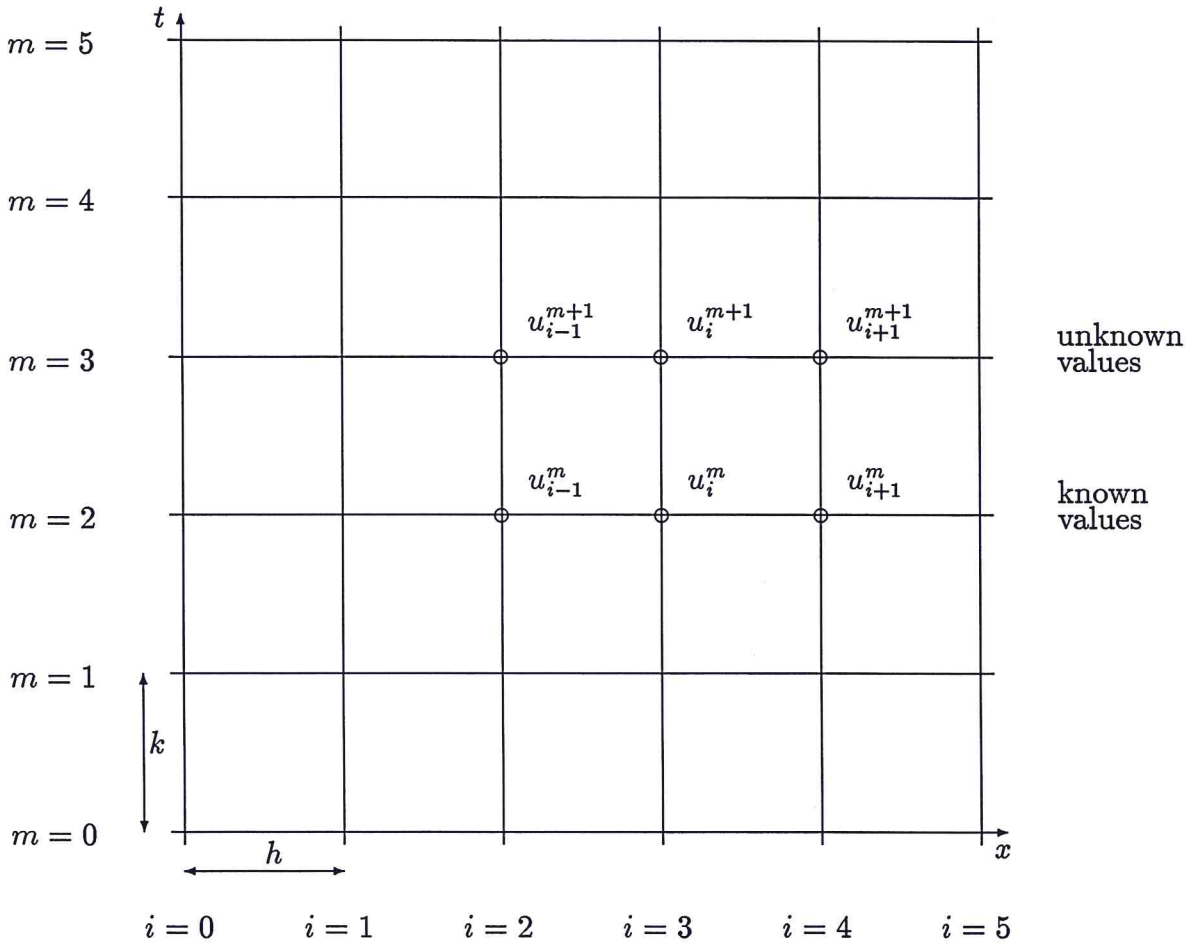
$$p^+ = k(\frac{\sigma^2}{4h^2} + \frac{(r - \frac{1}{2}\sigma^2)}{4h}) \tag{48}$$

$$\bar{p} = -u_i^m - \frac{k\sigma^2}{4h^2}(u_{i+1}^m - 2u_i^m + u_{i-1}^m) - \frac{k(r - \frac{1}{2}\sigma^2)}{4h}(u_{i+1}^m - u_{i-1}^m) \tag{49}$$

Equation (45) is again an implicit expression for the values of u at $m+1$ and therefore requires the solution of a tridiagonal matrix equation. Figure 3 shows this diagramatically.

The accuracy of this method is $O(h^2 + (k/2)^2)$ which has improved accuracy with respect to the time step and the method is also unconditionally stable and convergent.

Figure 3: The Crank-Nicholson Finite Difference Method



The Crank-Nicholson Approximation can be generalised to the Weighted Average Approximation which is unconditionally stable and convergent for $\frac{1}{2} \leq \theta \leq 1$

$$(u_{xx})_i^{m+\theta} = \frac{1}{h^2}[\theta(u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}) + (1-\theta)(u_{i+1}^m - 2u_i^m + u_{i-1}^m)] \quad (50)$$

4 Iterative Point Methods for Implicit Finite Difference Approximations

These methods are typically used where the number of unknowns is large but the relationships between them involve only a few of the unknowns.

To illustrate their use we will consider a simple example

$$u_t = u_{xx} \quad (51)$$

The Crank-Nicholson approximation to this is as follows

$$\frac{(u_i^{m+1} - u_i^{m+1})}{k} = \frac{1}{2h^2}[(u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}) + (u_{i+1}^m - 2u_i^m + u_{i-1}^m)] \quad (52)$$

Equation (52) can be rewritten by grouping the known values at time m into a constant b_i and dropping time superscripts

$$u_i = \frac{1}{2}R(u_{i-1} - 2u_i + u_{i+1}) + b_i \quad (53)$$

$$b_i = u_i^m + \frac{1}{2}R(u_{i+1}^m - 2u_i^m + u_{i-1}^m) \quad (54)$$

We can consider equation (53) to be an iterative relationship for computing the value of u_i. Denoting the successive approximations to u_i as $u_i^{(0)}$, $u_i^{(1)}$, ... , we have

$$u_i^{(n+1)} = \frac{1}{2}R(u_{i-1}^{(n)} - 2u_i^{(n)} + u_{i+1}^{(n)}) + b_i \tag{55}$$

This converges very slowly however, a better iterative procedure is

$$u_i^{(n+1)} = \frac{1}{2}R(u_{i-1}^{(n)} - 2u_i^{(n+1)} + u_{i+1}^{(n)}) + b_i \tag{56}$$

or

$$u_i^{(n+1)} = \frac{R}{2(1+R)}(u_{i-1}^{(n)} + u_{i+1}^{(n)}) + \frac{b_i}{(1+R)} \tag{57}$$

Equation (57) which computes the $(n+1)$th iterate in terms of the nth iterates is called a Jacobi iteration. Convergence can usually be improved by using new values as they are calculated (Gauss- Seidel iteration)

$$u_i^{(n+1)} = \frac{R}{2(1+R)}(u_{i-1}^{(n+1)} + u_{i+1}^{(n)}) + \frac{b_i}{(1+R)} \tag{58}$$

Gauss-Seidel iteration typically doubles the rate of convergence of the iteration process compared with Jacobi iteration. The rate of convergence can be improved further by using successive over- relaxation (see Smith (1975) for a good discussion of these techniques).

5 Stability and Convergence

We will now examine in more detail the analysis of the stability and convergence of finite difference methods.

Let U represent the exact solution of the partial differential equation and let u represent the exact solution of the finite difference equation. Then $u - U$ is called the discretisation error and is the error introduced by representing the partial differentials by finite differences. Typically this can be reduced by taking higher order terms in the finite difference approximations or by using smaller space and/or time steps.

If we now let N represent the actual computed solution then $N - u$ is called the round-off error and is the error introduced in the computational process by representing real numbers to a finite precision.

We say that the finite difference approximation is convergent if the discretisation error tends to zero as space and time steps tend to zero. The finite difference approximation is stable if the round-off errors are small and remain bounded for all time.

Convergence can be analysed by substitution of $u_i^m = U_i^m + e_i^m$ into the finite difference equation, expanding the $U_{i\pm1}^m$ about U_i^m using Taylor's Theorem. Typically the terms in U tend to the PDE as the limit is taken leaving a difference equation for the errors.

5.1 Stability under round-off errors

To illustrate the analysis of stability we will consider a simple example. Consider an explicit finite difference approximation to the following simple partial differential equation

$$u_t = u_{xx} \tag{59}$$

The explicit finite difference equation is

$$\frac{(u_i^{m+1} - u_i^{m+1})}{k} = \frac{(u_{i+1}^m - 2u_i^m + u_{i-1}^m)}{h^2} \tag{60}$$

Which can be re-written in terms of the mesh ratio R

$$u_i^{m+1} = Ru_{i-1}^m + (1 - 2R)u_i^m + Ru_{i+1}^m \tag{61}$$

In matrix form, writing the time superscripts as subscripts, we have

$$u_{m+1} = Au_m \tag{62}$$

Simple matrix algebra gives

$$u_m = Au_{m-1} = A(Au_{m-2}) = A^m u_0 \tag{63}$$

Now let

$$u_0^* = u_0 + e_0 \tag{64}$$

From equation (63) we have

$$e_m = u_m - u_m^* = A^m(u_0 - u_0^*) = A^m e_0 \tag{65}$$

Now $Av_s = \lambda_s v_s$, where v_s are the eigenvectors and λ_s are the eigenvalues of A.

We can write the error vectors as linear combinations of the eigenvectors

$$e_0 = \sum_{s=1}^{I-1} c_s v_s \tag{66}$$

Where I is the number of space intervals.

Therefore, substituting into equation (65), we obtain

$$e_m = A^m e_0 = A^m \sum c_s v_s = \sum c_s \lambda_s m v_s \tag{67}$$

Therefore the method will be stable if $|\max(\lambda_s)| \leq 1$ or the modulus of the maximum eigenvalue is less than or equal to unity.

Now it can be shown that

$$\lambda_s = 1 + R[-4\sin^2(\frac{s\pi}{2I})] \tag{68}$$

Which leads to the standard result

$$R \leq \frac{1}{2} \tag{69}$$

6 Finite Difference Schemes and Discrete Stochastic Processes

It has been shown by Brennan and Schwartz (1978) that finite difference schemes are equivalent to approximating the movements of the stock price by a discrete stochastic process.

Consider the logarithmically transformed Black-Scholes equation

$$\frac{1}{2}\sigma^2 U_{xx} + (r - \frac{1}{2}\sigma^2)U_x + U_t - rU = 0 \tag{70}$$

From which we obtain the explicit finite difference approximation, evolving backwards in real-time

$$u_i^m(1 + rk) = p^- u_{i-1}^{m+1} + p u_i^{m+1} + p^+ u_{i+1}^{m+1} \tag{71}$$

$$p^- = k(\frac{\sigma^2}{2h^2} - \frac{(r - \frac{1}{2}\sigma^2)}{2h}) \tag{72}$$

$$p = 1 - \frac{k\sigma^2}{h^2} \tag{73}$$

$$p^+ = k(\frac{\sigma^2}{2h^2} + \frac{(r - \frac{1}{2}\sigma^2)}{2h}) \tag{74}$$

The value of the option at time m may be regarded as its expected value at time $m + 1$ discounted at the riskless rate, r.

The expected value can be considered to be obtained by assuming the logarithm of the stock price follows the discrete stochastic process

$$dx = \begin{matrix} p^+ & +h \\ p & 0 \\ p^- & -h \end{matrix} \tag{75}$$

With expectation and drift given by

$$E = E[dx] = p^+(+h) + p^-(-h) = (r - \frac{1}{2}\sigma^2)k \tag{76}$$

$$Var[dx] = p^+(h - E)^2 + p(-E)^2 + p^-(-h - E)^2 \tag{77}$$

$$Var[dx] = k\sigma^2 - (r - \frac{1}{2}\sigma^2)^2k^2 \tag{78}$$

In the diffusion limit of $k \to 0$ we have

$$dx = (r - \frac{1}{2}\sigma^2)dt + \sigma dz \tag{79}$$

Where dz is a Wiener process, $E[dz] = 0$, $E[dz^2] = dt$.

Implying the diffusion limit of dS is

$$\frac{dS}{S} = rdt + \sigma dz \tag{80}$$

Therefore the explicit finite difference approximation is equivalent to approximating the diffusion process (80) by the discrete stochastic process (75).

Note that the variance of the discrete stochastic process is a downward biased estimate of the diffusion process the upper bound of the bias being σ^4.

Brennan and Schwartz (1978) go on to show that the implicit finite difference approximation is equivalent to a generalised discrete stochastic process in which the stock price may jump to a infinity of possible future values and the variance of which is upward biased.

Courtadon (1982) shows that the Crank-Nicholson approximation is equivalent to a generalised discrete stochastic process but with an unbiased variance the bias in the variances is related to the miscentering of the explicit and implicit methods. Recently Hull and White (1989) have shown that this relationship can be used to adjust the explicit finite difference coefficients such that the method is unconditionally stable.

7 Two-Dimensional Methods

7.1 The Alternating Direction Implicit Method

For two-dimensional problems the explicit approach is impractical since the stability condition requires the step sizes to be very small. A Crank-Nicholson approach, although unconditionally stable and convergent, yields a large set of simultane-

ous equations which cannot be solved by a simple recursive technique as in the one-dimensional case.

The alternating direction implicit (ADI) method, first developed by Peaceman and Rachford (1955), overcomes these problems. To illustrate its advantage over the Crank-Nicholson approach consider the simple partial differential equation

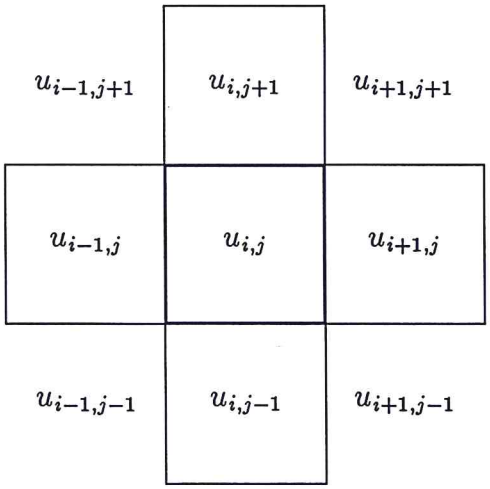
$$u_t = u_{xx} + u_{yy} \tag{81}$$

Where $0 \leq x \leq a$, $0 \leq y \leq b$, $Ih = a$ and $Jh = b$.

A Crank-Nicholson approximation gives

$$\frac{(u_{i,j}^{m+1} - u_{i,j}^{m})}{k} = \frac{1}{2}[(u_{xx} + u_{yy})_{i,j}^{m} + (u_{xx} + u_{yy})_{i,j}^{m+1}] \tag{82}$$

This can be visualised more easily with the computational molecule at $m + 1$



Where the solid lines indicate cells on which the above method depends, this requires the solution of $(I-1)(J-1)$ simultaneous equations given that the boundary values are known.

The ADI method replaces each of the second order derivatives by their implicit approximations in two stages

$$\frac{(u_{i,j}^{m+\frac{1}{2}} - u_{i,j}^{m})}{k/2} = \frac{(u_{i+1,j}^{m+\frac{1}{2}} - 2u_{i,j}^{m+\frac{1}{2}} + u_{i-1,j}^{m+\frac{1}{2}})}{h^2} + \frac{(u_{i,j+1}^{m} - 2u_{i,j}^{m} + u_{i,j-1}^{m})}{h^2} \tag{83}$$

Which gives a tridiagonal matrix equation for $u^{m+\frac{1}{2}}$, then

$$\frac{(u_{i,j}^{m+1} - u_{i,j}m + \frac{1}{2})}{k/2} = \frac{(u_{i+1,j}^{m+\frac{1}{2}} - 2u_{i,j}^{m+\frac{1}{2}} + u_{i-1,j}^{m+\frac{1}{2}})}{h^2} + \frac{(u_{i,j+1}^{m+1} - 2u_{i,j}^{m+1} + u_{i,j-1}^{m+1})}{h^2} \tag{84}$$

Which gives a tridiagonal matrix equation for u^{m+1}. Figures 4 and 5 illustrate this two-stage process.

Each step alone is unstable but together they are unconditionally stable.

7.2 Hopscotch

This method was introduced by Gourlay (1970) and it has similarities with the Crank-Nicholson and ADI methods in that it relies on mixing explicit and implicit techniques together with alternating the rows of the lattice to which the sub-stages are applied. Two variations exist, the odd-even and line methods. We will consider line-hopscotch as it is the simpler of the two methods.

Consider the partial differential equation

$$u_t = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y \tag{85}$$

Explicitly we have

$$\frac{(u_{i,j}^{m+1} - u_{i,j}^{m})}{k} = a\frac{(u_{i+1,j}^{m} - 2u_{i,j}^{m} + u_{i-1,j}^{m})}{h^2} +$$

Figure 4: Alternating Direction Implicit Method: Step one

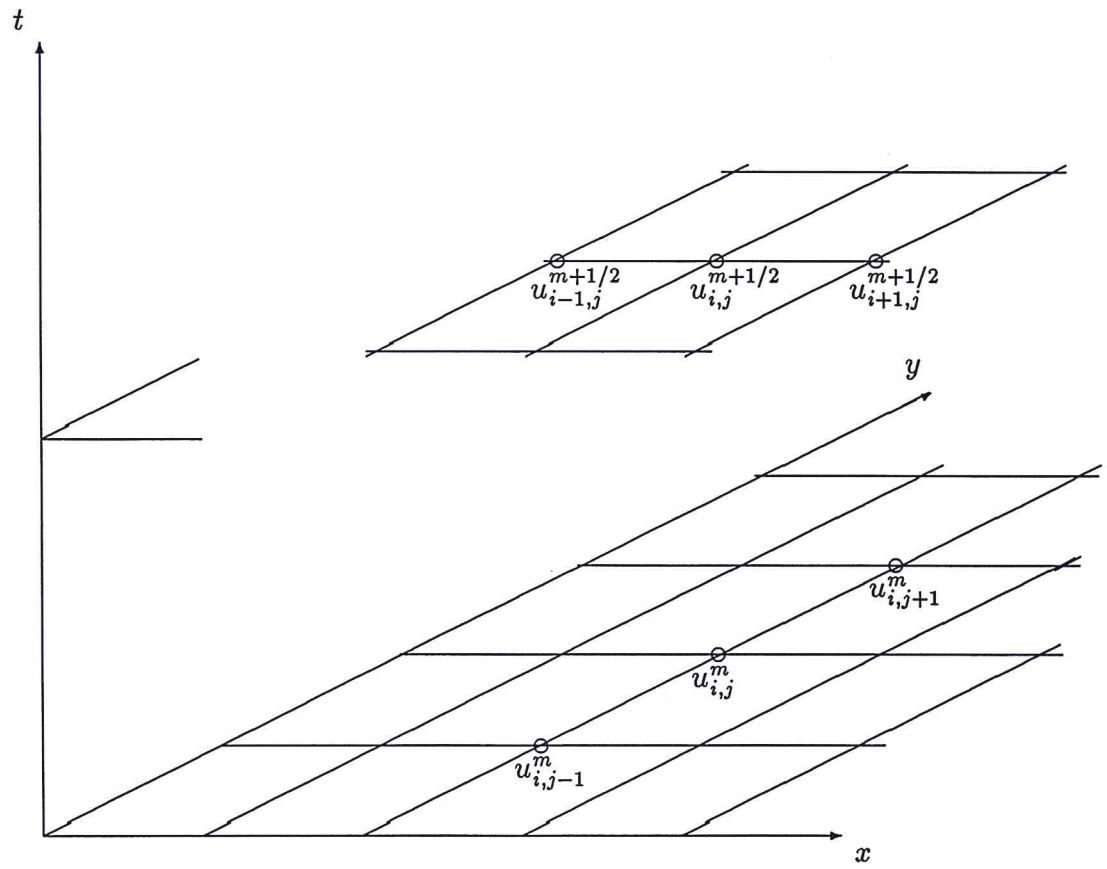
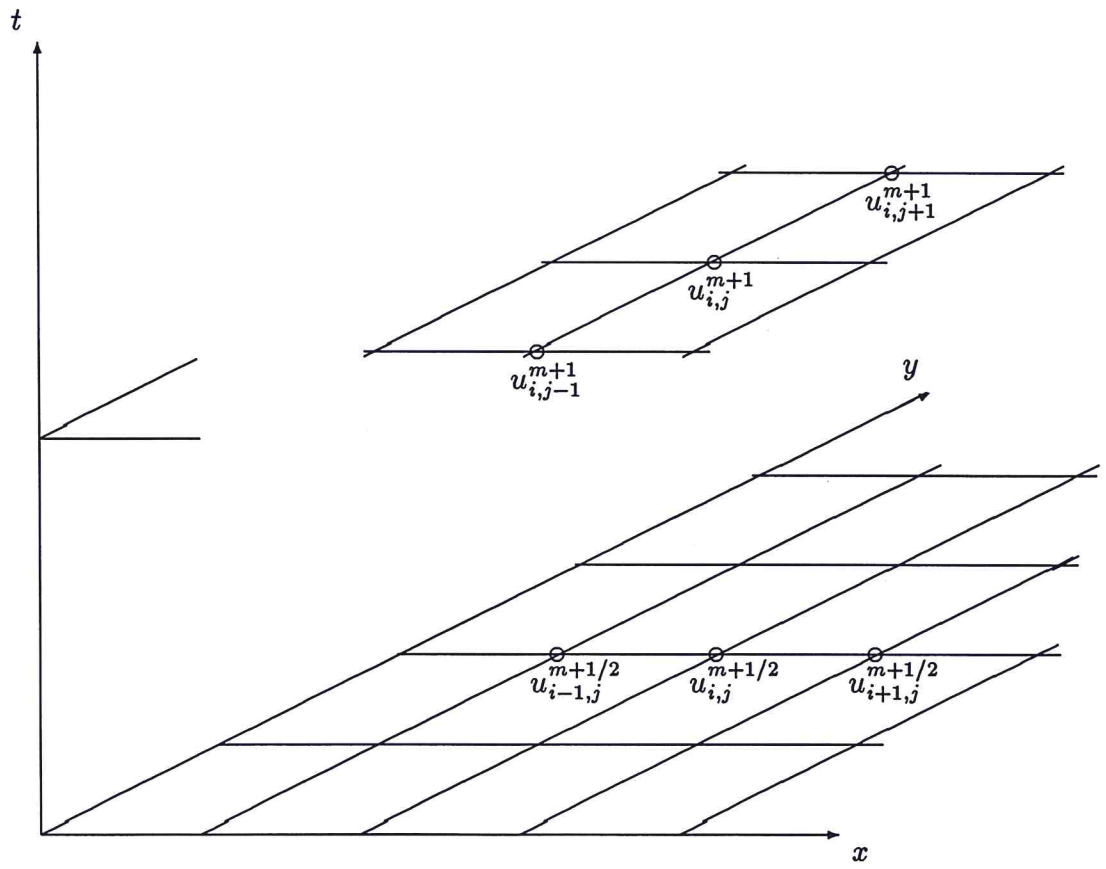


Figure 5: Alternating Direction Implicit Method: Step two

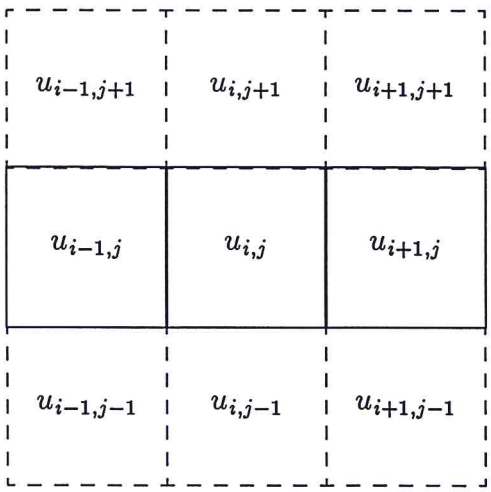


$$2b\frac{(u^m_{i+1,j+1}-u^m_{i-1,j+1}-u^m_{i+1,j-1}+u^m_{i-1,j-1})}{4h^2}+ \tag{86}$$
$$c\frac{(u^m_{i,j+1}-2u^m_{i,j}+u^m_{i,j-1})}{h^2}+d\frac{(u^m_{i+1,j}-u^m_{i-1,j})}{2h}+e\frac{(u^m_{i,j+1}-u^m_{i,j-1})}{2h}$$

Implicitly we have

$$\frac{(u^{m+1}_{i,j}-u^m_{i,j})}{k}=a\frac{(u^{m+1}_{i+1,j}-2u^{m+1}_{i,j}+u^{m+1}_{i-1,j})}{h^2}+$$
$$2b\frac{(u^{m+1}_{i+1,j+1}-u^{m+1}_{i-1,j+1}-u^{m+1}_{i+1,j-1}+u^{m+1}_{i-1,j-1})}{4h^2}+ \tag{87}$$
$$c\frac{(u^{m+1}_{i,j+1}-2u^{m+1}_{i,j}+u^{m+1}_{i,j-1})}{h^2}+d\frac{(u^{m+1}_{i+1,j}-u^{m+1}_{i-1,j})}{2h}+e\frac{(u^{m+1}_{i,j+1}-u^{m+1}_{i,j-1})}{2h}$$

The computational molecule at $m+1$ is



Now if we apply the explicit method for $m+j$ even the dashed cells will be known. We may then apply the implicit method at $m+j$ odd giving a tridiagonal matrix equation to solve. At the next time step the previously implicitly solved rows are solved explicitly and vice-versa. We therefore have only half the number of tridiagonal matrix equations to solve compared with the ADI method.

Line-hopscotch is unconditionally stable if there are no first order terms (i.e $d = e = 0$). Empirical investigations have shown that if first order terms are present quite severe instabilities can occur. It is therefore necessary to transform a general PDE with first order terms (for example one obtained for an option under stochastic volatility) to remove the first order terms. This will typically considerably alter the variables from their fundamental nature and complicate the boundary conditions. Consequently making it difficult to maintain an intuitive understanding of the behaviour of the solution and its results. We therefore believe that the Hopscotch technique is not in general suitable for contingent claim valuation problems. However, more established finite difference techniques, such as the ADI method, do not suffer this problem.

Empirical investigations also show that line-hopscotch is,

(1) Sensitive to a non-uniform space grid.

(2) Sensitive to variable coefficients. Note that variable coefficients lead to variable rates of convergence across the space grid for all finite difference techniques.

(3) Sensitive to poorly constructed derivative boundary conditions.

8 Conclusions

We have presented a development of finite difference techniques for one and two-dimensional problems dealing with the critical issues of stability and convergence as they arise. We described transformations of the PDE which simplify it and facilitate the application of finite difference techniques. The one-dimensional methods of explicit, implicit and Crank-Nicholson were then presented and their stability

and convergence properties noted. Iterative methods which allow successive refinement of the solution were then described. We then gave an introduction to the analysis of stability and convergence. The relationship of finite difference schemes to discrete stochastic processes was described. The two-dimensional techniques of alternating direction implicit and hopscotch were described and we presented some theoretical and empirical analysis of their stability and convergence properties.

One remaining question is that of which is the most efficient method. This is a difficult question as it depends on the trade-off between accuracy and speed which is required. The accuracy criterion is complicated because the different methods will produce the same accuracy of values for different lattice resolutions. Geske and Shastri (1985) make a detailed empirical study of the comparative efficiency of the finite difference and binomial methods.

We conclude by noting that care must be exercised in applying finite difference techniques to option valuation problems especially in the two-dimensional case. It may often prove fruitful to investigate other solution techniques.

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