Interest-Rate Derivatives:

Evolutionary Valuation and Hedging

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I Introduction and aims

Our aim in this article is to show how to apply the evolutionary model of the term structure of interest rates to the valuation and hedging of interest-rate caps in the LIBOR-based money market. We will use the formulation of the evolutionary model as given in Carverhill (1991) and briefly reviewed in Section III below; this model can be regarded as a reformulation of Heath *et al.* (1987) or of Ho and Lee (1986). Also, we will concentrate on the single-factor version of the model, and will make the assumption, associated with the Vasicek model (which is an 'Equilibrium Model' in the terminology of Carverhill (1991)), that the evolution of the term structure as prescribed by the model is independent of the level of the interest rates.

The evolutionary model is contrasted in Carverhill (1991) with the equilibrium model which is represented in such papers as Vasicek (1977) and Cox *et al.* (1985). The basic difference is that the equilibrium model seeks to characterise the term structure at any time, assuming that it is in economic equilibrium, whereas the Evolutionary Model concentrates on the evolution of the term structure from an exogenously given initial condition, rather than on the term structure itself. This basic difference leads to the contrasting strengths and weaknesses of the models; the equilibrium model predicts a shape for the term structure which may not accurately match the actual term structure at any given time, but which one hopes to be a reasonable match over all time; whereas the prediction of the evolutionary model will initially be perfect, but will decrease in accuracy as it is pushed further into the future.

In Section II of this paper we will build up the money-market term structure from LIBOR rates going out to 1 year and swap rates going out to 10 years. We will use data for the year 1988. Also we will not be concerned with liquidity questions, and for each swap rate we will take the average between bid and offer.

In Section III we will briefly review the evolutionary model, and we will show that 84% of the money-market term structure for 1988 can be accounted for by a single factor in this model. It follows from this that it is reasonable to value a cap of term 5 to 10 years using a single factor in the evolutionary model, and to hedge it using a swap of the same 'vintage' as the cap, i.e. a swap that was written at the same time as the cap, and which has the same maturity as the cap. The rate on this 'second-hand swap', will be such that it had zero net present value (NPV) when it was written; however, the current term structure is unlikely to be such that it currently has zero NPV.

In Section IV we show explicitly how to value the cap and hedge it with the second-hand swap as above, in our version of the evolutionary model. Our approach is to break down the cap as a strip of its constituent 'caplets', each of which is an option on a single interest payment, and which can also be regarded as an option to exchange one pure discount bond for another. Each caplet can thus be valued in the evolutionary model using the ideas of Margrabe (1978), which also prescribes a hedge involving the two discount bonds. Finally in Section IV we show how to replace the discount bonds in the hedge by an amount given by the appropriate 'gearing factor' of the secondary swap.

In Section V are some remarks intended to be of use in the practical implementation of this article.

II The UK LIBOR-based money market and its term structure

In this section we show how to build up the money-market term structure from LIBOR rates and swap rates. We will use LIBOR rates for 3 months, 6 months and 1 year, and swap rates for 2, 3, 4, 5, 7 and 10 years, and we will restrict attention to data for the year 1988. Also we will ignore issues of liquidity, and we will take each swap rate to be the average between bid and offer. All our LIBOR and swap data refers to quarterly compounding of the interest payments.

LIBOR (the London Interbank Offer Rate) is the rate at which banks are prepared to lend money to each other, and hence it represents the default-free fixed money-market interest rate. The swap rate is the rate of fixed-interest payments that can be swapped for floating LIBOR payments, and hence the swap rate also represents the fixed rate which can be obtained in the money market, because such a fixed-rate loan can be manufactured from a floating LIBOR loan and a swap. Note that a swap is a zero-NPV transaction; it does not cost any money. However a 'second-hand' swap (i.e. a swap which has only a part of its term left to run, and which is purchased in a secondary market) will usually have a non-zero NPV, because its rate is unlikely to be equal to the swap rate which would prevail in the market, to run till the maturity time of the second-hand swap.

Thus the LIBOR rates and the swap rates together make up the fixed-interest rates on borrowing or lending which are available over the various terms, in the money market. But term-structure models are more conveniently formulated in terms of prices of pure-discount bonds (i.e. which pay just £1 when they mature) over the various terms. To go from one to the other, note that borrowing or lending at a given rate is equivalent to selling or buying a bond whose coupons are equal to the interest payments, and such that the bond is traded at par. Thus to obtain the money-market term structure from our data we first construct the money market 'par yield curve' by interpolating between our data on a quarterly basis, and then we obtain the corresponding bond prices recursively via the formula

$$1 = \left\{ \sum_{i=1}^{n} P(t_0, t_i) \frac{1}{4} c(t_0, t_n) \right\} + P(t_0, t_n). \tag{1}$$

In this formula, t_0 is the current time, and the times t_0, t_1, t_2, are quarterly spaced. Also $c(t_0, t_n)$ is the interest rate for borrowing at t_0 to be paid back at t_n and $P(t_0, t_n)$ is the price at t_0 of the pure-discount bond to mature at time t_n. The par yield curve at time t_0 is represented by the function $c(t_0, t_1)$, $c(t_0, t_2), \ldots,$ and the term structure is represented by the function of prices $P(t_0, t_1)$, $P(t_0, t_2), \ldots$ or equivalently of rates $p(t_0, t_1)$, $p(t_0, t_2), \ldots,$ where

$$p(t_0, t_n) = \frac{-1}{t_n - t_0} \ln P(t_0, t_n).$$

As we have said, a second-hand swap will usually have a non-zero NPV. If the swap relates to future payments at quarterly times $t_1, t_2, \ldots, t_n$, and corresponds to receiving fixed payments of $c/4$ and paying floating LIBOR, and if the current time t is a proportion α_t of the way past the previous payment date, then the swap value is

$$s(t, t_n, c) = \left\{ \left(\frac{c}{4} \right) \sum_{i=1}^{n} P(t, t_i) \right\} + \{P(t, t_n) - 1\} - \alpha_t L_{t_1}. \tag{2}$$

The last term arises because if one lends from time t to t_n at floating LIBOR with payments at times $t_1, t_2, \ldots, t_n$, then the first payment will be a proportion $(1 - \alpha_t)$ of the LIBOR rate L_{t_1}. (Also this payment will be known at time t. Note that we are pro-rating the interest payment, which might be considered crude.) But the swap corresponds to paying the whole of L_{t_1} and receiving $(c/4)$, at time t_1.

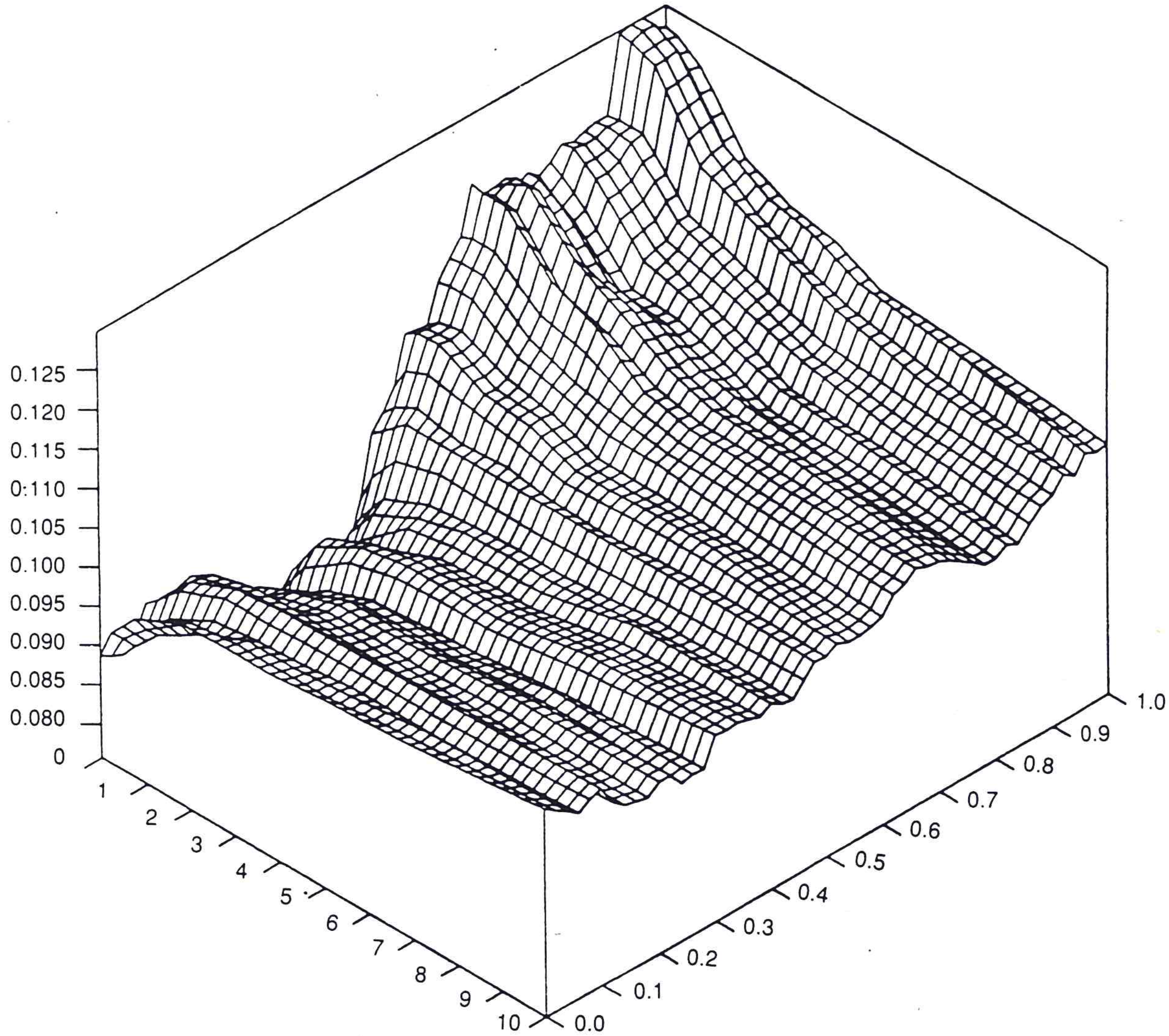

Figure 9.1 The British money-market term structure in 1988

Figure 9.1 represents the term structure calculated on a weekly basis over 1988.

III **The evolutionary model applied to the LIBOR money market**

We will now briefly review the evolutionary model of the term structure of interest rates. As already said, we will restrict attention to the single-factor version of the model. This is reasonable in view of the fact, explained below, that 84% of the evolution can be accounted for by a single factor in the model. Also we will assume that the evolution of the term structure, as prescribed by the model, is independent of the level of the interest rates. The evolution is characterised by the model in terms of the drift and volatility of the rates or prices, as functions of the term to maturity, and the assumption that these functions do not depend on the level of rates does lead to the theoretically unsound conclusion that the rates can become negative. However, as argued in Carverhill (1991), the probability of this is too small to be detrimental to option values, and so this assumption is worth making in view of the great technical simplifications which it makes possible.

Let us denote by P_t^q the price at time t of the pure-discount bond to mature at time q, and by p_t^q the associated spot rate, i.e.

$$p_t^q = \frac{1}{(q-t)} \ln P_t^q.$$

Then the evolutionary model can be formulated in any of the three following equivalent ways, as explained in Carverhill (1991):

1. (Price-based formulation)

$$\frac{\mathrm{d}P_t^q}{P_t^q} = (v_t^q + r_t)\,\mathrm{d}t + \mu_t^q\,\mathrm{d}B_t$$

   where r_t is the short rate at time t, and B_t is a standard Brownian motion.

2. (Rate-based formulation)

$$\mathrm{d}p_t^q = \left(\frac{1}{q-t}\right)(p_t^q - r_t)\mathrm{d}t + \chi_t^q\,\mathrm{d}t + \psi_t^q\,\mathrm{d}B_t.$$

3. (Rate-based formulation with time difference instead of differential)

$$p_{t+\varepsilon}^q = g_{t\,t+\varepsilon}^q + \chi_t^q\,\Delta_t^{t+\varepsilon}t + \psi_t^q\,\Delta_t^{t+\varepsilon}t\,B,$$

   where

$$\Delta_t^{t+\varepsilon}B \equiv B_{t+\varepsilon} - B_t \sim N(0,\varepsilon),$$

$$\Delta_t^{t+\varepsilon}t \equiv (t+\varepsilon) - t \equiv \varepsilon,$$

$$g_{ts}^q = \frac{-1}{q-s} \ln G_{ts}^q,$$

   where, $G_{ts}^q = P_t^q / P_t^s$. Thus g_{ts}^q is the forward rate obtainable at time t, to cover the period $[s, q]$.

In these formulations v_t^q and μ_t^q are respectively the drift and volatility of the pure-discount bond price as functions of the term to maturity, and χ_t^q and ψ_t^q are the drift and volatility of the rate. Our Vasicek-type assumption is that these functions do not depend on the term structure itself. We will also make the time-homogeneity assumption that these functions depend only on $(q-t)$. These functions are related via the equations

$$v_t^q = -(q-t)\,\chi_t^q + \frac{1}{2}(q-t)^2\,(\psi_t^q)^2,$$

$$\mu_t^q = -(q-t)\,\psi_t^q.$$

As explained in Carverhill (1991), these various formulations of the evolutionary model are useful in different situations.

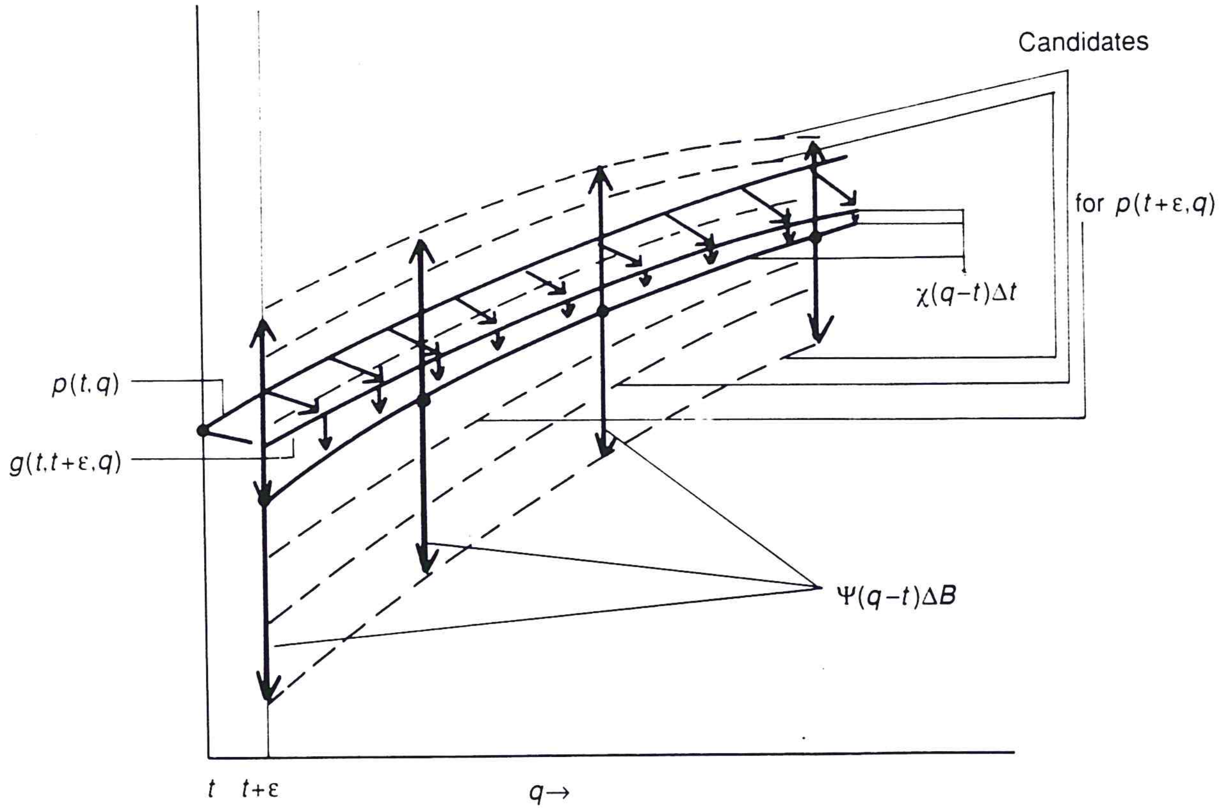

Figure 9.2 Graphical representation of the Evolutionary Model

tion is appropriate for empirical estimation of the model. Figure 9.2 is a graphical representation of this formulation.

In order to value and hedge an option via the evolutionary model it is necessary to know the volatility function ψ (or equivalently μ) but not the drift function χ (or equivalently υ). (See Section IV below.) This is fortunate because the volatility is much easier to estimate. To do this we will begin with the weekly differences between the term structures in 1988. These functions are represented in Figure 9.3. Each of the difference functions which make up this figure can be expressed as

$$\{p(t + \varepsilon, q + \varepsilon) - p(t, q): q \text{ runs from } t \text{ to } t + 10 \text{ years}\},$$

where ε is 1 week. Note that these difference functions are simplistic, though not to an empirically significant extent; to conform strictly to the evolutionary model one should work with

$$p(t + \varepsilon, q) - g(t, t + \varepsilon, q) \equiv p(t + \varepsilon, q) - p(t, q) - \frac{1}{q - t}\,(p(t, q) - r(t)).\ \varepsilon,$$

where $r(t)$ is the short rate.

To estimate the volatility function ψ we use the technique of principal-component analysis, as described in Carverhill (1991) (see also Lawley and Maxwell, 1971). This involves calculating the covariances of the difference functions between pairs of terms, to form a covariance matrix. Then the

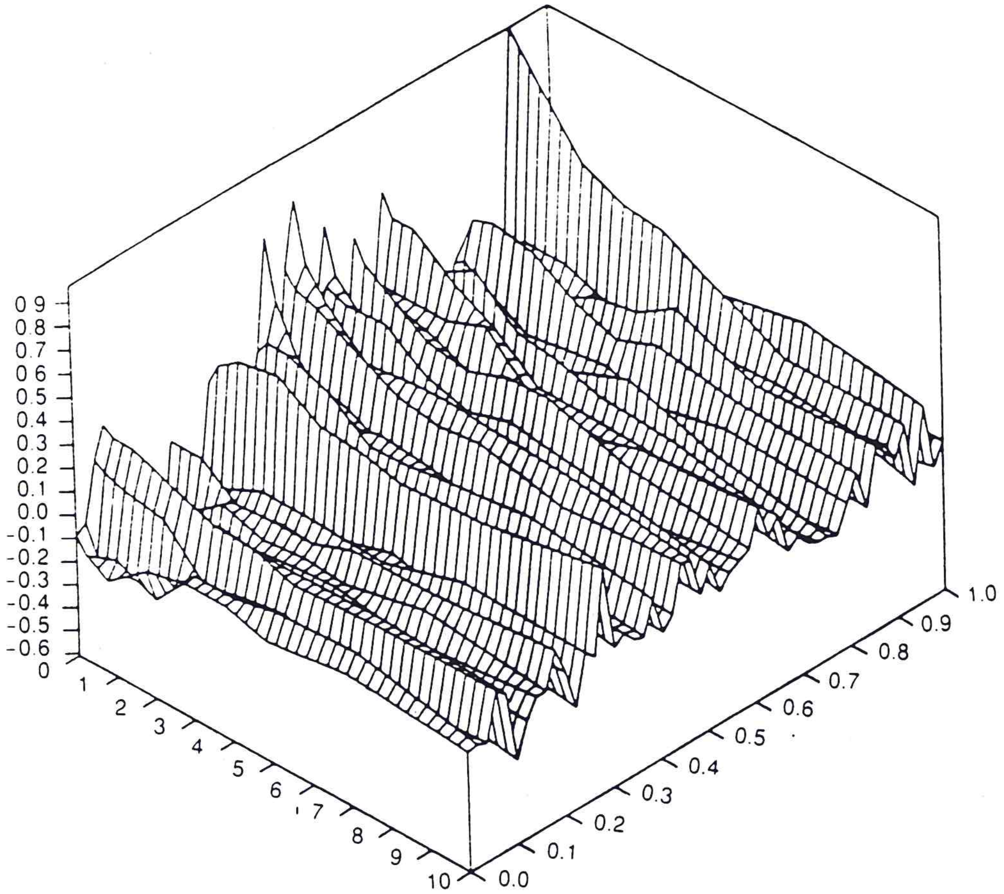

Figure 9.3 Weekly differences in the British term structure in 1988

principal component of the evolution of the term structure is the principal eigenvector of this matrix, and the principal eigenvalue (as a proportion of the sum of all the eigenvalues) tells us what proportion of the evolution is accounted for by the principal component. The second eigenvector and eigenvalue tell us about the second component of the evolution, and so on. The principal component is the most faithful representation of the evolution in terms of a single component ('most faithful' being in a least-squares sense), and so it is the best choice of volatility function ψ in the 1-factor evolutionary model.

The first two eigenvalues of the covariance matrix based on the data of Figure 9.3 are 13.77 and 1.70, and the sum of all the eigenvalues is 16.46. Thus, the principal component accounts for 84% of the term-structure evolution and the second factor accounts for 10%.

The graphs of the principal and second components are shown in Figures 9.4 and 9.5. These volatilities are expressed in terms of percentage of the principal amount of the loan, and on an annualised basis. Note that they are absolute, rather than proportional, volatilities.

Note that the principal component attenuates as term-to-maturity increases. This reflects the fact that the short rate moved more than the long rate in 1988, as the term structure became inverted. When valuing options

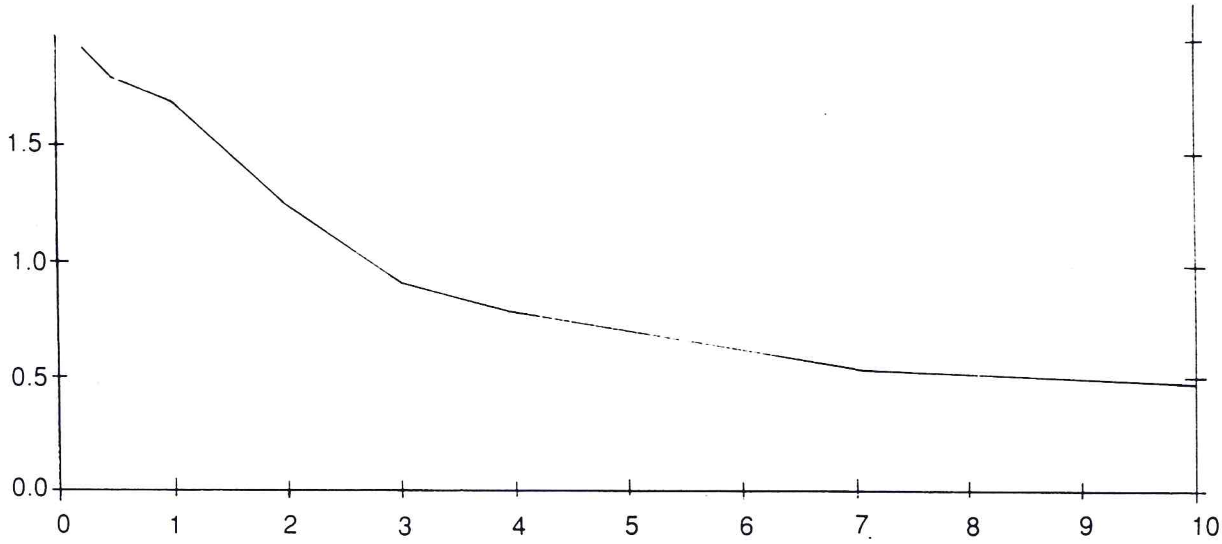

Figure 9.4 Principal component of the term-structure movement over 1988

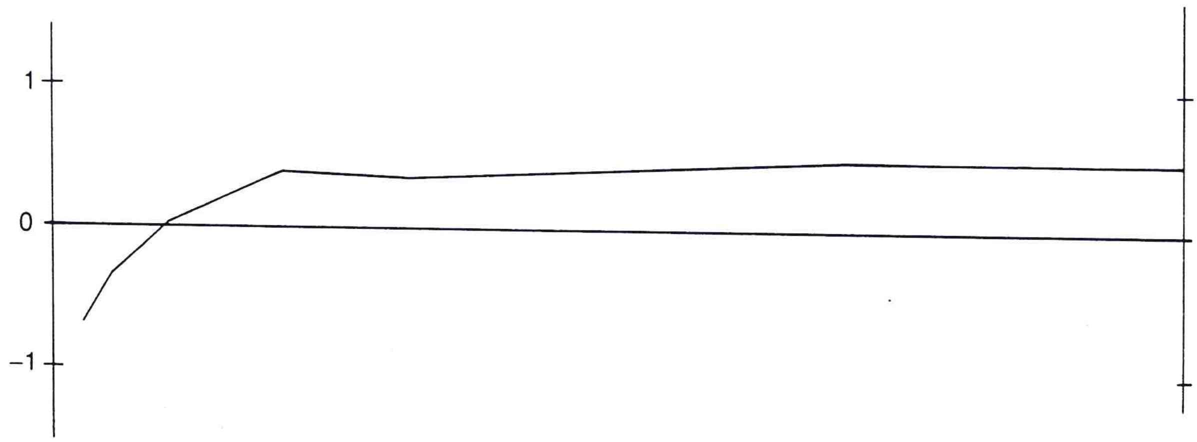

Figure 9.5 Second component of the term-structure movement over 1988

it might be wise to take a higher volatility than in Figure 9.4 for longer terms-to-maturity, because these have to predict the volatility into the quite distant future.

IV **Valuing and hedging an interest-rate cap via the evolutionary model**

A cap on 3-month LIBOR compensates its holder whenever the quarterly floating LIBOR interest payment on borrowing the appropriate principal amount rises above the strike rate of the cap. Thus, the cap is really a strip of call options on each of the interest payments.

As has been pointed out, for instance in Hull (1989), a call option on an interest rate can be thought of as a put option on a bond; if the call is European, matures at time s, and caps the simple interest rate at $\bar{r}$ for borrowing a principal amount K over the time interval $[s, q]$, then it is equivalent to a put option to sell at time s an amount $K[1 + (q - s)\bar{r}]$ of the pure-discount bond to mature at time q, at a price K (i.e. a price $1/[1 + (q - s)\bar{r}]$ per unit of the bond).

Let us verify this equivalence. The above interest-rate call will be exercised if the actual interest rate r (simply compounded, over the interval $[s, q]$) is greater than $\bar{r}$. If $r > \bar{r}$ then the actual and avoided cash flows are:

Time	s	q
Actual cash flow	K	$-K[1 + (q - s)\bar{r}]$
Cash flow in the absence of the call	K	$-K[1 + (q - s)r]$

This gives a saving of $K(q - s)(\bar{r} - r)$at time q, which is equivalent to $\{K(q - s)(\bar{r} - r)\}/\{1 + (q - s)r\}$ at time s.

The above bond put will be exercised if $P(s, q) < 1/[1 + (q - s)\bar{r}]$, i.e. if $\bar{r} > r$, and the actual and avoided cash flows are:

Time	s	q
Actual cash flow	K	$K[1 + (q - s)\bar{r}]$
Cash flow in the absence of the put	$\dfrac{K[1 + (q - s)\bar{r}]}{[1 + (q - s)r]}$	$K[1 + (q - s)\bar{r}]$

This again gives a saving of $\{K(q - s)(\bar{r} - r)\}/\{1 + (q - s)r\}$ at time s.

In fact this option can be thought of as the option to exchange one asset for another, and so it can be valued using a Black–Scholes–Margrabe formula (see Margrabe, 1978). Our option is equivalent to the option to give at time s an amount $K[1 + (q - s)\bar{r}]$ of the q-maturity bond, in return for receiving an amount K of the s-maturity bond. Using the equation of Section III for the price processes of these bonds (with q replaced by s for the s-maturity bond) then Margrabe's formula for the value at time t of this option is

$$KP(t, s)\, N(d_1) - K[1 + (q - s)\bar{r}]\, P(t, q)\, N(d_2), \tag{3}$$

where

$$d_1 = \left[\log\left\{\frac{P(t, s)}{[1 + (q - s)\,\bar{r}]\, P(t, q)}\right\} + G/2\right] / \sqrt{G}, \tag{4}$$

$$d_2 = d_1 - \sqrt{G},$$

$$G = \int_{p=t}^{s} [\mu(q-\rho) - \mu(s-\rho)]^2 d\rho. \quad (5)$$

(Recall that μ is the proportional bond volatility in terms of term-to-maturity, and it is related to the absolute interest-rate volatility ψ via $\mu(s-\rho) = (s-\rho)\ \psi(s-\rho)$.)

The Margrabe hedging procedure for this option is to take an amount $-KN(d_1)$ of the s-bond, and an amount $+\ K[1 + q - s)\bar{r}]\ N(d_2)$ of the q-bond.

The conclusion thus far concerning valuing and hedging an interest-rate cap via the 1-factor Evolutionary Model is the following. The value of the cap is the sum of the value of its constituent calls, and each of these can be valued using the Black–Scholes–Margrabe formula (3–5). Each of these calls relates to an interest-rate payment (in arrears) concerning a period, say $[s, q]$, and the call can be hedged using the pure-discount bond to mature at time s, and that to mature at time q, if these bonds exist in the market.

If the cap is sufficiently short-lived so that each of its constituent calls can be hedged with a short sterling future, then its value and hedging procedure can be obtained by elementary Black–Scholes techniques, and these will be very similar to the above (see Carverhill, 1990).

We now show how to hedge the cap with a single hedging instrument, via the 1-factor evolutionary model. According to this theory all price movements of interest-rate instruments are perfectly correlated with one another, because the term structure is driven by a single random input. Therefore, having chosen any hedging instrument, we can substitute any other hedging instrument for it, and we should adjust the hedge ratio by dividing the 'gearing' factor between the two hedging instruments. By 'gearing' between two instruments we simply mean their relative sensitivity to the noise; if the prices of the instruments evolve via

$$dP^1/P^1 = \alpha^1 dt + \sigma^1\ dB_t,$$

$$dP^2/P^2 = \alpha^2 dt + \sigma^2\ dB_t,$$

then the gearing of P^2 against P^1 is σ^2/σ^1. To hedge the cap with a single instrument, take the hedge ratio to be the sum of the hedge ratios for each underlying bond, adjusted by the gearing factor against the chosen instrument.

As stated in Section I, our aim is to hedge the cap with a secondary swap, whose initial and maturity times are the same as those of the cap. If the current time is t and the swap and cap refer to future interest payments at times $t_1, t_2, \ldots, t_n$, and the rate for this swap is c, then from equation (2) its value satisfies the equation

$$dS(t, t_n, c) = \frac{1}{4} c \sum_{i=1}^{n} \{P(t, t_i)\,[v(t_i - t) + r(t)]\ dt + \mu(t_i - t)\ dB_t\}$$

$$+ \{P(t, t_n)\,[v(t_n - t) + r(t)]\,dt + \mu(t_n - t)\,dB_t\} - L_{t_i}.$$

Also the gearing of this instrument against the t_j-maturity bond is given by

$$\frac{\frac{1}{4}c\sum_{i=1}^{n}\{P(t, t_i)\,\mu(t_i - t)\} + \{P(t, t_n)\,\mu(t_n - t)\}}{P(t, t_j)\,\mu(t_j - t)}. \tag{6}$$

The hedge ratio of the cap against the swap of the same vintage is thus obtained by taking the above hedge ratios against the pure-discount bonds, dividing by the appropriate version of the gearing factor (6), and then summing.

V Some remarks concerning implementation

In order to bring out some points concerning the practical implementation of our procedure for hedging a cap with a swap having the same initial and maturity dates, we will consider just one caplet of the cap, namely the one that matures in $\frac{1}{4}$ year. This caplet is thus an option to give away in $\frac{1}{4}$ year's time an amount $K(1 + \bar{r}/4)$ of the bond which pays 1 in $\frac{1}{2}$ year's time, in return for receiving an amount K of the bond to mature in $\frac{1}{4}$ year's time, i.e. at the same time as the option.

We will present the results of a numerical comparative statics exercise and show how the values change and the hedge performs when the term structure is flat (i.e. constant as a function of term to maturity), and changes in jumps of 0.00125 as time evolves. This amount represents a jump of 1 daily standard deviation if the annual volatility of the term structure is flat at 2%. Also, in this exercise, we take $K = 1$, and the strike rate and rate on the hedging SWAP both to be 10%. The results are shown in Table 9.1.

The first three columns of the table show the interest rate and the corresponding option value and hedge ratio (which indicates the number of units of the swap to hold in order to hedge the option). Column 4 shows the value of the swap. Finally, columns 5 and 6 show the change in the option value and hedge portfolio which result from a 0.125% increase in the interest rate to the rate shown. It can be seen that the portfolio captures most of the change in the option value, but that there is a hedging error of roughly half the change in the (column 5) option value. This is what we would expect as attributable to the gamma (i.e. the sensitivity to hedging) of the option. When the option is at-the-money this error is about 1% of the value of the option and 5% of the change in this value. This hedge with the swap is almost identical to the hedge with the pair of discount bonds.

Table 9.1 Table showing the performance of the swap for hedging the cap

r (interest rate)	*Option value*	*Hedge ratio using the swap*	*Swap value*	*Change in the options value*	*Change in the hedge portfolio using the swap*
0.09000	0.000256	−0.006708	−0.058547	0.000053	0.000048
0.09125	0.000320	−0.008050	−0.049936	0.000063	0.000058
0.09250	0.000395	−0.009544	−0.041416	0.000075	0.000069
0.09375	0.000482	−0.011184	−0.032984	0.000087	0.000080
0.09500	0.000582	−0.012956	−0.024641	0.000101	0.000093
0.09625	0.000697	−0.014842	−0.016386	0.000115	0.000107
0.09750	0.000826	−0.016822	−0.008216	0.000129	0.000121
0.09875	0.000971	−0.018868	−0.000132	0.000144	0.000136
0.10000	0.001130	−0.020953	−0.007869	0.000159	0.000151
0.10125	0.001304	−0.023047	−0.015785	0.000174	0.000166
0.10250	0.001493	−0.025123	−0.023620	0.000189	0.000181
0.10375	0.001695	−0.027151	−0.031373	0.000203	0.000195
0.10500	0.001911	−0.029107	−0.039046	0.000216	0.000208
0.10625	0.002139	−0.030970	−0.046639	0.000228	0.000221
0.10750	0.002379	−0.032724	−0.054153	0.000239	0.000233
0.10875	0.002628	−0.034357	−0.061859	0.000249	0.000243
0.11000	0.002887	−0.035861	−0.068949	0.000258	0.000253

Note

I would like to thank Chiang Fu for valuable discussions relating to the work presented here; also Les Clewlow for help with computing work; and Hill Samuel and Midland Montagu for providing the interest-rate data that I have used.

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