A Survey of Elementary Techniques for Pricing Options on Bonds and Interest Rates

Andrew P Carverhill

Senior Research Fellow
Financial Options Research Centre
University of Warwick

April 1990

I would like to thank Simon Babbs for clarifying to me many of the details and issues which are included in this article.

I would like to emphasise that, notwithstanding the help I have received in my preparation of this paper, I take responsibility for all errors and peculiarities of emphasis that are almost impossible to eradicate in an article such as this.

Financial Options Research Centre
School of Industrial and Busines Studies
University of Warwick
Coventry
CV4 7AL
Phone: 0203 523606

FORC Preprint: 90/9

A Survey of Elementary Techniques for Pricing Options on Bonds and Interest Rates

Andrew Carverhill

1. Introduction

The pricing techniques that we will survey in this article are the most commonly used by practitioners who write and trade options on bonds and interest rates, and they are all based on the techniques of Black and Scholes as applied to the more elementary problem of pricing options on equities. To implement these techniques we only need to make an appropriate assumption about the volatility of the bond or interest rate in question. We will distinguish between, and we will treat separately, options on bonds and options on interest rates. The former relate to actual Government securities or their associated futures contracts and the latter relate to short money market interest rates such as 3–month or 6–month LIBOR, or their associated futures contracts.

The pricing procedures of this article should be contrasted with the more sophisticated procedures which are based on mathematical models of the term structure of interest rates, ie, models of the comovements of the interest rates over the whole spectrum of maturities. In a planned sequel to this article we will survey these more sophisticated pricing procedures and compare them with the procedures of this article. The term structure models for these more sophisticated valuations are separated into two classes, which we refer to as 'equilibrium' models (represented by the models of [V] and [CIR]), and 'evolutionary' models (represented by the models of [HL], [HJM], [C_1] and [C_2]).

The plan of this article is as follows:

In Section 2 we give a survey of the options on bonds and interest rates which are commonly available either over the counter or on an exchange. To do this we must first survey the underlying instruments on which the options are written. We go into some detail in this section, as is necessary to understand the subtleties of the pricing procedures.

In Section 3 we show how to price the interest rate options.

In Section 4 we discuss some aspects of the Government Securities Market, including the concepts of duration, volatility and the yield curve. (NB, this sense of 'volatility' is different from the option–theoretic sense of volatility, as we will explain below, and this nomenclature is actually restricted to the UK. In the US it would be known as the sensitivity of the bond price to parallel shifts of the yield curve.) These concepts will be used in our final Section 5, in which we show how to price bond options.

2. A Survey of Options on Bonds and Interest Rates

In this section we give a survey of the most important of these instruments. In order to do this we must first survey the underlying primary instruments, and we must go into some institutional detail, which will be important later when we discuss valuation procedures.

We will deal separately with options on interest rates and options on bonds. By "options on interest rates" we mean caps, floors, LIBOR futures options ("short sterling options" in the UK) etc, which are settled against "money market" interest rates such as LIBOR. When valuing these it is appropriate to look at the "money market term structure" of interest rates, ie, the interest rates across the spectrum of maturities that are determined by the "money market" instruments such as LIBOR, LIBOR futures, swaps, FRA's, etc. By "options on bonds" we mean options which involve Government Securities or their associated futures contracts. For valuing these the "Government Securities term structure" is appropriate. Since the money market and government securities term structures are not well correlated in the UK, it might be rather inefficient to hedge across these term structures, ie, to offset the risks of a money market instrument by buying (or selling short) a government securities instrument, or vice versa. These term structures are better correlated in the US.

The Money Market Debt (Interest Rate) Instruments

First we survey the primary money market debt instruments on which the money market options are based. The simplest of these is LIBOR, which is the rate for borrowing money for 3, 6, 9 or 12 months. The LIBOR rates are widely advertised and they refer to borrowing at 'spot', ie, (virtually) immediately, with the interest payment in arrears, ie, at the end of the period. Going beyond LIBOR is the Forward Rate Agreement (FRA), which is made OTC (ie, 'over the counter', privately between two counterparties, rather than via an exchange). The purpose of an FRA is to guarantee the rate on borrowing a nominal sum over a certain time and to begin at a certain date in the future. There is no transfer of cash when the FRA is made, but if the rate turns out to be different from that of the agreement when the period of borrowing begins, then one party pays to the the other the difference in the interest payments on the nominal sum.

Interest rate futures are exchange traded at LIFFE in the UK, and for sterling they are called "short sterling futures". The futures contract refers to 3 month LIBOR and its price is quoted as 100–LIBOR interest rate which is being fixed by the contract. The counterparties to the contract are legally trading with the exchange, and their contracts are marked to market in the following way: when the contract is purchased (sold short) there is no cash transaction, but at the end of that day the purchaser (seller) pays the exchange the amount by which the contract has lost (gained) since the transaction. Thus, the exchange will send cash to the purchaser (seller) if the contract gains (loses), and the net effect is that the cash is transferred between buyer and seller. On subsequent days the purchaser (seller) pays the amount by which the contract loses (gains) during the day, and the maturity value is taken to be 100–LIBOR. The exchange requires the counterparties to put up a margin which might be adjusted during the time the contract is held, and if this margin requirement cannot be met by a counterparty, then their position (but not that of the other counterparty) is transferred to a new counterparty. These trading rules ensure that the counterparties are not at credit risk from each other, and that taking a position does not require large cash transactions. This has led to the futures market being very liquid and efficient, with bid–ask spreads of only one 'tick', ie $\frac{1}{100}$ of a percent of the nominal principal.

Some more detail about the short sterling future: one future contract refers to borrowing a nominal sum of £$\frac{1}{2}$m for 3 months. If a counterparty is long one of these contracts then he will receive a variation margin payment of £12.50 for each 'tick' (ie, 0.01) by which the price increases over each day. This amount £12.50 is 0.01% of interest on £$\frac{1}{2}$m, not compounded, over 3 months. If the futures price is say 89.23, then his long position will effectively guarantee an interest rate of 100–89.23 = 10.77% on borrowing £$\frac{1}{2}$m for 3 months. In order to clarify the LIFFE margining system for futures and options we give in the Appendix some examples which illustrate the transactions involved in taking a position on LIFFE.

Finally we mention interest rate swaps, which are agreements to swap fixed and floating interest rate payments on a nominal sum. As with FRA's, these agreements are made OTC, and there is no cash transfer when the swap is begun. Swaps are often available for periods beyond 5 years. The 'swap rate' for a particular period is indicated on brokers screens, and it is the rate of fixed payments that can be swapped for LIBOR over that period. This swap rate can be taken simply as the fixed interest rate at which the market allows money to be borrowed or lent over this period. To see this note that if one borrowed at this fixed rate, then one could cancel this transaction completely by lending the money at LIBOR and entering into a swap.

Options on Interest Rates

The main exchange traded interest rate option is the option on the future interest rate contract, which (future) is described above. In the UK, the 'short sterling option', which is an option on the short sterling future contract, is exchange traded on LIFFE and is liquid up to about a year into the future.

A given short sterling option refers to a specific future contract, which matures at the same time as the option, and the option can be exercised at any time up to this maturity time. (Note that these options are rarely exercised early, and it can be we shown theoretically that early exercise is never advantageous.) The purchaser of the option is entitled to buy or sell the future contract from or to the seller at a price which is fixed by the option contract. Also the option marks to market in the same way as the future itself, and the price of the option is transferred between buyer and seller when the option is exercised, and not when the option is purchased. Thus, there is no cash transaction when the option is written, but there may be a positive or negative cash flow between writer and buyer as it marks to market, and so both parties have to maintain margin accounts with LIFFE. If held to maturity the short sterling option fixes a ceiling or floor on the interest rate for a 3 month loan in the future. In the Appendix we give an example illustrating the transactions involved in taking a short sterling options position on LIFFE. (NB. A short sterling call (put) pays off if the price of the future contract ends up higher (lower) than the strike price. Therefore, since a higher futures price corresponds to a lower interest rate, then a short sterling call (put) is really a put (call) on the interest rate.)

Many interest rate options trade OTC, for example options on FRA's, but the most important OTC options are caps, floors, collars and 'swaptions' (ie, options on swaps). If a loan is taken out on which the interest is paid say every 3 months, and floats with LIBOR, then a cap gives insurance against this rate exceeding a given rate (the cap rate), by paying the difference if LIBOR is higher. A floor similarly guarantees to a lender that the interest rate payments do not fall below the floor rate. An interest rate collar guarantees that the rate is constrained between two bounds, and is thus equivalent to a cap together with a short floor. If the provider of the loan is also providing the associated option, then the price of the option may be charged as an interest rate premium. Otherwise it will usually be charged as an up–front payment.

A swaption is the option to enter into a swap at a future date. It may be European or American, and if American, then either the term from the exercise date may be fixed in the swaption contract, or the end date of the swap. Finally, a swap can be extendible or terminable, and both of these features can be thought of in terms of swaptions. Note that a swaption can be thought of as the option to borrow or lend in the future at a given interest rate.

Government Security Debt Instruments and Related Options

The basic difference between government securities and the money market instruments described above, is that since the former are issued by the Government, there is thought to be no default risk associated with them. The default risk of the money market interest rate instruments is reflected in their rates being slightly higher.

The basic government security is the government bond ('gilt' in the UK). The government bond with coupon c percent will pay this proportion of the principal or face value of the bond, usually in the form of two six–monthly payments of $\frac{c}{2}$, and at maturity it will pay this together with the principal. In the UK there are about 100 such bonds with coupons ranging between 3% and 15%, and maturity dates which are currently up to 30 years into the future. The Government Bond Market is very sophisticated, with analysts doing arbitrage among them. The effect of this is that the market is very efficient, and it gives rise to a tightly determined Government Securities term structure, ie, for any given time to maturity (ie, 'term') there is a tightly determined interest rate. Options can be traded on any of these bonds.

In addition to these bonds there are index linked bonds in the UK whose coupons float with an inflation index. Also there are consul bonds ('perpetuities') which pay a coupon indefinitely, except that the Government has the right to redeem them at par value.

There are two UK Government Securities futures contracts which are exchange traded on LIFFE. These are the short and long gilt futures. We will describe the long gilt future; the other is similar, but this contract has an option contract associated with it. For the future and the option, the mark–to–market rules are applied, as described above for the short sterling option. Also the option is paid for not when it is purchased, but when it is exercised. (See the Appendix for more details on the long gilt future.) When the future contract expires the question arises of which long gilt should be delivered and how much. Some time before the future matures, LIFFE publishes a list of the stocks that can be delivered on each day of the month to satisfy the contract. It also publishes a conversion factor for each stock on the list, such that the current clean price of the stock multiplied by the conversion factor gives the clean price that the stock would have if its coupon were 9%. The writer of the gilt futures contract has the option to deliver any of the gilts listed for each day, and on any day in the delivery month. He will choose the 'cheapest–to–deliver' gilt, and the question of which is cheapest to deliver (the 'quality option') is an important and perhaps difficult one. Note that this question impinges only indirectly on the problem of valuing the option on the futures contract itself because the value of cheapest–to–deliver option is already impounded into the futures price.

3. Pricing Options on Interest Rates

As we have said already, the techniques that we will describe in this section are the most widely used in practice, and they involve the elementary Black–Scholes ideas, which are applied taking the interest rate itself as the pricing variable. Our plan for this section is to obtain several versions of the Black Scholes equation and corresponding Black Scholes solution formulae, which apply in various situations, and then to discuss which situations apply to each of the examples of interest rate options surveyed in Section 2.

Our basic assumption about the interest rate on which the option is written (and which we will denote by r_t) is that it has constant proportional volatility, ie, it satisfies the stochastic equation

$$dr_t = \alpha(r_t)\,dt + \sigma r_t\,dB_t \tag{3.1}$$

where the proportional volatility parameter σ is a positive constant, and in which the form of the drift function α is not relevant to option valuation.

Some notes concerning volatility:

i. The assumption of constant proportional volatility (ie, equation (3.1)) with a sensible choice of α, has the virtue of predicting that the interest rate will never become negative, as well as allowing an orthodox application of the Black–Scholes ideas. However, in Section 5 we will assume that the bond prices have constant proportional volatility (equation (5.1)), and this corresponds to the yield and interest rate having constant absolute volatility. This allows us to apply the orthodox Black–Scholes ideas to bond options, but it is inconsistent with equation (3.1). Actually, the Black–Scholes ideas can easily be adapted to dealing with constant absolute volatility. Also over a short time (say up to a year) it would be difficult to favour on empirical grounds constant proportional volatility over constant absolute volatility.

ii. As we have explained, interest rate futures prices are quoted as 100% – interest rate that they are guaranteeing. Some practitioners will apply the Black–Scholes formula directly to the futures price, and this corresponds to assuming that 100% – interest rate has constant proportional volatility. In fact since the interest rate is unlikely to go above say 20%, then this assumption is close to the assumption of constant absolute volatility.

Interest Rate Option Valuation in General

The Black–Scholes equation for the value ϕ of the option that paid for up–front when it is purchased, and that is written on the interest rate r_t of equation (3.1), is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} r^2 \sigma^2 \frac{\partial^2 \phi}{\partial r^2} - r \bar{r} \frac{\partial \phi}{\partial r} + \bar{r} \phi. \qquad (3.2)$$

In this equation $\bar{r}$ is the short interest rate which we assume to be continuously compounded, and which must be assumed to be constant in order to apply the Black–Scholes ideas. The Black–Scholes Formula associated with (3.2) (ie, its solution for European options) is

$$C_t(r) = r\, N(d_1) - e^{-\bar{r}\,(T-t)}\, x\, N(d_2) \text{ for a European call,}$$
$$(3.3)$$
$$P_t(r) = x\, e^{-\bar{r}(T-t)}\, N(-d_2) - r\, N(-d_1) \text{ for a European put.}$$

where

T is the maturity time,

x is the strike rate,

N(–) is the univariate cumulative normal distribution function,

$$d_1 = \left[\ \ln(r/x) + (\bar{r} + \sigma^2/2)\,(T-t)\right] / \left[\sigma \sqrt{T-t}\,\right],$$

$$d_2 = d_1 - \sigma \sqrt{T-t}.$$

Note that in obtaining equation (3.2) and formula (3.3) we are thinking of the interest rate r_t as the value of an asset. We can take it to be the actual interest payment for borrowing a nominal sum of money over a certain period of time. If we do this then the payoff of the option can also be thought of in terms of an interest payment on the nominal sum. Thus, a call at strike rate x pays on exercise the interest payment r_t minus x, and so it in effect guarantees a ceiling (or 'cap') of x on the interest rate for borrowing this nominal sum. Similarly a put guarantees a minimum rate (or 'floor') on lending the nominal sum.

Note also that we assume that the 'asset value' r_t obeys equation (3.1) even though we also assume that the short rate $\bar{r}$ is constant. This is clearly inconsistent, because if the short rate were constant then all interest rates would be equal; actually it would be more reasonable to assume that the short rate were equal to r_t itself. In fact, this inconsistency does not affect the option valuation as seriously as one might imagine, provided the option is not too long lived (say less than a year); it merely introduces a small discrepancy into the discounting. This assertion is justified in [C_3].

Finally, note that in deriving equation (3.2) and formula (3.3) one has to assume (see below) that the option premium is paid up front when it is purchased, and its payoff comes when it is exercised, and both of these assumptions are inappropriate in many situations. For instance a LIFFE style option is paid for not when it is purchased ut when it is exercised (see the Appendix); also if the payoff comes in the form of an adjustment to an interest rate which is paid in arrears, then the payoff comes after the option is exercised.

In order to prepare for remarks to be made later, we now mention some of the detail in deriving the Black–Scholes equation (3.2). The key to this is to form the hedged portfolio comprising one option and a quantity δ ('delta') of the underlying, whose value is r_t. The value of the hedged portfolio at time t is

$$\phi_t + \delta r_t, \tag{3.4}$$

and over a short time Δt this value changes by

$$\frac{\partial\phi}{\partial t}\Delta t + \frac{\partial\phi}{\partial r}\Delta r + \frac{1}{2}r^2\sigma^2\frac{\partial^2\phi}{\partial t^2}dt + \delta\,\Delta r, \tag{3.5}$$

where Δr is the increment $r_{t+\Delta t} - r_t$. To obtain (3.5) from (3.4) we must use the Ito Formula to obtain the increment ϕ_t, which is a function of r. To be hedged the increment (3.5) must be free of the stochastic term Δr, and for this we require $\delta = \frac{-\partial\phi}{\partial r}$, so that the increment is just

$$\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}r^2\sigma^2\frac{\partial^2\phi}{\partial t^2}\right]\Delta t \tag{3.6}$$

Now, since the hedged portfolio is riskless, it must earn the riskless interest rate $\bar{r}$ to prevent arbitrage, ie, we must have

$$\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}r^2\sigma^2\frac{\partial^2\phi}{\partial t^2}\right]\Delta t = \left[\phi_t - \frac{\partial\phi}{\partial r}r_t\right]\bar{r}\,\Delta t, \tag{3.7}$$

which is equivalent to the Black Scholes equation (3.2). From these remarks we see that the second term on the RHS of the Black Scholes equation (3.2) is the 'cost–of–carry' term for the underlying, ie, it corresponds to the interest at rate $\bar{r}$ forgone when cash is invested in the underlying part of the hedged portfolio; also the last term on the RHS of (3.2) is the 'cost–of–carry' term for the option itself.

Now we value an option which is paid for up–front when it is purchased, and is written on the forward or future contract corresponding to the interest rate r_t given by equation (3.1). Recall that a forward contract is made OTC and its premium is paid when it matures, and a future contract is exchange traded and marks to market. First note that an elementary arbitrage argument shows that if the short rate is a constant $\bar{r}$ and if there are no transactions costs, then the price f_t^T of the forward contract on r_t to mature at time T is given by

$$f_t^T = e^{\bar{r}(T-t)} r_t \qquad (3.8)$$

(NB, this price must be paid at time T. The present value of the forward is $e^{-\bar{r}(T-t)} f_t^T \equiv r_t$, and this is why no arbitrage is possible. Note that the futures prices f of this section correspond to 100–f in the LIFFE convention for quoting futures prices.) Also, a not so elementary arbitrage argument (see [Hull p60]) shows that if the short rate $\bar{r}$ is constant then the forward and future prices will be the same, and these being the same is borne out empirically to high accuracy. (NB, the future price will be paid via daily variation margin payments and a final payment at time T. See the Appendix.) Thus, when valuing options on both futures and FRA's, participants in the market can estimate volatility etc from the futures prices. In fact this will be the usual practice because being a very liquid and efficiently traded instrument, the future price is most easily visible. Finally, if we assume equation (3.8) and apply to Ito Formula to it, then we can deduce that the volatility of the forward or future contract is the same as that of the underlying interest rate r_t itself. Let us denote by σ the proportional volatility of future, forward and spot interest rate r_t.

The Black Scholes equation for an option on the forward or future interest rate, paid for up front is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} f^2 \sigma^2 \frac{\partial^2 \phi}{\partial f^2} + \bar{r}\,\phi \tag{3.9}$$

and the solution for European options (the Black Scholes Formula) is

$$\begin{aligned} C_t(f) &= e^{-\bar{r}(T-t)} \left[f\, N(d_1) - x\, N(d_2) \right] \quad \text{for a European call,} \\ P_t(f) &= e^{-\bar{r}(T-t)} \left[x\, N(-d_2) - f\, N(-d_1) \right] \quad \text{for a European put,} \end{aligned} \tag{3.10}$$

where the terms are as in equation (3.3) except that

$$\begin{aligned} d_1 &= \left[\ln (f/x) + (\sigma^2/2)\,(T-t) \right] / \sigma \sqrt{T-t}, \\ d_2 &= d_1 - \sigma \sqrt{T-t}. \end{aligned}$$

Note that equation (3.9) differs from equation (3.2) in that (3.9) has no cost of carry term for the underlying forward contract, reflecting the fact that this contract must be paid for when it matures. Also note that the Black–Scholes Formulae (3.3) and (3.10) agree with each other if (3.8) holds, reflecting the fact that an option on the forward must be equivalent to an option on the spot if the option can be exercised only when the forward matures. The valuation of options on forward contracts goes back to the paper [B] and formula (3.10) is often called 'Black's formula'.

Now we value an option on the forward or future contract, which is paid for when the option is exercised or matures. As we have indicated above and will justify more fully below, this covers the example of LIFFE style futures options. The Black–Scholes equation for this instrument is just

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} f^2 \sigma^2 \frac{\partial^2 \phi}{\partial f^2}. \tag{3.11}$$

and the solution for a European option is

$$\begin{aligned} C_t(f) &= f\, N(d_1) - x\, N(d_2) \quad \text{for a call,} \\ P_t(f) &= x\, N(-d_2) - f\, N(-d_1) \quad \text{for a put,} \end{aligned} \tag{3.12}$$

where

$$d_1 = \left[\ln (f/x) + \sigma^2/_2 \, (T-t) \right] / \left[\sigma \sqrt{T-t} \right],$$

$$d_2 = d_1 - \sigma \sqrt{T-t}.$$

In equation (3.11) the cost of carry terms are missing for both the underlying and the option itself, and (3.11), (3.12) are equivalent to (3.9), (3.10) with $\bar{r} = 0$. Note that the value $\phi_t(f)$ is the fair price that should be agreed for the option, to be paid when it is exercised. The present value of the option at time t (and if the futures price then is f) is given by $e^{-\bar{r}(T-t)} \phi_t(f)$, if we know that the option will be exercised at the maturity time T. In fact it can be shown (see [C_3]) that it is never rational to exercise a LIFFE option early and in practice early exercise is rare. The criterion explained in [C_3] for non–early exercise is that under the so–called risk–neutral probability distribution of the underlying, the present value of the payoff of the option evaluated along the random trajectory of the underlying, should be a sub–martingale. (A stochastic process is said to be a sub–martingale if its expected value never decreases. In the context of American options this translates to saying that is always logical to wait before exercising.)

Valuing the Interest Rate Options of Section 2

First, the LIFFE Short Sterling option (and 'LIFFE–style' options generally) should be valued using the Black–Scholes Formula (3.12), because both the option premium and the underlying premium are essentially paid when the option expires. (NB. In formula (3.12) f represents 100% – quoted futures price.) Let us justify the applicability of Formula (3.12) rigorously, in a way which also does not gloss over the difference between forwards and futures and marking to market of the option. (See the appendix for details of how these contracts marks to market.) The key observation is that the present value of a LIFFE–style future or option is always zero, because the cost of taking the position in terms of cash is zero if we neglect the initial margin, which is not really a payment. Therefore the present value of the hedged portfolio comprising one LIFFE option and 'delta' of the future must also be zero, and in earning the riskless return it must earn zero. This observation leads to equation (3.11) (which has no cost of carry term) and hence (3.12), and we see that the net variation margin payment on the hedged portfolio should also be zero.

An option on an FRA will usually be European, and it should be priced by formula (3.10) if it is paid for when it is written, or (3.12) if it is paid for when exercised. As explained above, the volatility of the future contract is appropriate in the valuation. If the option is purchased to guarantee a ceiling as a loan, then the option premium may be paid in the form of an extra component of the interest rate payments on that loan.

In this case the premium payments should be such that they are equal to formula (3.12) when discounted to the maturity date of the option. The rate at which this discounting should be done will be the rate which is expected to obtain over the period of the FRA itself; a good candidate for this if obtainable is the current rate which is available (at zero present value) for the period of the underlying FRA.

Now we value caps, floors and collars. As we explained in Section 2, a cap is a series of calls on the forward interest rate, and a floor is a series of puts. Each of these options can be valued by Formula (3.10) if paid for up–front, with the volatility being taken from the futures contract. The value of the cap or floor is just the sum of these values. If the premium is paid in the form of an extra component on the interest rate, then these payments should equal the up–front payment when discounted to the present at the current swap rate for the period of the cap. This rate for discounting is appropriate because (as we explained Section 2) it is equivalent in terms of present value, to discounting at the floating LIBOR rate. To value a collar, note that it is just a cap plus a short floor.

As we said in Section 2, the futures contract extends about two years into the future, and beyond that the cap floor or collar will usually be hedged against swap contracts or FRAs etc. In fact it will be appropriate to value the cap, floor or collar off the instruments that will be used for hedging, using a version of the Black–Scholes Formula as above. The precise procedure for doing this must be chosen by the hedger, taking account of the liquidities etc, of the candidates for hedging instruments, and the correlations between them.

Finally, a swaption can be valued and hedged against the forward swap instrument, which might be constructed from ordinary swaps, FRAs etc, depending on the skill of the hedger.

4. The Government Securities Market: Duration, Volatility, and the Yield Curve

In this section we will briefly describe the way the Government Securities Market operates and the concepts that it uses. This will be necessary for understanding the valuation techniques of Section 5 for bond options. A fuller description of this market can be found in [P]. Note that the Government Securities Market also indirectly sets the prices of the Money Market Instruments; the Money Market Rates are higher than the Government Security rates, reflecting the default risks in the money market.

Buying a Government Security is in effect lending money to the Government at a fixed interest rate and for a fixed period. For an ordinary bond, this interest rate and period are formulated by the market participants as "(gross redemption) yield" and "duration"; they depend on the bonds coupon, redemption date, and market price.

The yield is the internal rate of return on the bond, ie, that interest rate y which would make the sum of the discounted future payments represented by the bond equal to its current market price. Thus, the yield y is given by the formulae

$$\left.\begin{aligned} P &= ({}^{c}/_{2})v^{\alpha} + ({}^{c}/_{2})\, v^{\alpha+1} + \dots + ({}^{c}/_{2})v^{\alpha+(q-2)} + ({}^{c}/_{2} + 100)v^{\alpha+(q-1)} \\ v &= {}^{1}/_{(1+{}^{y}/_{200})} \end{aligned}\right\} \quad (4.1)$$

where

q is the number of coupons to be paid in the future of the bond's life;

α is the proportion of the six month period to go until the next coupon payment;

c is the coupon expressed in percentage terms, and we are assuming that it is paid in two six–monthly parts. Also, to be consistent with this we are taking the yield y to be compounded half yearly and expressed on an annual basis;

P is the 'dirty' price of the bond, ie, including the accrued interest $^{(1-\alpha)c}/_{2}$, which must be paid when the bond is bought (nb, the 'clean' price of the bond is sometimes quoted; this does not include the accrued interest. Note that the clean price of the bond will fall by the amount of the coupon payment when it is paid, but the dirty price will not. Note also that we are following the market convention and taking P to be the price of an amount of the bond with nominal (face) value of 100.)

The duration of a bond is just the average time to go to receive the payments, weighted by the amount of the payments, these amounts being discounted to the present via the yield on the bond. Thus, for the above bond, the duration D is given in terms of half years by the formula:

$$D = \frac{\left\{ \alpha \left[\left(^{c}/_{2}\right)v^{\alpha}\right] + (\alpha+1)\left[\left(^{c}/_{2}\right)v^{\alpha+1}\right] + \ldots + \left(\alpha+(q-2)\right)\left[\left(^{c}/_{2}\right)v^{\alpha+(q-2)}\right] + \left(\alpha+(q-1)\right)\left[\left(^{c}/_{2}+100\right)v^{\alpha+(q-1)}\right] \right\}}{\left[\left(^{c}/_{2}\right)v^{\alpha}\right]+\left[\left(^{c}/_{2}\right)v^{\alpha+1}\right] + \ldots + \left[\left(^{c}/_{2}\right)v^{\alpha+(q-2)}\right]+\left[\left(^{c}/_{2}+100\right)v^{\alpha+(q-1)}\right]} \qquad (4.2)$$

Note that we can write this formula as

$$D = {}^{v}\left[{}^{\partial P}/_{\partial v}\right]/_{P} \qquad (4.3)$$

where P is given by equation (4.1).

The yield curve is the graph of available interest rates offered by Government Securities, in terms of their times to maturity. For a given time to maturity this rate might be taken to be the yield on a bond with that time to maturity and which is trading at par, ie, for which the current market price is 100. Since for a given time to maturity there is unlikely to be a bond at par, this yield has to be estimated by interpolation. Note that if a bond does trade at par then its coupon and yield are the same. To see this put c=y in equation (4.1) to deduce that $p=100(1+c/200)^{(1-\alpha)}$. A more elementary construction of the yield curve would be to plot the yields vs times to maturity for the bonds, and fit a curve through these points. This construction glosses over the fact that different coupons might correspond to different yields.

The yield curve can be said to represent the state of the Government Securities market, and the main concern of the market is with the behaviour of this curve. Speculators are gambling on this and hedgers are seeking to protect themselves against adverse movements of the yield curve, and both might use options to do this. Its shape is determined by supply (by the Government) and demand (by investors) for fixed interest lending over the various terms to maturity and also by intra–market trading which seeks to exploit arbitrage opportunities among the securities. This intra–market trading is described in Chapter 7 of [P]. It largely involves comparing the yields of pairs of bonds and switching investment from the lower to the higher if the difference seems to be an anomaly. Another technique is to compare the yield of a bond with the market as a whole by predicting what it should be using the yield curve. The effect of this intra–market trading is that the yield curve is tightly determined and that arbitrage opportunities are small and transient.

The volatility (nb, this is UK nomenclature) of a bond is the gearing factor which gives the proportional change in the price of the bond in terms of a small change in its yield. Thus the volatility v is given by the formula

$$v = {}^{-1}/_{P} \frac{\partial P}{\partial y} \tag{4.4}$$

and comparison of (4.4) and (4.3) gives

$$v = {}^{D}/_{(200+y)} \tag{4.5}$$

Note that this concept of volatility is different from the option–theoretic concept, which simply gives the amplitudes of the erratic movements of the price. However, we will show in Section 5 that these two types of volatility are proportional if the yield curve moves in a parallel fashion. This fact justifies the technique of constructing an 'immunised' portfolio of bonds, whose return is immune from (parallel) shifts in the yield curve. The technique is simply to construct the portfolio to have zero weighted average duration/volatility (such a portfolio will involve short (negative) holdings of some bonds, so that the duration/volatility of the portfolio as a whole will be zero.

It is an empirical fact that the primary aspect of movement of the yield curve is parallel displacement (see [C] and references therein). Therefore the immunised portfolio will be largely immune from changes in the market. Note however, that it might not be immune from a transition of the yield curve between being normal and being inverted.

5. Pricing Bond Options

In order to carry out the Black–Scholes valuation for a bond option we must estimate the (option–theoretic) volatility σ_t^P in the following equation for the bond price P_t:

$$dP_t/P_t = \mu_t^P dt + \sigma_t^P dB_t. \tag{5.1}$$

In this equation the coefficients μ_t^P and σ_t^P should be allowed to depend on time. Also μ_t^P must depend on the bond price itself, though this does not affect our work because μ_t^P does not enter into the option valuation.

We will show that under the assumption that shifts in the yield curve are parallel (see Section 4), then this volatility σ_t^P is proportional to the Government Securities Analysts volatility, and hence to the duration of the bond. This proportionality is studied and verified empirically in the article [SS]. Also it will allow us to price bond options in terms of the duration of the bond and the volatility (in the sense of 'sigma') of the yield curve. So, let us assume that the yield curve does move in a parallel fashion, and write

$$dy_t = \mu^y dt + \sigma^y dB_t, \tag{5.2}$$

where y_t is the yield at time t for any given term to maturity. Then, noting that the bond price is a function of time and yield and using the Ito formula, we have

$$dP_t/P_t = {}^1/_{P_t}\left[\frac{dP_t}{dt} + \tfrac{1}{2}(\sigma^y)^2\frac{d^2P}{dy^2} + \mu^y\frac{dP_t}{dy}\right]dt + \left({}^{\sigma^y}/_{P_t}\right)\frac{dP_t}{dy}dB_t. \tag{5.3}$$

Comparing (5.1) and (5.3) and using (4.4) and (4.5) we see that

$$\sigma_t^p = \sigma^y \begin{bmatrix} \text{Government} \\ \text{Securities} \\ \text{Volatility} \end{bmatrix} = \frac{\sigma^y . D_t}{(200+y)} \tag{5.4}$$

as required.

Now let us write $\sigma_t^y/(200+y) \equiv \rho$ and assume that this is constant, so that we can write the volatility σ_t^p of the bond as ρD_t. Note that this corresponds to σ_t^y being (almost!) constant, ie to the yield and hence interest rates having constant absolute volatility. This contradicts our assumptions of Section 3 (see equation (3.1)), but as we mention there, this discrepancy may not be significant. Assuming that ρ is constant, then the Black–Scholes equation for the value V_t of an option (paid for up–front) on the bond is

$$\frac{\partial V_t}{\partial t} = \bar{r}V_t - \bar{r}P\frac{\partial V_t}{\partial P} - \tfrac{1}{2}\rho^2 D_t^2 P^2 \frac{\partial V_t}{\partial P^2} \tag{5.5}$$

where $\bar{r}$ is the short interest rate, assumed constant. Also, if we assume that the duration is constant over the life of the option, then (5.5) leads to the following Black–Scholes Formula for European options:

$$\left.\begin{aligned} \text{Call}_t(P) &= P\ N(d_1) - e^{-\bar{r}(T-t)}\ X\ N(d_2) \\ \text{Put}_t(P) &= Xe^{-\bar{r}(T-t)}\ N(-d_2)\ - P\ N(-d_1) \end{aligned}\right\} \tag{5.5}$$

where

X is the strike price,

$$d_1 = \left[\ln (p/X) + (\bar{r} + \rho^2 D^2/2)(T-t)\right] / \left[\rho D \sqrt{T-t}\right],$$

$$d_2 = d_1 - \rho D \sqrt{T-t}.$$

The assumption that the duration is constant is reasonable if the time to maturity of the option is much shorter than that of the bond. This will usually be the case in practice because the bond price must converge to par as it approaches maturity, and so an option which matured only a short time before the bond itself would not be a useful instrument. If the time–to–maturity of the option is a significant proportion of that of the underlying bond, then it is appropriate to replace the duration squared in Formula (5.6) by its average value over the life of the option. This can be justified using the work of [M], which is actually in advance of the work of [BS]. When valuing bond options it is reasonable to estimate the sigma–volatility using the prices of many bonds, and the equation

$$dP_t / P_t = \mu_t^P \, dt + \rho \, D_t \, dB_t \tag{5.7}$$

and this is done in [SS].

LIFFE Gilt Options have the same marking–to–market and settlement procedures as the LIFFE Short Sterling Options (see the Appendix), and hence the same alterations must be made in the Black–Scholes equation and formula (ie, we must set $\bar{r} = 0$). Also the σ–volatility should be taken from the price of the underlying futures contract rather than the duration, although under the assumption that the yield curve moves in a parallel fashion, these should give the same value.

Appendix

LIFFE Style Margining for Futures and Options

We will illustrate the LIFFE mark–to–market procedures with two simple example, in a similar way to the publication [LIFFE], except that the option of their example does not relate to an interest rate.

So, suppose a counterparty buys one Short Sterling (3 month) Future at 89.93, with only the rest of the day and two more days to go until the future matures. Also suppose that the prices at the ends of the remaining days are 89.92, 90.05, 89.97. This last price must correspond to LIBOR = 100 – 89.97 = 10.03. At the end of each day the buyer must make variation margin payments of

89.93 – 89.92 = +0.01 (+ 1 tick = +£12.50)

89.92 – 90.05 = –0.13 (– 13 ticks = –£152.50)

90.05 – 89.97 = +0.08 (8 ticks = +£100.00)

The contract is based on a nominal principal sum of £500,000, and so one tick represents 3 months' interest at 0.01 percent on this sum, not compounded, which is £12.50. In buying the futures contract the counterparty wants to lock himself into the interest rate 100 – 89.93 = 10.07 percent on borrowing the principal sum of £500,000. In fact, he still pays LIBOR on the sum (ie, 10.03 percent) but he has also had to pay the difference 10.07 – 10.03 = 0.04 percent in variation margin.

As well as these variation margin payments, the counterparty must also maintain an initial margin account, which always contains enough to cover (within reason) the next days variation margin payment. The requirements for the initial margin account are also explain in detail in the publication [LIFFE]. Since LIFFE pays interest at (almost) LIBOR on the initial margin account, the initial margin requirements do not affect the valuation of the future or the option, and we will not go into details about the initial margin requirements.

Now consider a counterparty who buys a Short Sterling Put on the above future with strike 90.00, also with two clear days to go before maturity. (NB, the Short Sterling option matures at the same time as the underlying future, but this is not the case for all LIFFE options, in particular for the long gilt option described below.) Suppose the premium of the option is 0.11 and its values at the ends of the remaining days are 0.12, 0.03, 0.03. This last value is fixed by LIFFE to be (strike price) – (final futures price), ie, LIBOR – (100 – strike) if this is positive, or zero otherwise. Note that the other values are as the Black–Scholes formula (3.12) prescribes if $\sigma = 15\%$, though to obtain σ as an implied volatility from these prices would be an ill conditioned calculation since the option is so short–lived.

At the end of each day the buyer of the call must make variation margin payments of

0.11 – 0.12 = –0.01	(–1 tick = –£12.50)
0.12 – 0.03 = +0.09	(+39 ticks = +£487.50)
0.03 – 0.03 = –0.01	(–0 ticks = –£00.00)

When the option matures its final payoff value is 0.03. Also the premium must be paid for it. Of this premium, which originally was 0.11, the net variation margin payments make up 0.08, and the rest, ie 0.03, cancels with the payoff, and so there is no cash flow when it matures. The buyer of the option has in effect paid the original premium of 0.11 to guarantee that the interest rate on borrowing £500,000 for 3 months does not rise above 10%. In fact he still pays LIBOR = 10.03%, but the option payoff makes up the difference.

Next we consider the long gilt future and option. In contrast to the short sterling future and option, the long gilt option matures before the associated future, and so the settlement procedure when the option matures is different. In fact as we explain in Section 2 the maturity day of the future varies according to which long gilt is delivered against it.

So, suppose one long gilt call option is bought, with strike price 100 and premium $^6/_{64}$, and with the rest of the day and two more days to go before it matures. Suppose the closing prices for the future contract for these three days are 99 $^{31}/_{32}$, 100 $^7/_{32}$, 101$^2/_{32}$, and for the option they are $^5/_{64}$, $^{23}/_{64}$, 1$^4/_{64}$. Note that each future contract refers to £50,000 nominal of underlying long gilt, and the quoted price of both future and option refers to 100 nominal of the underlying. Also, the future price is quoted to the nearest $^1/_{32}$ (ie, $^1/_{32}$% of £50,000, which is £15,625), but the option price is quoted as a multiple of $^1/_{64}$(ie, £7.8125). The final option price is as it must be, ie (futures price) – (strike price).

At the end of each day the buyer of the option must make variation margin payments of

$$^6/_{64} - {}^5/_{64} = {}^1/_{64} \quad (= 1 \times £7.8125),$$
$$^5/_{64} - {}^{23}/_{64} = {}^{-18}/_{64} \; (= -18 \times £7.8125 = -£140.625),$$
$$^{23}/_{64} - 1^4/_{64} = {}^{-45}/_{64} \; (= -45 \times £7.8125 = -£351.5625).$$

Also when the option matures he must pay the remaining premium of $^{14}/_{64}$, which makes his total payment equal to the original premium of $^{6}/_{64}$. The payoff of the option is that he is able to buy the future contract for 100, when it is actually worth $101^{2}/_{32}$. This payoff has value equal to the final premium installment on the option.

References

[B] Black F, *The Pricing of Commodity Contracts*, Journal of Financial Economics 3, pp 167–179, 1976.

[BS] Black F and Scholes M, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy 81, pp 637–659, 1973.

[C_1] Carverhill A P, *A Primitive Theory of the Term Structure of Interest Rates*, FORC Pre–print 89/4, University of Warwick, 1989.

[C_2] Carverhill A P, *Valuing Interest Rate Options via a Primitive Theory of the Term Structure*, FORC Pre–print 89/5, University of Warwick, 1989.

[C_3] Carverhill A P, *Some Things you were Afraid to Ask about Options Theory*, in Preparation.

[CIR] Cox J C, Ingersoll J E and Ross S A, *A Theory of the Term Structure of Interest Rates*, Econometrica 53, pp 385–407, 1985.

[HJM] Heath D, Jarrow R, Morton A, *Bond Pricing and the Term Structure of Interest Rates: A new methodology*, Pre–print, Cornell University, NY 14853, 1988.

[HL] Ho T S Y, Lee S B, *Term Structure Movements and Pricing Interest Rate Contingent Claims*, Journal of Finance 41, pp 1011–1029, 1986.

[LIFFE] *Futures-Style Margining for Options; A Guide to the Margining of LIFFE Options*, LIFFE Publication, 1989.

[M] Merton R C, *Theory of Rational Option Pricing*, Bell Journal of Economics and Management Science 4, pp 141–183, 1973.

[P] Phillips P, *Inside the New Gilt-Edged Market*, Cambridge, Woodhead, Faulkner, 1987.

[SS] Schaefer S M and Schwartz E S, *Time Dependent Variance and the Pricing of Bond Options*, European Options colloquium, 1987.

[V] Vasicek O, *An Equilibrium Characterization of the Term Structure*, Journal of Financial Economics 5, pp 177–188, 1977.