The Term Structure of Interest Rates and Associated Options; Equilibrium vs Evolutionary Models

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1. Introduction

Among models of the term structure of interest rates we distinguish two classes, which we will refer to collectively as the "Equilibrium Model" and the "Evolutionary Model". The Equilibrium Model is represented by the papers [CIR 1985] (Cox, Ingersoll and Ross), [Vasicek 1977], [Courtadon 1982]. It seeks to characterise the term structure by assuming that it is in economic equilibrium, and is determined ('driven') by a given set of parameters. This set of parameters might typically comprise the long and short interest rates, or just the short rate, and being in economic equilibrium means that there are not arbitrage opportunities among the interest rate instruments whose prices make up the term structure. The Evolutionary Model is represented by the papers [HL 1986] (Ho and Lee), [HJM 1987] (Heath, Jarrow and Morton), [Carverhill 1989a], [Carverhill 1989b], [Babbs 1990]. It concentrates on the evolution of the term structure, rather than the term structure itself, from an empirically given initial shape. Again, it assumes that these are no arbitrage opportunities among the interest rate instruments. These fundamental differences give rise to the contrasting strengths and weaknesses of the two models: the Equilibrium Model predicts a shape for the term structure which may not accurately match the actual term structure at a given time, but which one expects to be a reasonable match for any time; the Evolutionary Model predicts a term structure which decreases in accuracy as it is pushed further into the future.

Our aim in this paper is first to present the two models and the pricing procedures for contingent claims which they entail, from a unified perspective. This enables us to discover the similarities, differences, and conflicts between these models. Also we discuss the work of some other authors (notably [Dybvig 1989], [Dybvig Ingersoll Ross 1989],[Hull White 1990a], [Jamshidian 1989]) and its relationship to the ideas of this paper. Finally, our perspective allows us to clarify and extend much of the work to which we make reference. This paper includes the material of the two preprints [Carverhill 1989a] and [Carverhill 1989b].

A plan of the paper is as follows:

In Section 2, we establish some notation and basic assumptions concerning the term structure of interest rates. In Sections 3 and 4 we present the equilibrium and Evolutionary Models from our perspective. For the sake of clarity and simplicity, we restrict our attention in these sections (and throughout the paper until Section 9) to the single factor versions of the models, ie, to the case where there is just one factor of randomness driving the model.

In Section 4 the Evolutionary Model is formulated in terms of the dynamics of the bond prices. However, other formulations are also needed in subsequent sections of the paper, namely in terms of spot interest rates and (instantaneous) forward rates, and these are presented in Section 5. The instantaneous forward rate formulation is the same as that of [HJM 1987] and [Babbs 1990]. In all of these formulations the model is presented in terms of the drift and volatility as a function of term to maturity, of the prices or rates. In the Evolutionary Model these functions are the 'basic ingredients', to be determined or chosen exogenously, but in the Equilibrium Model they are determined by the model itself.

In Section 6 we discuss the conflicts and possible reconciliations between the models. We see that the Evolutionary Model can be regarded as a generalisation of the Equilibrium Model; any version of the Equilibrium Model can be 'differentiated' to give a version of the Evolutionary Model. However, the converse does not hold; not all Evolutionary Models are 'integrable'.

In Section 7 we discuss the stability of the Evolutionary Model. The criteria for this is the attenuation of the volatility as term to maturity increases. This criterion also implies that the long rate is constant. The paper [Webber 1990] discusses related questions; in particular it discusses the implications of this criterion for the short rate.

Section 8 deals with the valuation of term structure contingent claims (options) in the Evolutionary Model. In view of Section 6, this also gives the valuation of options in the Equilibrium Model, in terms of the drift and volatility of the term structure as functions of term to maturity. In the Evolutionary Model we cannot use a Black–Scholes type equation for general option valuation; rather, we use the risk–neutral expectation approach for this. Also we use the ideas of [Margrabe 1978] for valuing the option to exchange one asset for another. However, our option valuations are often surprisingly close to what one obtains in the Equilibrium Model.

In Section 9 we discuss some empirical aspects of the behaviour of the term structure, and the desirability and technique for introducing more factors of randomness into the models and associated option valuations.

Finally, in Section 10 we summarise our conclusions. In this section we argue that the extra generality that the Evolutionary Model has over the Equilibrium Model, does make it more realistic as a vehicle for describing the behaviour of the term structure.

2. The Term Structure of Interest Rates

Broadly speaking, the term structure of interest rates at any given time is the collection of rates that are available for borrowing or lending at that time, as a function of the various terms to maturity over which the loan can be taken. In our technical work, we will take the term structure at time t to be any of the functions $\{P_t^q\}_{q\epsilon[t,\infty)}$, $\{p_t^q\}_{q\epsilon[t,\infty)}$ or $\{f_t^q\}_{q\epsilon[t,\infty)}$ of q, where

P_t^q is the price of time t of the pure discount bond which pays 1 at time q;

p_t^q is the spot rate associated with this bond, ie,

$$p_t^q = \frac{-1}{q-t} \log P_t^q;$$

f_t^q is the instantaneous forward rate associated with the term structure at time t, ie

$$f_t^q = \frac{\partial P_t^q}{\partial q} \Big/ P_t^q \equiv \frac{-\partial}{\partial q} (\log P_t^q) \tag{2.2}$$

Note also that

$$p_t^q = \frac{1}{q-t} \int_{\rho=t}^{q} f_t^\rho \, d\rho \tag{2.3}$$

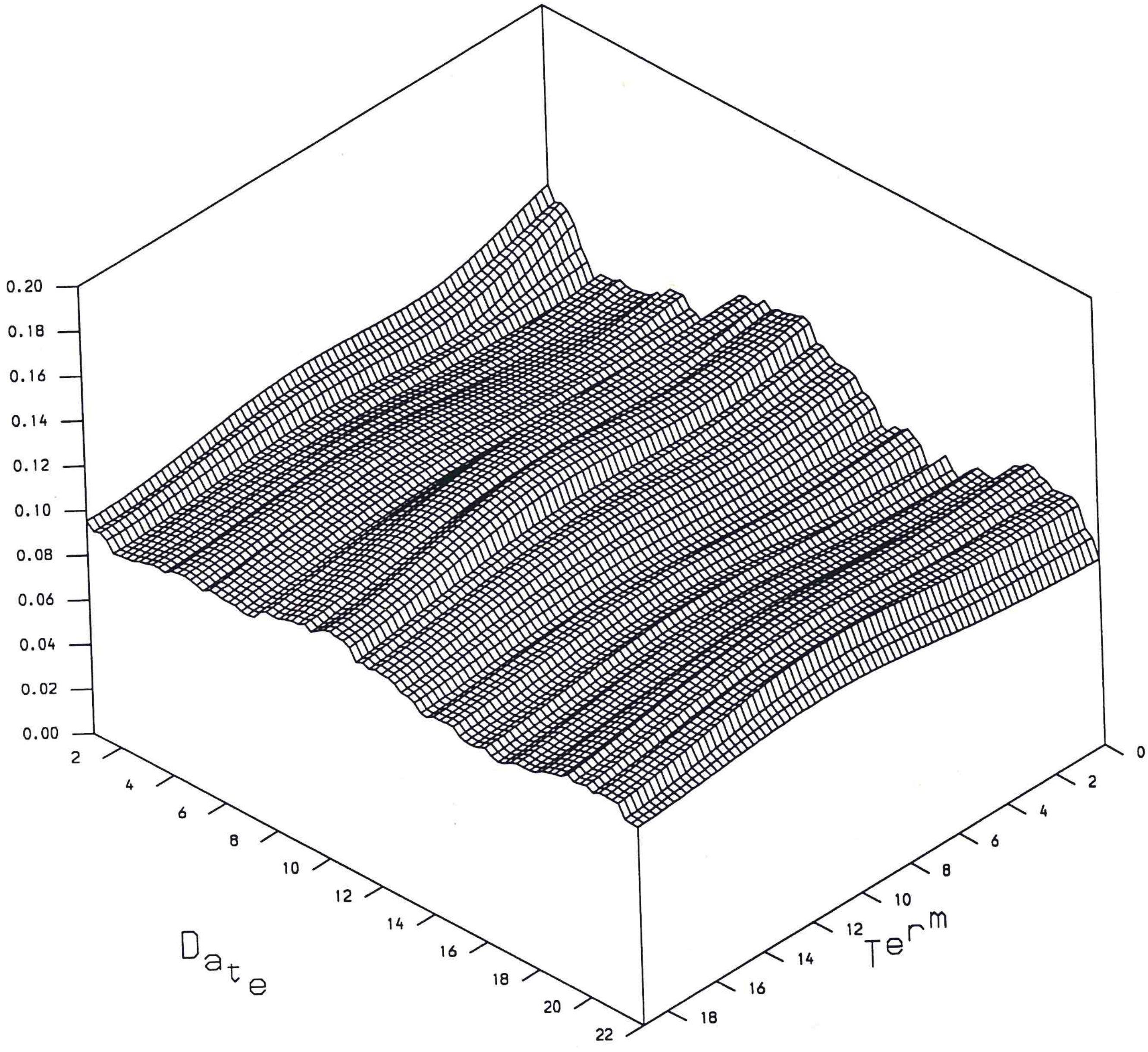
In order to make these alternative characterisations work, we must assume that P_t^q is appropriately smooth in the q variable. Also, we define the short rate r_t by

$$r_t \triangleq f_t^t \equiv \frac{-\partial}{\partial q} P_t^q \Big|_{q=t} \tag{2.4}$$

Of course, such a continuum of pure discount bonds does not exist in the market, and so the first task in any empirical work must be to estimate the term structure, ie, to estimate the virtual prices $\{P_t^q\}_{q\in[t,\infty)}$ from the prices of the bonds that we actually observe in the market. This estimation is itself the subject of many research articles, for instance [Schaefer 1981], [Steeley 1991], which generally show (at least for UK and US Government Bonds) that the market is efficient, and behaves as though the pure discount bonds do exist.

This paper will confine itself to discussing only the broad aspects of the empirical issue. This discussion will be based largely on the UK term structure as estimated by the technique of [Steeley 1991]. Steeley first expresses the pure discount bond price function in terms of cubic B–splines, and then adjusts the spline coefficients so as to minimise the sum of the squared errors when the actual coupon bond prices are reconstructed from this function. He restricts himself to high coupon bonds in order to avoid tax effects on the prices.

The following 3–dimensional graph shows the evolution of the term structure according to Steeley's procedure; the slice of the graph for fixed t gives the rate–based term structure $\{p_t^q\}_{q\geq t}$, in terms of the 'term', ie, (q–t).


0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
2
4
6
8
10
12
14
16
18
20
22
Date
0
2
4
6
8
10
12
14
16
18
Term

(Figure 2.5)

3. The Equilibrium Model

As we have said, in this section (and up till Section 8) we confine our selves to the single factor version of the Equilibrium Model. The assumptions are:

(EQ1) The short rate r_t is driven by the autonomous stochastic equation

$$dr_t = \xi(r_t)dt + \eta(r_t)dB_t \tag{3.1}$$

in which dB_t is the increment of the standard Brownian motion. We do not want to be specific here about the form of the coefficients ξ and η, except to say that they are functions of the short rate (and possibly the current time t) alone; this is what we mean when we say that the equation is 'autonomous'. The notation ξ, η is chosen to prevent a clash later in the paper.

(EQ2) The entire term structure at any time is determined by the short rate at that time. Thus we can express the term structure as a function $P_t^q(r)$; this represents the value at time t of the q–maturity bond, if the short rate at time t is r. We assume that the function P is appropriately smooth in r, so that we can apply the Ito formula to give

$$dP_t^q = \frac{\partial P_t^q}{\partial t}dt + \frac{\partial P_t^q}{\partial r}dr_t + \frac{1}{2}\frac{\partial^2 P_t^q}{\partial r^2}\left\langle dr_t, dr_t\right\rangle$$

$$= \Big(\text{(using Equation (3.1)}\Big)$$

$$= \left[\frac{\partial P_t^q}{\partial t} + \xi\frac{\partial P_t^q}{\partial r} + \frac{1}{2}\eta^2\frac{\partial^2 P_t^q}{\partial r^2}\right]dt + \left[\eta\frac{\partial P_t^q}{\partial r}\right]dB_t \tag{3.2}$$

(EQ3) There are no arbitrage opportunities among the bonds. This is the Economic Equilibrium condition, and given Assumption (EQ2), it is easily shown to be equivalent to the risk premium γ_t^q defined below being independent of q (although it can depend on the short rate). If we write

$$dP_t^q / P_t^q = (v_t^q + r_t)dt + \mu_t^q \, dB_t \tag{3.3}$$

then the risk premium γ_t^q is defined by

$$\gamma_t^q = v_t^q / \mu_t^q . \tag{3.4}$$

Thus γ_t^q is the expected short term return that the bond offers, in excess of the short rate r_t, for each unit of the risk μ_t^q that the bond entails. (NB. We often suppress the r dependence in our notation p_t^q, ξ, η, v_t^q, μ_t^q, γ_t^q.)

The 'basic ingredients' of the Equilibrium Model are thus the functions ξ, η and γ discussed above; these must be estimated or chosen in some way when applying the model. Most generally they can be functions of current time t and short rate r, though we will emphasise the time homogeneous situation where there is no t dependence. Choosing the basic ingredients so that they vary with t amounts to including a change in the nature of the term structure dynamic into the model. This is appropriate for instance if one expects the market to become more risk averse at some given time in the future.

To solve the Equilibrium Model proceed as follows:

First, compare Equations (3.2) and (3.3) to yield

$$\left.\begin{aligned} \nu_t^q &= \left[\frac{\partial P_t^q}{\partial t} + \xi\,\frac{\partial P_t^q}{\partial r} + \tfrac{1}{2}\eta^2\,\frac{\partial^2 P_t^q}{\partial r^2}\right] / P_t^q - r \\ \mu_t^q &= \eta\,\frac{\partial P_t^q}{\partial r} \Big/ P_t^q \end{aligned}\right\} \qquad (3.5)$$

Now substitute into Equation (3.4) using Equation (3.5) to obtain the Black–Scholes Equation

$$\frac{\partial P_t^q}{\partial t} = -\tfrac{1}{2}\eta^2\,\frac{\partial^2 P_t^q}{\partial r^2} + (\gamma\eta - \xi)\,\frac{\partial P_t^q}{\partial r} + rP_t^q \qquad (3.6)$$

The price P_t^q is then the solution to this equation with initial (final!) condition

$$P_q^q \equiv 1 \qquad (3.7)$$

It seems that this Black–Scholes Equation cannot, in its full generality, be solved analytically. However, it can be solved using standard numerical procedures for diffusion equations, with time t evolving backwards from the final condition at time q. Also, the papers [Vasicek 1977] and [CIR 1985] make reasonable choices of the basic ingredients ξ, η and γ, and for these they are able to find analytic solutions. These solutions are important because they easily lend themselves to studying comparative statics and other aspects of the Equilibrium Model.

The choices of [Vasicek] and [CIR] for Equation (3.1) and the risk premium γ are respectively

$$dr_t = \tilde{\alpha}(\tilde{\gamma} - r_t)dt + \tilde{\rho}dB_t,\ \gamma = \tilde{q},\ \text{([Vasicek] choice)} \qquad (3.8)$$

$$dr_t = \tilde{\kappa}(\tilde{\theta} - r_t)\, dt + \tilde{\sigma} \sqrt{r_t}\, dB_t, \quad \gamma(r) = \tilde{\lambda} \sqrt{r} / \tilde{\sigma}, \text{ ([CIR] choice)} \tag{3.9}$$

where the symbols with tildas are all positive constants, and we have added the tildas to their notations to prevent clashes with our own. These choices for the short rate equation both give mean reverting processes, which is empirically reasonable, and the CIR choice has the advantage that the short rate can never become negative. Their solutions of the model are respectively

$$\left.\begin{aligned} P_t^q(r) = \exp \Big[& {}^1/_{\tilde{\alpha}} \left(1 - \exp\{-\tilde{\alpha}\,(q-t)\}\right) \left(R(\infty) - r\right) - (q-t)\, R(\infty) \\ & - \tilde{\rho}^2 / 4\tilde{\alpha}^3 \left(1 - \exp\{-\tilde{\alpha}\,(q-t)\}\right)^3 \Big] \\ \text{where} \quad & \\ R(\infty) = {} & \tilde{\gamma} + \tilde{\rho}\, \tilde{q}/\tilde{\alpha} - \tfrac{1}{2}\, \tilde{\rho}^2/\tilde{\alpha}^2, \ (\equiv \text{long rate}) \end{aligned}\right\} \text{ ([Vasicek])} \tag{3.10}$$

and

$$P_t^q(r) = A_t^q \exp\{-rB_t^q\}$$

where

$$A_t^q = \left[\frac{2\tilde{\gamma}\exp\{(\tilde{\kappa} + \tilde{\lambda} + \tilde{\mu})\ (q-t)\ /\ 2\}}{(\tilde{\gamma}+\tilde{\kappa}+\tilde{\lambda})\left(\exp\{\tilde{\lambda}\ (q-t)\} - 1\right) + 2\tilde{\gamma}}\right]^{2\tilde{\kappa}\tilde{\theta}/\tilde{\sigma}^2} \qquad \{[CIR]\}\ (3.11)$$

$$B_t^q = \frac{2\left(\exp\{\tilde{\gamma}\ (q-t)\} - 1\right)}{(\tilde{\gamma} + \tilde{\kappa} + \tilde{\lambda})\left(\exp\{\tilde{\gamma}\ (q-t)\} - 1\right) + 2\tilde{\gamma}}$$

$$\tilde{\gamma} = \left((\tilde{\kappa} + \tilde{\lambda})^2 + 2\,\tilde{\sigma}^2\right)^{\frac{1}{2}}$$

In the sequel we will base ourselves heavily on volatility functions related to the function μ as defined in Equation (3.3). For the CIR and Vasicek models, μ can be calculated explicitly using formula (3.5). If we do this, then we see that, just as for the short rate volatility, μ depends only on the term to maturity (q–t) in the Vasicek model, but it also has a factor $\sqrt{r}$ in the CIR model. We will find it convenient in the sequel to have our volatility functions independent of the term structure, despite the fact that this makes negative rates possible, and therefore we will refer back to the Vasicek model rather than the CIR model. The Vasicek volatility is the simple function

$$\mu_t^q = -\tilde{\rho}\,/_{\tilde{\alpha}}\left(1 - e^{-\tilde{\alpha}(q-t)}\right). \qquad (3.12)$$

Actually, we do not regard the non–zero probability of negative interest rates as a strong reason for rejecting the Vasicek model in favour of the CIR model, because for reasonable choices of the parameters, this probability will be small and will have only a small effect on contingent claim values. In fact the estimates $\tilde{\alpha} = 0.1531$, $\tilde{\rho} = 0.1449$ can be found in [Dybvig 1989], and for these and for current interest rates 5%, 7%, 9%, then [Babbs 1990, Chap 8] values the 10 year cap struck at zero, to be respectively 7.19, 0.45, 0.02 basis points. The cap valuation procedure that he uses is the same as we will present in Section 8 below for the Vasicek volatility; it will be clear then how this relates to the valuation based on the Vasicek model and why it does not require knowledge of the parameters $\tilde{\kappa}$, $\tilde{q}$. Also he assumes that the current term structure is flat. NB, the expression in terms of basis points means units per 10000 units of notional principal.

Apart from the possibility of negative interest rates, it is difficult to argue on empirical grounds in favour of either the CIR or the Vasicek model over the other. See [Brown Schaefer 1988], [Steeley 1989a], [Hull White 1990b].

The Vasicek and CIR models give similar predictions for the possible shapes of the term structure. These are 1 parameter families which are parameterised by the short rate (as they must be according to the general Equilibrium Model), and which converge to a long rate as term to maturity increases. This long rate is constant and independent of the short rate; it is given explicitly in formula (3.10) and implicitly in formula (3.11). However, the empirically derived figure (2.5) indicates that there is more than a 1 parameter family of possible term structures, and the long rate is not constant. This leads us to the development of the Evolutionary Model, and to the higher factor models which we discuss in Section 8.

4. The Evolutionary Model

We formulate the 1 factor version of the Evolutionary Model via the following assumptions:

(EV1) The term structure evolves via the stochastic equation

$$dP^q_t / P^q_t = (v^q_t + r_t)\, dt + \mu^q_t\, dB_t \qquad (4.1)$$

(EV2) There are not arbitrage opportunities among the pure discount bonds. As before, this translates to the risk premium γ^q_t defined by

$$\gamma^q_t = v^q_t / \mu^q_t \qquad (4.2)$$

being independent of q.

The 'basic ingredients' of the Evolutionary Model are the functions v^q_t, μ^q_t, γ_t; these must be estimated or chosen in some way when applying the model. Notice that equations (4.1) and (3.3) look the same. However, this equation plays a different role in each model; in the Equilibrium Model it merely defines the functions v and μ in terms of the model, whereas in the Evolutionary Model it is the basic equation. Comparing assumption (EV1) here with assumption (EQ2) for the Equilibrium Model reveals the basic difference between the models; the latter characterises the term structure itself, whereas the former characterises its evolution. Note that there is no Evolutionary Model analogue to assumption (EQ1) for the Equilibrium Model; in the Evolutionary Model the behaviour of the short rate is derived from the model (see Section 5 below).

In the most general formulation of the model, for given t these basic ingredients will be allowed to depend on the term structure at time t, and it would actually be admissible to allow them to depend also on previous term structures. As before, we will emphasise the time homogeneous situation, which is characterised as follows; the 'shape' of the term structure, ie expressed in terms of term to maturity, as the function $\left\{\tau \longmapsto P_t^{t+\tau}\right\}$, determines the shape of the basic ingredients, ie, the functions $\left\{\tau \longmapsto v_t^{t+\tau}, \mu_t^{t+\tau}, \gamma_t\right\}$ (in particular, γ is time independent given the shape of the term structure). In fact we will often specialise even further, to the case when the basic ingredients are time homogeneous and independent of the term structure, so that v_t^q and μ_t^q are simply functions of q–t, and γ is a constant. This assumptions corresponds to the Vasicek model in Section 3, and without it the empirical estimation of v, μ, γ is more difficult. This assumption also theoretically allows rates to become negative, but we have argued in Section 3, we do not regard this fact as a serious shortcoming of the assumption. Also, under this assumption we can write $v_t^{t+\tau}$ and $\mu_t^{t+\tau}$ simply as $v(\tau)$ and $\mu(\tau)$.

To 'solve' the Evolutionary Model we would like to write down the term structure at a future time s, given the current time t term structure and the random input $\{B_\rho: t \leq \rho \leq s\}$. We will do this in Section 8 (Proposition 8.6) as part of our treatment of contingent claim valuation. This cannot be done using the Black–Scholes technique of Section 3.

5. Alternative Formulations of the Evolutionary Model

In this section we present a number of equivalent alternatives to the formulation of Section 4 for the Evolutionary Model. These different formulations are useful in different contexts, and they serve to put the Model into the framework of [HL 1986], [HJM 1987], [Babbs 1990].

<u>*REFORMULATION 5.1*</u> *(in terms of rates rather than prices)*

For this we replace the basic evolution Equation (4.1) by

$$dp_t^q = \frac{1}{q-t}\left(p_t^q - r_t\right) dt + \chi_t^q \, dt + \Psi_t^q \, dB_t \tag{5.1}$$

in which the coefficients χ_t^q and Ψ_t^q play the same role as v_t^q and μ_t^q, and are related via

$$\left.\begin{aligned} v_t^q &= -(q-t)\,\chi_t^q + \tfrac{1}{2}(q-t)^2\,(\Psi_t^q)^2, \\ \mu_t^q &= -(q-t)\,\Psi_t^q \end{aligned}\right\} \tag{5.2}$$

<u>Proof</u>

Simply transform between rates and prices via the Ito Formula, and using equation (2.1). □

<u>*REFORMULATION 5.2*</u> *(in terms of differences over a time increment, rather than a time differential)*

Replace equation (4.1) by

$$p^q_{t+\varepsilon} = g^q_{t,t+\varepsilon} + \chi^q_t \Delta^{t+\varepsilon}_t t + \Psi^q_t \Delta^{t+\varepsilon}_t B, \qquad (5.3)$$

in which

χ^q_t and Ψ^q_t are given by equation (5.2);

$$g^q_{t,s} = \frac{-1}{q-s} \log G^q_{t,s}, \quad G^q_{t,s} = p^q_t / p^s_t \quad \text{for } t \le s \le q, \qquad (5.4)$$

so that $G^q_{t,s}$ is the forward price of the q–maturity bond at time t, to be purchased at time s, and $g^q_{t,s}$ is the associated forward rate. NB, this forward rate must not be confused with the instantaneous forward rate f^q_t, defined in Section 2. Note that $f^q_t = \lim_{s \to t} g^q_{t,s}$;

$\Delta^{t+\varepsilon}_t$ represents the appropriate difference over the time increment $[t,t+\varepsilon]$,

so that

$$\Delta^{t+\varepsilon}_t t = (t+\varepsilon) - t \equiv \varepsilon,$$

$$\Delta^{t+\varepsilon}_t B = B_{t+\varepsilon} - B_t \sim \text{normal (mean 0, standard deviation } \sqrt{\varepsilon}\text{)}.$$

Proof

This is immediate from Reformulation 5.1 and the observation that

$$g^q_{t,t+\varepsilon} = \frac{q-t}{q-(t+\varepsilon)} p^q_t - \frac{\varepsilon}{q-(t+\varepsilon)} p^{t+\varepsilon}_t$$

$$= p_t^q + \frac{\varepsilon}{q-(t+\varepsilon)} (P_t^q - p_t^{t+\varepsilon})$$

$$= p_t^q + \frac{1}{q-t} (p_t^q - r_t) + o(\varepsilon).$$

Technical purists will note that equation (5.3) actually holds only approximately, with error of small order in ε in the L^2 norm. □

This formulation is useful for empirically estimating the Evolutionary Model, which we discuss in Section 9 below.

Our final reformulation of the Evolutionary Model is just that of [HJM 1987]; also of [HL 1986] and [Babbs 1990]. This formulation works with the instantaneous forward rates, and can easily be integrated over time, so that is does not just deal with a small time increment $[t,t+\varepsilon]$, but a possibly large one $[t,s]$.

REFORMULATION 5.3 (the HJM Model)

The Evolutionary Model can be formulated simply as

$$df_\rho^\tau = \alpha_\rho^\tau \, d\rho + \sigma_\rho^\tau \, dB_\rho \tag{5.5}$$

where

$$\left.\begin{aligned} \alpha_\rho^\tau &= \frac{\partial}{\partial\tau}\left[(\tau-\rho)\,\chi_\rho^\tau\right] \equiv \frac{-\partial}{\partial\tau}\left[v_\rho^\tau - \tfrac{1}{2}(\mu_\rho^\tau)^2\right], \\ \sigma_\rho^\tau &= \frac{\partial}{\partial\tau}\left[(\tau-\rho)\,\Psi_\rho^\tau\right] \equiv \frac{-\partial}{\partial\tau}\left[\mu_\rho^\tau\right]. \end{aligned}\right\} \tag{5.6}$$

If we recall that

$$P_t^q = \exp\left\{-\int_{\tau=t}^{q} f_t^\tau \, d\tau\right\}, \tag{5.7}$$

then we can integrate equation (5.5) to obtain

$$p_s^q = P_t^q \Big/ P_t^s \; H_{t,s}^q \quad \text{for } t \le s \le q,$$

where (5.8)

$$H_{t,s}^q = \exp\left\{-\int_{\tau=s}^{q} \int_{\rho=t}^{s} \left[\alpha_\rho^\tau \, d\rho + \sigma_\rho^\tau \, dB_\rho\right] d\tau\right\}.$$

The reformulation is given by either of equations (5.5) or (5.8), together with equation (5.6).

<u>Proof</u>

First invert equation (5.6) to obtain

$$\chi_t^q = \frac{1}{q-t} \int_{\tau=t}^{q} \alpha_t^\tau \, d\tau,$$

(5.9)

$$\Psi_t^q = \frac{1}{q-t} \int_{\tau=t}^{q} \sigma_t^\tau \, d\tau$$

(The lower limit of integration in equation (5.9) is determined by the requirement that χ_t^q and Ψ_t^q be finite.) Now, multiplying equation (5.3) by (q–t) yields

$$\log P^q_{t+\varepsilon} = \log \left[{P^q_t}/{P^{t+\varepsilon}_t} \right] - (q-t)\, \chi^q_t\, \Delta^{t+\varepsilon}_t\, t - (q-t)\, \Psi^q_t\, \Delta^{t+\varepsilon}_t\, B$$

and then substituting with equation (5.9) yields

$$\log P^q_{t+\varepsilon} = \log \left[{P^q_t}/{P^{t+\varepsilon}_t} \right] - \int_{\tau=t}^{q} \int_{\rho=t}^{t+\varepsilon} \left[\alpha^\tau_\rho \, d\rho + \sigma_\rho{}^\tau \, dB_\rho \right] d\tau \tag{5.10}$$

(and these last two equations have error of small order in ε in the L^2 norm). Now, using equation (5.7) we can rewrite (5.10) as

$$\int_{\tau=t+\varepsilon}^{q} f^\tau_{t+\varepsilon} d\tau = \int_{\tau=t+\varepsilon}^{q} f^\tau_t \, d\tau + \int_{\tau=t+\varepsilon}^{q} \int_{\rho=t}^{t+\varepsilon} \left[\alpha^\tau_\rho d\rho + \sigma^\tau_\rho \, dB_\rho \right] d\tau \, , \tag{5.11}$$

and this yields equation (5.5), as long as f^τ_t is continuous in τ. (NB, equation (5.11) holds with $o(\varepsilon)$ error, and replacing t by $t{+}\varepsilon$ in the lower limit of the τ integral in the last term only introduces a further $o(\varepsilon)$ error). □

This instantaneous forward rate (HJM) formulation has a number of technical advantages, which arise from the fact that it can abe integrated over time. For instance it is clear from equations (5.5) and (5.7) that bond prices go to par as they mature. Also equation (5.5) yields the following equation for the short rate:

$$r_q \equiv f^q_q = f^q_t + \int_{\rho=t}^{q} \left[\alpha^q_\rho \, d\rho + \sigma^q_\rho \, dB_\rho \right] \tag{5.12}$$

Differentiating equation (5.12), we have

$$dr_q = \frac{\partial f_t^q}{\partial q} dq + \left\{ \int_{\rho=t}^{q} \left[\frac{\partial \alpha_\rho^q}{\partial q} d\rho + \frac{\partial \sigma_\rho^q}{\partial q} dB_\rho \right] \right\} dq$$

$$+ \alpha_q^q \, dq + \sigma_q^q \, dB_q \tag{5.13}$$

and note by equation (5.9) that

$$\alpha_q^q = \chi_q^q, \quad \sigma_q^q = \Psi_q^q \tag{5.14}$$

The following section will be devoted to comparisons and reconciliation between the Equilibrium and Evolutionary Models. We will see that any version of the Equilibrium Model (ie, any choice of the basic ingredients (ξ, η, γ)) gives rise to a version of the Evolutionary Model, but not vice–versa. For this Evolutionary Model. equation (5.13) can be reconciled with equation (3.1). The essential difference between these equations is that equation (5.13) gives the behaviour of the short rate r_q as time q evolves beyond t, and 'conditional on' (ie, given knowledge of) the entire term structure at time t; whereas equation (3.1) (with t replaced by q) gives the behaviour of r_q conditional on r_q itself.

For the moment, we show how to reconcile equation (5.13) and equation (5.1), which gives the spot rate formulation of the Evolutionary Model. From equation (5.1) we easily obtain that

$$d_t \, p_t^{t+\tau} = \frac{\partial p_t^{t+\tau}}{\partial \tau} dt + {}^1/_\tau \left(p_t^{t+\tau} - r_t\right) dt + \chi_t^{t+\tau} dt + \Psi_t^{t+\tau} \, dB_{t,}$$

and putting τ=0, that

$$d_t r_t = 2\left.\frac{\partial p_t^{t+\tau}}{\partial \tau}\right|_{\tau=0} dt + \chi_t^t\, dt + \Psi_t^t\, dB_t \qquad (5.15)$$

(NB, our notation d_t here means the differential with respect to t, with the other variable constant.)

Equation (5.13) with t=q is reconciled with equation (5.15) using the easily verified fact that

$$\left.\frac{\partial f_t^q}{\partial q}\right|_{q=t} = 2\left.\frac{\partial p_t^q}{\partial q}\right|_{q=t} \qquad (5.16)$$

Note that substituting q for t in equation (5.15) does not yield equation (5.13). The reason for this is that (5.13) gives dr_q given the term structure at time t, whereas (5.15) with q instead of t gives dr_q given the term structure at time q.

Finally in this section, we mention that the model of [Babbs 1990] is just the same as the HJM formulation of the Evolutionary Model; specifically, a and b of [Babbs] Part 4 correspond to α and σ of [HJM 1987]. Also, as is well known (see [HJM 1987], [Carverhill 1988]), the model of [HL 1986] in its continuous time formulation, corresponds to the HJM Model with σ_t^q ($\equiv \sigma$) constant; in our other formulations this corresponds to $\Psi_t^q \equiv \sigma$ being constant by equation (5.9) and $\mu_t^q = -(q-t)$ by equation (5.2).

6. When can the Evolutionary and Equilibrium Models be reconciled?

The Equilibrium Model is characterised by the set of its basic ingredients (ξ, η, γ), which comprise the short rate drift and volatility functions and the risk premium. The Evolutionary Model is characterised by the set of its basic ingredients (ν, μ γ), or equivalently (χ, Ψ, γ) or (α, σ, γ), which comprise the appropriate form of the term structure drift and volatility functions and the risk premium; but note that to be strict, the risk premium is redundant in this characterisation, because it is determined by the other ingredients. Now, any version of the Equilibrium, ie, any choice of the set (ξ, η, γ) for this model, gives rise to a version of the Evolutionary Model, for which (ν, μ, γ) are determined by equation (3.5) (and in this equation, P_t^q is the solution of the model corresponding to (ξ, η, γ)). In this case we will say that the Evolutionary Model with this (ν, η, γ) 'comes from' the Equilibrium Model with this (ξ, μ, γ).

In Proposition 6.1 below, we show that if a version of the Evolutionary Model comes from an Equilibrium Model in the above sense, then the two models are not in conflict. However, it will be clear from Proposition 6.3 that not all Evolutionary Models come from an Equilibrium Model, and hence one can say that the Equilibrium Model is a special case of the Evolutionary Model.

PROPOSITION 6.1

Suppose the Evolutionary Model characterised by some choice of the set (ν, μ, γ) comes from an Equilibrium Model characterised by (ξ, η, γ). Also, suppose the initial term structure is admissible with respect to the Equilibrium Model, ie, it satisfies the Black–Scholes equation (3.6) with (3.7). Then subsequent term structures as dictated by the Evolutionary Model are also admissible with respect to the Equilibrium Model, and so both models agree in their term structure predictions.

Proof

The evolution according to the Evolutionary Model is dictated by equation (4.1), and this suffices, together with the initial condition, to determine the evolution. But the evolution according to the Equilibrium Model must also satisfy equation (3.3), which is the same as equation (4.1). □

PROPOSITION 6.2

If the Evolutionary Model (χ, Ψ, γ) came from the Equilibrium Model (ξ, η, γ), then we must have

$$\eta_t^{(r)} = \Psi_t^t(r) \tag{6.1}$$

for any t, r.

Proof

This follows from the fact that equation (5.15) must agree with equation (3.1). The result of Proposition 6.2 is very natural, but its proof also yields the more obscure fact that

$$2 \left. \frac{\partial p_t^{t+\tau}}{\partial \tau} \right|_{\tau=0} + \chi_t^t(r) = \xi_t(r) \tag{6.2}$$

Lemma 6.4 below will serve to verify this fact directly for a special case of the Equilibrium Model which includes the Vasicek and CIR models.

PROPOSITION 6.3

Suppose the Evolutionary Model (ν, μ, γ) is time homogeneous and comes from the Equilibrium Model (ξ, η, γ), and the drift and volatility functions ν_t^q and μ_t^q depend only on (q–t), ie they are independent of the term structure. Then the Equilibrium Model must actually be the Vasicek Model (characterised by equation (3.8)), and μ_t^q must satisfy equation (3.12)..

Proof

Since we are assuming that the Equilibrium Model applies, we can work with the equations of Section 3. In particular equation (3.5) (together with (6.1)) yields

$$P_t^{q\,'} = \left(\mu_t^q/_{\mu}\right) P_t^q, \quad P_t^{q\,''} = \left(\mu_t^q/_{\mu}\right)^2 P_t^q, \tag{6.3}$$

where the dash indicates the derivative with respect to r and P_t^q is the solution of the Equilibrium Model. Using in the Black–Scholes equation (3.6) yields

$$\frac{\partial P_t^q}{\partial q} = \left[-\tfrac{1}{2}(\mu_t^q)^2 + (\gamma - \xi/_{\mu})\,\mu_t^q + r\right] P_t^q \tag{6.4}$$

which is easily integrated to give

$$P_t^q = \exp\left\{-\int_{\rho=t}^{q} \left[-\tfrac{1}{2}(\mu_\rho^q)^2 + (\gamma - \xi/_{\mu})\,\mu_\rho^q + r\right] d\rho\right\} \tag{6.5}$$

Now, differentiating (6.5) with respect to r yields

$$P_t^{q'} = \left\{ ({\xi'}/_{\mu}) \int_{\rho=t}^{q} \mu_\rho^q \, d\rho - (q-t) \right\} P_t^q \tag{6.6}$$

(NB, $\gamma = \nu/\mu$. Thus the only terms in (6.5) that depend on r are ξ and r itself), and comparing (6.6) with (6.3) yields

$$\xi' \int_{\rho=t}^{q} \mu_\rho^q \, d\rho \;=\; \mu_t^q + \eta(q-t) \tag{6.7}$$

Differentiating (6.7) with respect to t yields

$$-\xi' = \frac{\mu_t^q + \eta}{\mu_t^q}. \tag{6.8}$$

Now, in (6.8) the LHS is independent of t and q, and the RHS is independent of r, and so both must be constant, say equal to $\tilde{\alpha}$ (suggestive notation!). The solution for the RHS of (6.8) is then just

$$\mu_t^q = =\eta/_{\tilde{\alpha}} \left[1 - \exp\left(-\tilde{\alpha}\,(q-t) \right) \right] \tag{6.9}$$

(NB, $\mu_t^q = 0$ by (5.2)), which is the Vasicek volatility (3.12). Also the solution for the LHS of (6.8) is just the Vasicek mean reversion function. □

From Proposition 6.3 it is clear that if we have an Evolutionary Model whose basic ingredients are independent of the term structure but which does not have the Vasicek form (6.9), then it does not confirm to any Equilibrium Model. Also, if we have an Equilibrium Model which is not of Vasicek type, then the Evolutionary Model to which it gives rise must have term structure dependent ingredients.

The CIR Model falls into this category. However, if we take the Vasicek Model and alter the form of the short rate mean reversion function, then the corresponding Evolutionary model must have a volatility function $\Psi_t^q(r)$ which is independent of r for q=t (by Proposition 6.2) but dependent on r for some other values of q.

LEMMA 6.4

In the special case of the Equilibrium Model when μ/η is independent of r, then we have

$$\left.\frac{\partial f_t^q}{\partial q}\right|_{q=t} = (\xi - \gamma\eta). \tag{6.10}$$

Notes

This special case includes Vasicek and CIR, as can be verified by calculating μ. However, by the remarks after Proposition 6.3, it will not hold universally. To see that (6.10) agrees with (6.2) note that $\chi_t^t = \gamma\Psi_t^t$.

Proof

First note that by time homogeneity we have

$$\frac{\partial f_t^q}{\partial q} = \frac{-\partial f_t^q}{\partial t}$$

etc, and from equation (2.2) we easily obtain

$$\left.\frac{\partial f_t^q}{\partial q}\right|_{q=t} = \left.\left[\left[\dot{P}_t^q\right]^2 - \left[\ddot{P}_t^q\right]\right]\right|_{q=t} \tag{6.11}$$

where the dot denotes the t derivative.

Our technique for the proof is to calculate the RHS of (6.11) using the Black–Scholes equation (3.6) for t derivatives, and equation (6.3) for r derivatives. Thus,

$$\begin{aligned}\dot{P} &= -\tfrac{1}{2}\eta^2 P'' + (\gamma\eta - \xi)P' + rP \\ &= \left[-\tfrac{1}{2}\mu^2 + (\gamma\eta - \xi)\,(\mu/\eta) + r\right] P,\end{aligned}$$

Also

$$\begin{aligned}\ddot{P} = &\left[-\dot{\mu}\,\mu + (\gamma\eta - \xi)\,(\dot{\mu}/\eta)\right]P \\ &+ \left[-\tfrac{1}{2}\mu^2 + (\gamma\eta - \xi)\,(\mu/\eta) + r\right]^2 P,\end{aligned}$$

and so

$$(\dot{P})^2 - \ddot{P} = \dot{\mu}\,\mu - (\gamma\eta - \xi)\,(\dot{\mu}/\eta).$$

The result follows because $\mu_t^t = 0$ and $\dot{\mu}_t^t = \Psi_t^t = \eta$ by (5.2) and Proposition 6.2. □

PROPOSITION 6.5

Suppose the Evolutionary Model has basic ingredients (α, σ, γ) which are independent of the term structure, and that it comes from the Equilibrium Model (ξ, η, γ). Also, suppose that the initial forward rate curve $\{f_t^q : q\varepsilon[t,\infty)\}$ is asymptotically admissible with respect to the Equilibrium Model, ie, if $\{\tilde{f}_t^q : q \;\varepsilon\; [t,\infty)\}$ is admissible, then

$$\tilde{f}_t^q - f_t^q \rightarrow 0 \text{ as } q \rightarrow \infty.$$

Then the term structure evolves to become nearly admissible, ie, we have

$$\sup\left\{f_s^{s+\tau} - \tilde{f}_s^{s+\tau} : \tau \;\varepsilon\; [s,\infty)\right\} \rightarrow 0 \text{ as } s \rightarrow \infty,$$

where the initial conditions $\left\{f_t^q\right\}_{q\geq t}$ and $\left\{\tilde{f}_t^q\right\}_{q\geq t}$ evolve to $\left\{f_s^q\right\}_{q\geq s}$ and $\left\{\tilde{f}_t^q\right\}_{q\geq s}$ as t evolves to s.

Note

The Conditions of Proposition 6.5 imply that we are in the Vasicek Model.

Proof

Note that

$$f_s^{s+\tau} = f_t^{s+\tau} + \int_{\rho=t}^{s} \left[\alpha_\rho^{s+\tau}\, d\rho + \sigma_\rho^{s+\tau}\, dB_\rho\right].$$

The result follows simply by subtracting from this the corresponding equation for $\tilde{f}_s^{s+\tau}$ and noting that the integrals cancel because they are independent of the term structure. □

The main conclusion of this section is as follows: Suppose one is working with the Evolutionary Model, and has made the simplifying assumption that the drift and volatility functions are independent of the term structure. Then, unless the volatility has the particular form of Vasicek, ie, (3.12) or (6.9), then one cannot expect the evolution to conform to any Equilibrium Model. However, if it does have this form, then from any initial term structure with the appropriate long rate, the evolution will settle down to being admissible with respect to the Evolutionary Model.

7. Volatility, Stability, and the Long Rate

In this section we will mostly discuss the Evolutionary Model, though in view of Section 6, our conclusions will also apply to the Equilibrium Model.

First, note that in terms of spot rates, the no arbitrage equation (4.2) translates to

$$\chi_t^q = \gamma\, \Psi_t^q + \tfrac{1}{2}\,(q-t)\,(\Psi_t^q)^2 \qquad (7.1)$$

From this we see that the volatility must attenuate to zero for large term to maturity, if the model is to be stable; in fact we must have

$$\Psi_t^q \leq 0\left((q-t)^{-\frac{1}{2}}\right), \tag{7.2}$$

If this does not hold, then either the no arbitrage condition breaks down, or the drift χ_t^q grows unboundedly and the model explodes for large term to maturity. In fact the [HL] model violates condition (7.2) and this is why it is theoretically unstable.

The following proposition almost gives a converse to this criterion for instability; it essentially shows that if

$$\Psi_t^q \leq 0\left((q-t)^{-1}\right), \tag{7.3}$$

then the model is stable and the long rate is constant.

PROPOSITION 7.1

Suppose the drift and volatility in the Evolutionary Model are independent of the term structure, and attenuate for large term to maturity such that

$$\int_{\tau=t}^{q} \alpha_t^\tau \, d\tau \leq K, \quad \int_{\tau=t}^{q} (\sigma_t^\tau)^2 \, d\tau \leq K^2, \tag{7.4}$$

uniformly for some κ, for all t,q with $q \geq t$.

(i) Then we have

$$E\left[\,\left|f_s^q - f_t^q\right|^2\right]^{\frac{1}{2}} \leq 2K \tag{7.5}$$

for all t, s, q with $t \leq s \leq q$. Thus, the term structure does not drift indefinitely far away from the initial condition.

(ii) Suppose the initial term structure has a well defined long rate r_∞, in the sense that if the initial time is t, then

$$f_t^q \to r_\infty \text{ as } q \to \infty$$

Then this long rate does not move as the term structure evolves.

Notes

Assuming that the risk premium γ is constant, condition (7.3) and equation (7.1) easily yield condition (7.4). Also, in view of (2.3), (7.5) easily yields a similar result for the spot rate rather than the forward rate. The volatility $\Psi_t^q = (q-t)^{-\frac{1}{4}}$ falls into the gap between the stability and instability criteria above. This volatility easily gives an unstable model.

Proof

(i) Note that

$$E\left[\left| f_s^q - f_t^q \right|^2 \right]^{\frac{1}{2}} \leq E\left[\left| \int_{\rho=t}^{s} \alpha_\rho^q \, d\rho \right|^2 \right]^{\frac{1}{2}} + E\left[\left| \int_{\rho=t}^{s} \sigma_\rho^q \, dB_\rho \right|^2 \right]^{\frac{1}{2}}$$

$$\leq \left| \int_{\rho=t}^{q} \alpha_\rho^q \, d\rho \right| + \left| \int_{\rho=t}^{q} (\sigma_\rho^q)^2 \, d\rho \right|^{\frac{1}{2}}$$

$$= 2K$$

(ii) Take $\varepsilon > 0$ and then take q such that $|f_t^q - r_\infty| < \varepsilon$. Then for any $s \geq t$ we have

$$\left| f_s^q - r_\infty \right| \leq \left| f_s^q - f_t^q \right| + \left| f_t^q - r_\infty \right|$$

$$\leq \left| \int_{\rho=t}^{q} \alpha_\rho^q \, d\rho \right| + \left| + \left| \int_{\rho=t}^{q} (\sigma_s^q)^2 \, d\rho \right|^{\frac{1}{2}} = \varepsilon$$

$$= \left| \int_{\tau=q-t}^{q-s} \alpha(\tau) \, d\tau \right| + \left| \int_{\tau=q-t}^{q-s} \sigma(\tau)^2 \, d\tau \right|^{\frac{1}{2}} + \varepsilon$$

putting $\alpha_t^q = \alpha(\tau)$ and $\sigma_t^q = \sigma(\tau)$ for $\tau = q-t$. It is clear from (7.4) that for sufficiently large q, the integrals in this last expression will be arbitrarily small. □

The conclusion of this section is that attenuation of the volatility for large term to maturity is necessary for a stable model, and (in a slightly stronger form) it is also sufficient, and it leads to the long rate being constant. This conclusion is related to that of the paper [DIR], which deals with a very general arbitrage free model, and is entitled 'Long Forward Rates can Never Fall'. Their model subsumes ours, except that they prohibit negative interest rates. In Section 9 below, we will discuss these stability results in the light of some empirical considerations; notably the fact that the long rate is not actually constant.

A necessary criterion for stability in the Equilibrium Model is that the short rate be mean reverting; note that the Vasicek volatility does satisfy our sufficient criterion if the mean reversion parameter $\tilde{\alpha}$ is positive. Also note that, putting s = q in (7.5), the stability criterion of Proposition 7.1 implies that for the Evolutionary Model, the short rate does not drift indefinitely far away. The relationship between mean reversion of the short rate and attenuation of the volatility is studied in [Webber 1990].

8. Interest Rate Option Valuation

In the Equilibrium Model, the Black–Scholes Equation (3.6) can be used to value any term structure contingent claim, and not just the pure discount bonds as in Section 3. Unfortunately, the derivation of Equation (3.6) does not generalise to the Evolutionary Model because one essential ingredient is missing; namely that the entire term structure should be determined by the short rate. Our aim in this Section is to discuss contingent claim valuation in the Evolutionary context. This will be based largely on the very comprehensive risk neutral expected payoff formula for contingent claims, which we present in Proposition 8.1. We will see that the valuation of contingent claims depends on the volatility of the term structure, but not on its drift or the risk premium, and this is analogous to the basic Black–Scholes valuation for equity options. Our approach does lead to a Black–Scholes equation in the Evolutionary context (see Proposition 8.9), but this is less central than that in the Equilibrium context.

This section is organised as follows: first we discuss contingent claim valuation in the Evolutionary Model via a series of propositions; then we compare our ideas with those of some other papers, namely [Hull White 1990a] [Dybvig 1989], [Black Derman Toy 1990]; finally we mention some ways in which our propositions might be implemented in a practical context.

<u>*PROPOSITION 8.1*</u>

Suppose a contingent claim has value ϕ_q at time q. Then at any time t with $t < q$, its value ϕ_t is given by

$$\phi_t = \tilde{E}_t \left[\exp\left\{ -\int_{\tau=t}^{q} r_\tau \, d\tau \right\} \phi_q \right], \tag{8.1}$$

where $\tilde{E}_t$ is the 'risk neutral expectation' at time t. This expectation corresponds to the 'risk neutral (martingale) probability' which is characterised by the ('Girsanov transformation') formula

$$d\tilde{B}_t = dB_t - \gamma dt, \tag{8.2}$$

γ being the risk premium. To obtain the probability distribution of any process under the risk neutral probability, we take the stochastic equation for the process and substitute $d\tilde{B}_t$ for dB_t, as given by formula 8.2.

Notes and Explanation

As an example of the transformation to the risk neutral probability le us consider the pure discount bond price P_t^q itself. Substituting formula (8.2) into equation (4.1), we see that the risk–neutral distribution of P_t^q is the same as that of $\tilde{P}_t^q$ given by

$$d\tilde{P}_t^q / \tilde{P}_t^q = r_t\, dt + \mu_t^q\, dB_t \tag{8.3}$$

We see from equation (8.3) that $\tilde{P}_t^q$ discounted at the short rate is a martingale.

Proof of Proposition 8.1

The evaluation of contingent claims as risk–neutral expectations is a very general principle; see [Harrison Kreps 1979], [Harrison Pliska 1981], also [Ingersoll 1987]. In general the existence of the risk–neutral martingale probability is associated with the absence of arbitrage opportunities among the assets, and it is characterised by the condition that under it, the asset values should be martingales when discounted at the short rate. In the context of the

Evolutionary Model, it is clear that this probability must be characterised by formula (8.2), in order to render $\tilde{P}_t^q$ as in equation (8.3) a martingale when discounted at the short rate. Also, it is clear that the probability being well defined corresponds to γ being independent of the bond (ie of q), and hence to the absence of arbitrage possibilities. Also, in general a formula like (8.1) holds for the value of any 'attainable' contingent claim, ie, any claim that can be replicated using a dynamic portfolio of the underlying assets, because the value of such a claim must also be a martingale under the risk–neutral probability, and when discounted at the short rate. In the Evolutionary Model any contingent claim is attainable because the model is 'complete'; these are enough independent assets to 'cover' (ie, hedge) the dimension of uncertainty (namely 1) in the model. □

In order to use formula (8.1) we must know the risk–neutral distribution of the short rate. This is given by:

PROPOSITION 8.2

The distribution of the short rate r_q, as governed by the risk–neutral probability, and given knowledge of the term structure at a previous time t, is the same as the distribution of $\tilde{r}_q$ given by

$$\tilde{r}_q = f_t^q + \frac{\partial}{\partial q} \int_{\rho=t}^{q} \left[\tfrac{1}{2} (\mu_\rho^q)^2 \, d\rho - \mu_\rho^q \, dB_\rho \right] \tag{8.4}$$

$$\equiv f_t^q + \int_{\rho=t}^{q} \frac{\partial}{\partial q} \left[\tfrac{1}{2} (\mu_t^q)^2 \, d\rho - \mu_\rho^q \, dB_\rho \right] \tag{8.5}$$

Proof

For this we substitute $d\tilde{B}_t$ for dB_t in equation for r_q, namely (5.13). Thus

$$\tilde{r}_q = f_t^q + \int_{\rho=t}^{q} \left[\alpha_\rho^q \, d\rho + \sigma_\rho^q \, d\tilde{B}_\rho \right]$$

$$= f_t^q + \int_{\rho=t}^{q} \left[\frac{\partial}{\partial q} (v_\rho^q - \tfrac{1}{2} \mu_\rho^{q\,2}) \, d\rho - \frac{\partial}{\partial q9} (\mu_\rho^q) \, d\tilde{B}_\rho \right]$$

(using equation (5.6)). Now, to get equation (8.4) take the differentials from under the integrals, and apply equation (8.2), and then to get equation (8.5) put the differentials back again. The justification for swapping the differential and integral is the fact that $v_q^q = \mu_q^q = 0$, and the general formulae

$$\frac{\partial}{\partial q} \int_{\rho=t}^{q} \theta_\rho^q \, d\rho = \theta_q^q + \int_{\rho=t}^{q} \frac{\partial}{\partial q} \theta_\rho^q \, d\rho,$$

$$\frac{\partial}{\partial q} \int_{\rho=t}^{q} \theta_\rho^q \, d\rho = \theta_q^q \frac{dB_q}{dq} + \int_{\rho=t}^{q} \frac{\partial}{\partial q} \theta_\rho^q \, dB\rho. \qquad \square$$

THEOREM 8.3

If a contingent claim has value ϕ_q at time q, then its value ϕ_t at time t with $t < q$ is given by

$$\phi_t = E_t \left[\exp\left\{ - \int_{\tau=t}^{q} \tilde{r}_\tau \, d\tau \right\} \phi_q \right], \tag{8.6}$$

where $\tilde{r}_\tau$ is given by equation (8.4) or (8.5).

Proof

Clear from Propositions 8.1 and 8.2. □

COROLLARY 8.4 (to Proposition 8.2*)*

The risk–neutral distribution of the random variable $\exp\left\{-\int_{\tau=t}^{q} r_\tau \, d\tau\right\}$ given the term structure at time t, is given by the unadjusted distribution

$$\exp\left\{-\int_{\tau=t}^{q} \tilde{r}_\tau \, d\tau\right\} \equiv P_t^q \exp\left\{\int_{\rho=t}^{q}\left[\mu_\rho^q \, dB_\rho - \tfrac{1}{2}\mu_\rho^{q^2} \, d\rho\right]\right\} \tag{8.7}$$

Proof

Substitute for $\tilde{r}_\tau$ in the LHS of (8.7) using (8.4). □

Note that the RHS of (8.7) is an exponential martingale, and so it has expectation 1. Therefore, if we put $\phi_q \equiv 1$ in (8.1), then this formula correctly gives the time t value of the q–maturity bond, ie, P_t^q. Also, if the volatility μ_ρ^q only depends on (q–ρ) and not on the term structure, then (8.7) has lognormal distribution $P_t^q \exp N\left(-\tfrac{1}{2}\int_{\rho=t}^{q} \mu_\rho^{q^2} \, d\rho, \left[\int_{\rho=t}^{q} \mu_\rho^{q^2} \, d\rho\right]^{\frac{1}{2}}\right)$, where N(α,σ) denotes the normal distribution with mean α and standard deviation σ.

PROPOSITION 8.5

Take times $t < s < q$. Then given the term structure of time t, the spot interest rate p_s^q is random, and its risk–neutral distribution is the same as the unadjusted distribution $\tilde{p}_s^q$, given by

$$\tilde{p}_s^q = g_{ts}^q + \int_{\rho=t}^{s} \left\{ \frac{1}{2} \left[\frac{(\mu_\rho^q)^2 - (\mu_\rho^s)^2}{q-s} \right] d\rho - \left[\frac{\mu_\rho^q - \mu_\rho^s}{q-s} \right] dB_\rho \right\} \tag{8.8}$$

Proof

Similar to Proposition 8.2. Start from the formula

$$f_s^\tau = f_t^\tau + \int_{\rho=t}^{s} \left[\alpha_\rho^\tau \, d\rho + \sigma_\rho^\tau \, dB_\rho \right]$$

for $s \le \tau \le q$, which comes from formula (5.5). Then substitute for α, σ in terms of μ, ν using equation (5.6), take the differential from under the integral; substitute $d\tilde{B}_\rho$ for dB_ρ using (8.2), so that the ν term cancels, to obtain

$$\tilde{f}_s^\tau = \tilde{f}_t^\tau + \frac{\partial}{\partial \tau} \int_{\rho=t}^{s} \left[\frac{1}{2} (\mu_\rho^\tau)^2 \, d\rho - \mu_\rho^\tau \, dB_\rho \right]$$

Finally, act on this last equation with $\frac{1}{q-s} \int_{\tau=s}^{q} \ldots d\tau$, and note that

$$\tilde{p}_s^q = \frac{1}{q-s} \int_{\tau=s}^{q} \tilde{f}_s^\tau \, d\tau. \quad \square$$

Note that in the limit as s –>q, equation (8.8) gives us equation (8.5). Also Proposition 8.5 gives rise to the following Corollary, which is the basic for the option valuation formula of Theorem 8.7 below.

COROLLARY 8.6

Take times $t < s < q$. Then given the term structure at time t, the pure discount bond price P^q_s is random, and its risk–neutral distribution is the same as the unadjusted distribution

$$\tilde{P}^q_s = \frac{P^q_t \exp\left\{-\int_{\rho=t}^{s}\left[\frac{1}{2}(\mu^q_\rho)^2 d\rho - \mu^q_\rho\, dB_\rho\right]\right\}}{P^s_t \exp\left\{-\int_{\rho=t}^{s}\left[\frac{1}{2}(\mu^s_\rho)^2 d\rho - \mu^s_\rho\, dB_\rho\right]\right\}} \tag{8.9}$$

Proof

Use (8.8) in the equation $\tilde{P}^q_s = \exp\left\{-(q-s)\,\tilde{p}^q_s\right\}$. □

Note that (8.9) does not give a martingale as s evolves, but with equation (8.7) it does make the following equation work:

$$P^q_t = E_t\left[\exp\left\{-\int_{\tau=t}^{s}\tilde{r}_\tau\, d\tau\right\}\tilde{P}^q_s\right] \tag{8.10}$$

One might think it a paradox that equation (8.9) does not give the equation

$$d\tilde{P}^q_s \Big/ \tilde{P}^q_s = r_s\, ds + \mu^q_s\, dB_s \quad \text{for } t < s < q. \tag{8.11}$$

The resolution of this paradox is that (8.9) tells us how $\tilde{P}^q_s$ behaves given the term structure at time t with t < s, but (8.11) tells us how $\tilde{P}^q_s$ behaves given the term structure at time s itself. One could easily obtain non–risk–neutral versions of (8.9) and (8.11), and a similar paradox would hold. Also, this paradox is essentially the same as that which exists between (5.13) and (5.15) with t replaced by q.

The following Theorem gives a formula for option valuation in the Evolutionary model. At the end of this section we will briefly discuss the applications of this formula for practical option valuation.

THEOREM 8.7

Consider the option to buy (or sell) at time q and at strike price X, a bond with payments $C_1, C_2, \ldots C_n$ at times $q_1, q_2, \ldots, q_n$ beyond time q. The value consistent with the Evolutionary Model at time t of this bond, is given by

$$E_t\left[\left\{\pm\left\{\left(C_1 P^q_t R_1 + \ldots + C_n P^{q_n}_t R_n\right) - X P^q_t R_0\right\}\right\}^+\right]$$

where (8.12)

$$R_i = \exp\left\{-\int_{\rho=t}^{q}\left[\tfrac{1}{2}(\mu^{q_i}_\rho)^2\, d\rho - \mu^{q_i}_\rho\, dB_\rho\right]\right\} \text{ for } i = 0, 1, \ldots, n;\ q_o \equiv q$$

and $\left\{\ldots\right\}^+$ denotes the greater of the expression in the brackets and zero. (Also take '+' for 'buy' and '–' for 'sell'.)

Proof

Substitute into equation (8.6) using (8.7), (8.9) and taking

$$\phi_q \left\{ \pm \left\{ \text{(Bond price at time q)} - X \right\} \right\}^+ . \quad \square$$

The next proposition presents the elegant approach of [Margrabe 1978], which is also implicit in [Merton 1973], for the option to exchange one asset for another. it is applicable to an option on a pure discount bond, because the option say to sell at time q and at a price X, an amount C_1 of the pure discount bond to mature at time q_1, is equivalent to the option to give away X of the q maturity pure discount bond at time q, in return for C_1 of the q_1 maturity pure discount bond. This Margrabe approach is an alternative (for options on pure discount bonds) to the risk–neutral expectation approach of the rest of this section.

PROPOSITION 8.8

Suppose the asset prices P_t and Q_t obey the equations

$$dP_t / P_t = v_t^P \, dt + \mu_t^P \, dB_t^P,$$

$$dQ_t / Q_t = v_t^Q \, dt + \mu_t^Q \, dB_t^Q,$$

$$\left\langle dB_t^P, dB_t^Q \right\rangle = \delta_t^2 \, dt \; (\delta_t \text{ nonrandom})$$

(ie, the correlation of their returns is δ_t, which is nonrandom).

Then the option to receive at time q asset Q in return for asset P has value at time t

$$\left.\begin{aligned} &Q_t\, N(d^+) - P_t\, N(d^-) \\ &\text{where} \\ &d^{\pm} = \left[\log\left[Q_t/P_t\right] \pm G/2\right] \Big/ \sqrt{G} \\ &G = \int_{\rho=t}^{q} \left[(\mu_\rho^P)^2 + (\mu_\rho^Q)^2 - 2\delta_\rho^2\, \mu_\rho^P\, \mu_\rho^Q\right] d\rho \end{aligned}\right\} \quad (8.13)$$

This option can be perfectly hedged by an amount $-N(d^+)$ of asset Q together with $+N(d^-)$ of asset P. If the two assets are perfectly correlated ($\delta_t \equiv 1$) then either asset can alone hedge the option. If hedged by Q, then one should take an amount $-N(d^+) + G(Q{:}P)\,N(d^-)$, where $G(Q{:}P)$ is the 'gearing ratio', ie, the relative sensitivity to the random factor of the assets; here we have $G(Q{:}P) = (\mu_t^P/\mu_t^Q)\,(P_t/Q_t)$.

<u>Proof</u>

See [Margrabe 1978] or [Merton 1973]. □

The fact that (8.13) agrees with (8.12) under the conditions of Proposition 8.8 is clear from discussions about the elementary Black–Scholes formula; see for example [Carverhill, Webber 1988]. Also, the option valuation according to the Vasicek Equilibrium Model would be the same as Formula (8.12) or (8.13) (with the appropriate volatility function, ie, equation (3.12)), except that we would use the current bond prices given by the model rather than given empirically. Thus, one might naïvely guess these formulae as modifications of the Vasicek Model which conform to the current term structure. Note that in the paper [Jamshidian 1989] the Vasicek value of a coupon bond option has a similar form to our Formula (8.12).

We now mention the issue of hedging options on coupon bonds in the Evolutionary Model. Our approach is to calculate using formula (8.12) the sensitivity of the option value to the random factor in the model. Thus, suppose that over a short time interval [t, t+ε] the Brownian Motion increment $\Delta_t^{t+\varepsilon} B$ is either of ± B. Then the option of Theorem 8.7 has sensitivity

$$\frac{1}{2B} \sum_{\{\tau=+1 \text{ or } -1\}} E\left[\left\{ \pm \left\{ (P_t^{q_1} + r_t\, \varepsilon + \tau\, \mu_r^{q_1} B)\, R_1 + \ldots - (P_t^{q} + r_t\, \varepsilon + \tau\, \mu_t^{q} B) R_o \right\} \right\}^+ \right],$$

which will be virtually independent of B if B is sufficiently small. The sensitivity of the coupon bond with payments $D_1, \ldots D_m$ at times $t_1, \ldots t_m$ is just $\mu_t^{t_i} D_1 + \ldots + \mu_t^{t_m} D_m$, and the ratio of these sensitivities is the hedge ratio if we hedge using the bond.

The next proposition presents a Black–Scholes equation in the Evolutionary context. Note that it is not as central as is equation (3.6) in the Equilibrium context; it is much more restrictive in its application; and we arrive at it by a completely different route.

PROPOSITION 8.9

Consider a contingent claim that pays off at time q an amount which depends only on the short rate at that time. Write $\phi_q(r)$ for the payoff if the short rate then is r. Then the time t value ϕ_t of this option is the solution to the Black–Scholes equation

$$\frac{\partial \phi_\tau}{\partial \tau} = -\frac{1}{2}\tilde{\eta}_\tau^2 \frac{\partial^2 \phi_\tau}{\partial r^2} + \xi_\tau^2 \frac{\partial \phi_\tau}{\partial r} + r\phi_\tau \text{ for } t \le \tau \le q \tag{8.14}$$

with the final condition ϕ_q given, and where $\tilde{\eta}\tau$ and ξ_τ are the coefficients in the equation

$$d\tilde{r}_\tau = \xi_\tau \, d\tau + \tilde{\eta}_\tau \, dB_\tau, \tag{8.15}$$

which is obtained by differentiating equation (8.5).

Proof

This follows because formula (8.6) is the Feynman Kac formula corresponding to the reverse time diffusion equation (8.14). (See [Oksendal 1985]. □

The assumption that the option payoff depends only on the short rate at the payoff time is a good approximation if it refers to a short lived bond, but it is not exact in the Evolutionary Model. Also, for τ strictly between t and q the solution to the Black–Scholes equation (8.14) is $E_\tau\left[\exp\left\{-\int_{\rho=\tau}^{q} \tilde{r}_\rho \, d_\rho\right\} \phi(\tilde{r}_q) \;\middle|\; \tilde{r}_\tau = r\right]$, but this might not be the value at that time of the contingent claim; in fact this value may not be determined by the short rate at time τ. Therefore equation (8.14) cannot be used for valuing American Options.

We now compare the ideas that we have presented in this section with those of the papers [Hull White 1990a], [Dybvig 1989], [Black Derman Toy 1990].

The goal of [Hull White 1990a] is to adapt the Equilibrium Model so as to avoid what they regard as its main shortcoming, namely that it does not fit the current term structure. They do this by altering the drift in their short rate equation (which corresponds to our equation (3.1)), so that the solution to the correspondingly altered Black–Scholes equation fits the current term structure. (They can also fit the 'term structure of volatility' which corresponds to our function μ or Ψ.) Via this approach some of the advantages of the Equilibrium Model are retained, in particular the relatively easy option valuation procedures.

Our Evolutionary Model can also be interpreted in terms of altering the drift in the short rate equation. The risk–neutral equivalent of equation (3.1) is

$$d\tilde{r}_t = (\xi - \gamma\eta)\, dt + \eta\, dB_t, \tag{8.16}$$

and substituting the coefficients of this equation into equation (8.14), rather than the coefficients of (8.15), gives the Equilibrium Black–Scholes equation (3.6) rather than the Evolutionary Black–Scholes equation. Note also that if the Evolutionary Model comes from an Equilibrium Model, then $\tilde{\eta}$ of (8.15) is the same as η of (8.16), and so the Evolutionary Model corresponds to taking the drift coefficient $\tilde{\xi}$ of (8.15) rather than $(\xi - \gamma\eta)$ of (8.16).

However, the modified Equilibrium Model of [Hull White 1990a] does not agree with our Evolutionary Model; the drift alterations are different. From our perspective, their drift alteration seems ad–hoc, and its interpretation and stability over time are unclear. On the other hand the actual option valuation formula at least for their modified Vasicek model does agree with our Evolutionary Model; their justification for their formula is just like our Proposition 8.8.

The criticisms in [Dybvig 1989] are essentially the same as those we have just made. However, they are meant to apply to the Ho–Lee model, which can also be thought of as the modified (in the sense of [Hull White 1990a]) 'degenerate Vasicek Model' in which the volatility function Ψ_t^q is constant. In this case the 'ad–hoc' drift alteration of [HW] does agree with the Evolutionary Model, and so from our perspective it is not ad–hoc. The current–term–structure–consistent model that [Dybvig 1989] proposes is essentially the same as the higher–factor Evolutionary model that we describe in the following section.

The paper [Black Derman Toy 1990] gives a Binomial algorithm for bond option evaluation, which matches the initial term structure and volatility function. The model which underlies their algorithm is made explicit in [Hull White 1990a], and it fits into their framework. Therefore our criticisms of [Hull White 1990a] also apply to [Black Derman Toy 1990], and their option valuations are in conflict with ours.

Finally in this section, we will briefly discuss practical option valuation, based on the Evolutionary Model. This topic will be developed further in future research. Also, for the moment, we will restrict ourselves to the 1 factor model. This valuation is greatly simplified if we assume that the volatility function μ_t^q depends only on (q–t), and not on the term structure. Also, we have already argued that this is a reasonable assumption.

For a European option on a coupon bond, the valuation can be based on formula (8.12). Also, if we assume that the volatility function μ_t^q depends only on (q–t) and is moreover as in formula (3.12), then our Evolutionary Model is coming from the Vasicek model, and we can apply the trick of [Jamshidian 1989]. This trick reduces the coupon bond option evaluation to a series of pure discount bond option evaluation, one corresponding to each coupon, and each one being given by a version of formula (8.13). To apply the [Jamshidian] trick, note that for this volatility, if we take the current bond prices P_t^q, $P_t^{q_i}$, ...

$p_t^{q_n}$, to be given by the Vasicek Model rather than empirically, then formula (8.12) agrees with the Vasicek valuation. Therefore the transition from the Vasicek Model to the Evolutionary model can be achieved formally merely by adjusting the constants X, C_1, ... C_n. From our perspective, the [Jamshidian] trick requires that the time q prices of the bonds which correspond to the coupon payments, given the time t term structure, should be perfectly correlated with each other. In the notation of Theorem 8.7, these prices are $(P_t^{q_i} R_i / P_t^q R_o)$ for i=1, ..., n, and the i^{th} and j^{th} bond prices are related simply via the factor $e^{-\tilde{\alpha}(q_i - q_j)}$. For more general volatility functions these bond prices might not be perfectly correlated.

If μ_t^q depends only on (q–t) but is not necessarily as in equation (3.12), then to evaluate formula (8.12) one can use a binomial (or more accurately, a trinomial/explicit finite difference) type procedure. For this note that each random variable R_i in the formula can be obtained as the solution at time q to the stochastic equation $dr_\rho^i = r_\rho^i \, \mu_\rho^{q_i} \, dB_\rho$ with initial condition $r_t^i \equiv 1$, and that these r_ρ^i's have perfectly correlated increments. For the procedure one should replace each of these processes r_ρ^i by a trinomial random walk with appropriate variance.

For American coupon bond options Theorem 8.7 and Proposition 8.9 will not hold, and the trinomial procedure outlined above is not adaptable to this situation. Any such procedure would have to be based on a version of equation (8.11) rather than formula (8.12).

Interest rate options such as caps are strips of European options, each of which can be thought of as applying to a pure discount bond, and amenable to Proposition 8.8. The valuation and efficient hedging of interest rate caps will be dealt with in the paper [Carverhill 1991].

9. Some Empirical Considerations: Introducing more Random Factors into the Models

So far we have included just 1 factor of Brownian Motion in our models. For both the Equilibrium Model and the Evolutionary model this implies that the evolution over short times of all prices or rates are perfectly correlated. This correlation is supported empirically if attention is restricted to a relatively short term structure. [Dybvig 1989] reports that for the 9 month or 5 year term structure a single factor can account for about $98\frac{1}{2}\%$ of the evolution (in the sense of principal components, which we describe below). In this paper, the 9 months analysis is based on monthly US data between June 1964 and December 1987, and the 5 year analysis is based on annual US data between 1952 and 1987. Between the 1 factor models we favour the Evolutionary over the Equilibrium, on the grounds that empirically the term structure is not determined just by the short rate. This empirical assertion is supported by the 3 dimensional graph of Section 2, Figure 2.5.

There are a number of motivations for including more factors of Brownian Motion into the Evolutionary or Equilibrium Model. One is to capture more of the comovements of the rates or prices; [Steeley 1991] reports that for the 18 year UK term structure a single factor can only account for about 75% of the behaviour. Another motivation is to avoid the conclusion, which is drawn in Section 7 above but is also at odds with Figure 2.5, that the long rate should be constant.

Thus, it is reasonable to set up a 2–factor Equilibrium Model, in which the determining factors are the short rate and the long rate. This has been done for instance, in the paper [Schaefer Schwartz 1984]. The development of the higher factor model is very similar to that of the 1 factor model of Section 3; having specified the 'basic ingredients', ie, the process for the determining parameters and the risk parameters, one obtains a Black–Scholes Equation for bond prices and any other contingent claim value. Note that in [Schaefer

Schwartz 1984] the long rate is actually taken to be the 'consul' rate, which is the reciprocal of the price of the 'consul bond', which pays out continuously at unit rate. Therefore this model does not correspond directly to any of the formulations of the present paper, though it does to the spirit of our Equilibrium Model. Also this paper appeals to the empirically supported assumption (see references in the paper, also [Steeley 1989]) that the short rate and the spread between short and long rate are orthogonal mean reverting processes.

This 2 factor Equilibrium Model still suffers from the empirical fact, again supported by Figure 2.5, that the term structure is not determined just by the short and the long rate, and thus we are led to favour the 2 factor Evolutionary Model over it. Recall also from Section 6 that the Evolutionary Model is more general, ie, the Equilibrium Model can be regarded as a special case of it. In our concluding section we will argue philosophically in favour of the Evolutionary Model, and against the assertion that introducing more factors into the Equilibrium Model will make it as accurate as we desire.

We now describe how to develop and estimate a higher factor Evolutionary Model using Principal Components Analysis. This technique is described in general in [Lawley Maxwell 1971], [Steeley 1989b]. Just as with the Equilibrium Model, the more difficult aspect of this is to specify the basic ingredients of the model; having done this the development is very similar to that of the single factor model. We take as our point of departure Formulation 5.2 of the model. Our basic equation for the n–factor model is

$$p^q_{t+\varepsilon} = g^q_{t,t+\varepsilon} + \chi^q_t \, \Delta_t^{t+\varepsilon} t + \sum_{i=1}^{n} \Psi^{q,i}_t \; \Delta_t^{t+\varepsilon} B^i \tag{9.1}$$

the notation being as in Section 5. Thus the basic ingredients are the functions $\chi(s)$, $\Psi^1(s)$, ..., $\Psi^n(s)$, where $\chi(s) \equiv \chi^q_t$ for $s = t-q$ etc; thus we are assuming time homogeneity in the model.

We estimate the volatility functions $\Psi^1, \ldots, \Psi^n$ using Principal Components Analysis. First, restrict attention to the discrete time grid $\{t_o < t_1 < t_2 \ldots \}$, in which $t_{i+1} - t_i = \varepsilon$ for all i and for a small ε, say ε being 1 month. Then estimate the term structures as in Section 2, in terms of spot rates, corresponding to these times. Next, calculate the following collection of time series, each one corresponding to a given term–to–maturity

$$\Delta_{t_0}^{t_1} p(t_2), \quad \Delta_{t_1}^{t_2} p(t_3), \quad \Delta_{t_2}^{t_3} p(t_4), \ldots \qquad \text{(term } \varepsilon\text{)}$$

$$\Delta_{t_0}^{t_1} p(t_3), \quad \Delta_{t_1}^{t_2} p(t_4), \quad \Delta_{t_2}^{t_3} p(t_5), \ldots \qquad \text{(term } 2\varepsilon\text{)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta_{t_0}^{t_1} p(t_n), \quad \Delta_{t_1}^{t_2} p(t_{n+1}), \quad \Delta_{t_2}^{t_3} p(t_{n+2}), \ldots \qquad \text{(term } n\varepsilon\text{)}$$

(where $\Delta_{t_i}^{t_{i+1}} p(t_j) = p_{t_{i+1}}^{t_j} - g_{t_i t_{i+1}}^{t_j}$)

Now form the matrix $C \equiv \{c_{ij}\}$ i,j = {1, ... , n} in which each entry c_{ij} is the covariance between the series corresponding to term iε and the series corresponding to term jε. Next, calculate the eigenvalues, denoted $\lambda^1, \lambda^2, \ldots$ in descending order, and corresponding orthonormal basis of eigenvectors, denoted $\phi^1, \phi^2, \ldots$ Then the volatility functions are given by $\Psi^i = \sqrt{\lambda^i}\phi^i$. Also, their relative importance can be ascertained from the sizes of the corresponding eigenvalues; in fact the first say m volatilities will account for a proportion $(\lambda^1 + \ldots + \lambda^m) / (\lambda^1 + \ldots + \lambda^n)$ of the term structure evolution in a least squares sense, if the model is driven by equation (9.1).

This analysis has been carried out by a number of researchers. As mentioned above, for relatively short term structures just 1 factor is sufficient to account for virtually all of the evolution. Also [Steeley 1989b], working with the 18 year UK term structure, over the period October 85–October 87, concludes that taking 1, 2 and 3 factors accounts respectively for 87%, 94%, 98% of the evolution. We agree that 2 or 3 factors is enough to take in this model; beyond this number we are constrained by the shortcomings of the model itself, in particular by the time homogeneity assumption, and so it is better to refine the model rather than just include more factors. In his paper, Steeley also presents graphs of these 3 principal factors.

Note that the Principal Components estimation of the volatility functions implicitly assumes that these functions do not depend on the term structure; this corresponds to the Vasicek–type assumption that we have made earlier in this paper, but which theoretically leads to the (small) possibility of negative interest rates.

One can estimate the drift function χ in terms of the expectations associated with the above time series, corresponding to each time to maturity. Note that theoretically each volatility function Ψ^i is associated with a risk premium γ^i and we have

$$\nu = \gamma^1 \mu^1 + ... + \gamma^n \mu^n \tag{9.2}$$

where ν, μ^i are related to χ, Ψ^i as in Equation (5.2). However, these risk premia are difficult to estimate even in the single factor model and in fact they and the drift are not needed in option valuation.

Using Equation (9.2) we can show theoretically just as in Section 7, that for stability of the model we should have attenuation to zero of all the volatility functions, and constancy of the long rate. Thus, theoretically introducing more factors into the model does not solve the paradox of the long rate. Our feeling about this is that the 'long rate' should be allowed to correspond to a long but not infinite time to maturity, beyond which the model breaks down due to the sparsity of available bonds. Note that these remarks apply to the Equilibrium Model as well as the Evolutionary Model. Also, even the first factor presented in [Steeley 1989b] does not attenuate to zero as it theoretically should.

Valuing options in the higher factor Evolutionary Model is essentially the same as in the single factor model. If we split the drift function ν into the sum $\nu^1 + \ldots + \nu^n$ where $\nu^i = \gamma^i \mu^i$, then the factors remain separate in all the calculations, and Theorem 8.7 applies but with

$$R_i = \exp\left\{ - \sum_{j=1}^{n} \int_{\rho=t}^{q} \left[\tfrac{1}{2} \left(\mu^j(q_i-\rho)\right)^2 d\rho - \mu^j(q_i-\rho)\, dB_\rho^j \right] \right\} \tag{9.3}$$

10. Conclusions

Our aim in this paper has been to develop and compare from a unified perspective, the various models of the term structure of interest rates and associated procedures for option valuation. We have collected these models into two groups, which we have called the Equilibrium Model and the Evolutionary Model. The essential difference between these two collective models is that the Equilibrium Model seeks to characterise the shape of the term structure at any instant, assuming that this is determined by some set of parameters, whereas the Equilibrium Model seeks to characterise the evolution of the term structure, starting from an exogenously given initial shape. Thus, the Equilibrium Model predicts a shape for the term structure which may not accurately match the actual term structure at any given time, but which one hopes to be a reasonable match over all time; whereas the Evolutionary Model predicts a term structure which decreases in accuracy as the prediction is pushed further into the future.

Our first presentation of the models is given in Sections 3 and 4, and it makes them look quite different from one another. The Equilibrium Model takes as its basic ingredients the short rate process and the risk premium, and the bond prices satisfy a Black–Scholes type equation. On the other hand, the Evolutionary Model takes as its basic ingredients the drift and volatility of the prices or rates, as functions of their term–to–maturity. However, we see in Section 6 that the Equilibrium Model can be cast into the framework of the Evolutionary model by deriving the evolutionary equation within the Equilibrium Model. Then if the initial term structure is admissible with respect to the Equilibrium Model (ie, it is a shape which is allowed by the model), then the two models will not be in conflict in their predictions for the subsequent behaviour of the term structure.

One can also ask whether any given version of the Evolutionary model can be obtained in this way from a version of the Equilibrium Model. In Section 6, we answer this in the negative. In fact, if the Evolutionary Model has drift and volatility independent of the term structure, and it comes from a version of the Equilibrium Model, then this has to be the Vasicek version of the Equilibrium Model. Thus, the Equilibrium Model can be regarded as a special case of the Evolutionary Model.

The assumption for the Evolutionary Model, that the drift and volatility functions of the term structure do not depend on the term structure itself, so that they are purely functions of the term–to–maturity, is made frequently in the paper. It corresponds to the Vasicek assumption in the Equilibrium Model, and it leads to the theoretical shortcoming that the model can give negative interest rates. However, we argue that the probability of negative rates is insignificantly low, and so it is reasonable to make this assumption for the sake of the great technical simplifications to which it leads. As we have already mentioned, this assumption taken together with the assumption that the Evolutionary Model has been obtained from an Equilibrium Model, implies that we actually have the Vasicek Model. The drift and volatility functions which this entails are also reasonable, and they also give rise to great simplification in the Evolutionary Model. For example, with this drift and volatility, even if the initial term structure deviates from being admissible with respect to the associated Equilibrium Model (ie, the Vasicek Model), then this deviation is transient, and the behaviour predicted by the Evolutionary Model will eventually settle down to being Vasicek–admissible. Thus, this Evolutionary Model also possesses the attractive features of the Equilibrium Model.

In Section 7 we discuss the stability of the Evolutionary Model. We see that for stability, the volatility must attenuate to zero for large term–to–maturity, and (in a slightly stronger form) this leads to the model predicting that the long rate is constant. These conclusions also hold for the Equilibrium Model, in view of 6; and for the Vasicek Model, they correspond to the short rate being mean reverting. These conclusions also hold if we include extra factors into the model, as discussed in Section 9, but they are not supported empirically. Our feeling is that the model breaks down for large term–to–maturity due to the sparsity of available bonds.

In Section 8 we deal with interest rate option valuation. We cannot extend to the Evolutionary Model the Equilibrium Model technique of using the Black–Scholes Equation for this; rather we use the risk–neutral expectation technique. Option values in the Evolutionary Model are determined by the volatility but not the drift or risk premium of the term structure. This is in contrast to the Equilibrium Model valuations, but analogous to the situation when using the elementary Black–Scholes approach to valuing equity options. However, many features of our final Evolutionary option valuation formulae are surprisingly similar to the formulae in the Equilibrium Model. In particular our Formula (8.12) for a European coupon bond option agrees with the formula in [Jamshidian 1989] which is based on the Vasicek Equilibrium Model, except that [Jamshidian] puts into this formula the current bond prices dictated by the model, whereas in (8,12) we put in the empirically estimated prices. In fact the drift and risk premium enter the option valuation in the formula of [Jamshidian 1989] via these model prices. Also, we give a Black–Scholes Analytic formula for European options on pure discount bonds in the Evolutionary Model, and if the volatility has the Vasicek form, then we can extend this to coupon bond options via the trick of [Jamshidian].

In Section 9 we discuss some empirical issues, and the inclusion of more random factors into the Equilibrium and Evolutionary Models. The main motivation for this is that the single factor model cannot account for the less than perfect correlations between the movements of the various rates, in particular between the short and long rates. We describe the technique of Principal Component Analysis, which optimally estimates the volatilities in the higher factor model. We see in Section 9 that the development of the models, in particular the option valuations that they entail, is not essentially altered when we include more factors.

Finally, we must opine in favour of the Equilibrium Model or the Evolutionary Model. As might have already been guessed, we favour the Evolutionary Model.

First, we believe that the Evolutionary formulation more accurately reflects the way the market behaves; the market is aware of the volatilities and correlations among the rates and prices for the various maturities.

Second, both models agree on the term structure dynamic and the arbitrage mechanism, but the extra ingredient possessed by the Equilibrium Model, namely that the term structure should be determined at any instant by a set of parameters, seems to us to be philosophically without foundation. In Section 6 we saw that any version of the Equilibrium Model can be 'differentiated' to make it look like a version of the Evolutionary Model, but that not every version of the Evolutionary Model can be 'integrated' to go in the opposite direction.

We see no philosophical reason why the volatility of the Evolutionary Model should be such that it is integrable in this sense. On the other hand there is a pragmatic reason for choosing an integrable volatility, in particular the Vasicek Volatility; it makes the model technically more tractable.

Third, option valuation is more reasonable in the Evolutionary Model. In fact the similarities in option valuation between the Models are surprisingly strong and as we have already explained, these make the advantages of the Evolutionary valuations clear.

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