A Family of Ito Process Models for the Term Structure of Interest Rates

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Abstract: A family of Ito process models is constructed for the dynamics of the term structure of interest rates on default-free bonds, consistent with whatever term structure is initially observed. The results of Harrison and Kreps [1979] are extended to cover term structure models. It is shown that the family of models constructed in this paper can be supported in general equilibrium; in particular arbitrage opportunities are absent. A general formula is provided for the valuation of contingent claims. Consideration of sub-families sheds fresh light, in a generalised setting, on the term structure dynamics under which conventional duration is the correct risk measure for bond portfolios. A number of other models are shown to be special cases.

1 INTRODUCTION

1.1 TERM STRUCTURE

The prices at which fixed interest obligations[1] are traded in financial markets - and the term structure of interest rates those prices embody[2] - have long been a focus of attention for investors, borrowers, governments, and academics, not only as providing, obviously, a schedule of current opportunities for the placement or raising of funds, but also as embodying market participants' anticipations of future opportunities.

1.2 DERIVATIVE INSTRUMENTS

Alongside markets in fixed interest obligations, markets have grown up over the last couple of decades in a range of derivative instruments[3]. Some of these securities, such as options on bond or interest rate futures contracts, are derivative twice over, in that they are defined in relation to what are themselves derivative securities.

Interest rate derivatives are important in a number of ways. Firstly some of them (eg the long gilt futures contract on LIFFE; and some Forward Rate Agreements) provide a superior means of price discovery, in that the bid/offer spread can be tighter than those in the fixed interest obligations themselves; moreover, in the case of exchange traded derivative securities, it is only necessary to obtain the bid/offer prices on the exchange floor, rather than to seek quotations from a number of traders. The former point may be particularly important if, for example, one wishes to know on what terms a forward position can be constructed. Secondly, it may be possible to infer from market

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[1] "Fixed interest obligations" refers to any security whose terms comprise the payment of one or more cashflows, known at the outset in respect of both date and amount, by the issuer of the security, the obligor, to the holder of the security. This definition includes both coupon-paying and pure discount bonds. We will assume that all contracts are default free.

[2] In the absence of default risk, and of certain types of market "frictions", a fixed interest obligation may be valued by summing the values of its component cashflows, ie by regarding the obligation as a bundle of pure discount bonds (see eg the start of Section VI of Merton [1974], or of Section I of Cox, Ingersoll and Ross [1981]). Thus the way the market prices fixed interest obligations at any time can be expressed by a function describing the price per unit nominal, or equivalently yield to maturity, of pure discount bonds of arbitrary tenor. This function is called "the term structure of interest rates", or often simply "the term structure". Therefore to model the term structure is precisely to model the dynamics of the prices of fixed interest obligations.

[3] ie instruments which are not fixed interest obligations, but which generate cashflows between their counterparties which are functions of the price paths of fixed interest obligations.

prices of certain derivative securities, notably options, via an appropriate model, the degree of uncertainty attaching to market participants' anticipations of future interest rates. Thirdly, the volume of trading in some derivative securities is enormous, and such trading constitutes a significant activity of a large number of financial institutions.

Derivative instruments are used in a wide variety of ways, for both speculative and hedging purposes. For instance, a considerable number of financial institutions use combinations of fixed interest obligations and derivative securities to hedge positions in other derivative securities; thus, for example, government securities, futures on government securities, and options on government securities, may be used in combination to hedge a portfolio of over-the-counter (OTC) options on government securities. The latter options themselves could be of varying times to expiry and relate to underlying securities of a wide range of maturities.

1.3 THE NEED FOR MODELS

From the foregoing, it is clear that there is a need for models of the dynamics of the term structure, both because of interest in the term structure itself, and also to provide a consistent basis for the valuation of a broad range of derivative instruments. In addition, to be of use to practitioners in the derivative instruments markets, the initial state of whatever processes are used to model the term structure must be consistent with the current term structure actually observed, so that, for example, the value of an immediately expiring in-the-money option to purchase a fixed interest obligation will equal the excess of the current price of that obligation over the exercise price of the option.[4]

1.4 LITERATURE REVIEW

Until the last few years, the main foci of the substantial academic literature on the term structure were measurement[5] and explanation.

A pivotal paper in the latter field is Cox, Ingersoll and Ross [1981]. In this paper, as its title, "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates", suggests, the authors reassess the theories previously advanced. Specifically, they consider the various forms of the Expectations Hypothesis, the Hicksian Liquidity Preference approach, and the Preferred Habitat

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[4] Hull and White [1990] have recently made a similar point.

[5] Recent papers include Prisman [1990] and Steeley [1989], who also give numerous references to earlier literature.

Theory of Modigliani and Sutch, under which last head they effectively subsume the Segmented Market Hypothesis of Culbertson. They also review a number of linear adaptive interest rate forecasting models.

The overall thrust of the paper can perhaps be summed up[6] in two propositions. Firstly, any hypothesis about the term structure not informed by rigorous use of the mathematical tools of stochastic processes might well find itself reduced to palpable absurdity (and this applied to a disturbing fraction of the previous literature). Secondly, any term structure model wishing to hold itself out as consistent with an economic equilibrium requires very careful establishment of its credentials in that regard.

Stochastic process models of the term structure, both earlier (eg Vasicek [1977], Richard [1978] and Dothan [1978]) and subsequent (eg Brennan and Schwartz [1982] and Cox, Ingersoll and Ross [1985]) supposed that the evolution of the term structure could be described in terms of one or two state variables, and a process (usually a constant parameter) describing the market price of interest rate risk. These models can be solved to produce theoretical initial term structures, and thus, of course, are generally inconsistent with the term structure actually observed.[7]

The first model of term structure dynamics to incorporate consistency with the initial term structure actually observed as a fundamental feature is that of Ho and Lee [1986b][8]. Ho and Lee adopt a discrete-time approach based on the construction of an arbitrage-free binomial lattice to describe the evolution of the entire term-structure from its observed initial state. By way of illustration of their binomial framework, they devote much space to an extreme special "state-time independent" case of their framework, characterised in terms of two parameters.

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6 Babbs [1990] pp61-3 discusses Cox *et al.* in greater depth.

7 A recent paper by Dybvig [1989], however, offers hope that it may be possible to "reconcile" a number of existing term structure models with the actual observed term structure, thus making them - for the first time - candidates as models of practical use for the modelling of interest rate derivative instruments.

8 The [1986b] paper is the published version of the [1986a] paper. The unpublished version describes the general form of the binomial framework more fully, whereas the other devotes more attention to the extreme special case in which the framework is time and state independent. At a mathematical level, however, the contents of the two versions are essentially identical.

The chief weakness of their work is that a discrete time binomial model of term structure movements is, in itself, unrealistic; its validity can reside only in representing an approximation to something more plausible. The immediate question is that of convergence: namely, as the length of the timestep tends to zero, to what if anything do the distribution of the term structure and the values obtained for derivative instruments converge. Unfortunately, Ho and Lee do not discuss such matters. The issues have however been addressed independently by a number of authors[9].

An obvious idea, building on the contribution of Ho and Lee, is to seek to construct a family of continuous time stochastic processes for the term structure, consistent with its observed initial state. This programme has been tackled independently by Babbs [1990] (Babbs) and by Heath, Jarrow and Morton [1989] (HJM).

Both Babbs and HJM model each instantaneous forward interest rate by an Ito process[10]. The constructions used are similar in outline, but take somewhat different courses. The approach of Babbs, to be presented in this paper, clarifies the role played by the various regularity conditions. In addition, Babbs goes on to show that the family of models he has constructed is viable, ie can be supported in general equilibrium. This is achieved, as we shall see in Section 3 of this paper, by adapting and extending the work of Harrison and Kreps [1979] to a term structure context, and allowing a rich class of trading strategies. HJM, like Babbs, obtain the key technical result of the existence of a "unique equivalent martingale measure" (EMM), a reassignment of probabilities under which normalised security price processes are martingales; however, HJM do not demonstrate viability, and adopt a heavily circumscribed definition of trading strategies. While Babbs and HJM use different numeraire securities in order to obtain their EMMs, their formulae for the value of contingent claims can be shown to be equivalent.

Pursuing a similar programme at a less generalised level, Hull and White [1990] have produced "extended Vasicek" and "extended CIR" models to reconcile the models of Vasicek [1977] and Cox, Ingersoll and Ross [1985] with the observed initial term structure.

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9 See eg: Babbs [1990], especially chapter 20, pp308-15; Hull and White [1990]; and Heath, Jarrow and Morton [1990].

10 In an earlier paper, HJM [1987] allowed the process followed by each forward rate to depend on the rate in question; HJM [1989] relaxed this in favour of general dependence on the history of the entire term structure.

1.5 PURPOSE AND STRUCTURE OF THIS PAPER

The aim of this paper is to provide a general continuous time framework for modelling term structure dynamics, for use: in research on the term structure itself; for analysis of bond portfolios; and for the pricing of contingent claims.

The first task is to present (Section 2) the construction of a family of Ito process models, given in chapter 4 of Babbs [1990] (Babbs). The initial state of each member of the family is consistent with whatever term structure is currently observed, and the price of each bond attains par at maturity.

Our construction makes use (subsection 2.1.1) of an argument found in Vasicek [1977] concerning absence of arbitrage opportunities between bonds of different maturities. The necessity of the restrictions imposed is more apparent than their sufficiency. The second task of this paper (Section 3) is therefore to demonstrate that all arbitrage opportunities have indeed been eliminated. This is achieved as a consequence of establishing the stronger result that the models are "viable" in the sense defined by Harrison and Kreps [1979] (HK), ie that they could be supported in a general equilibrium in an economy populated by rational agents.[11] The key technical concept used in HK is that of an "equivalent martingale measure" (EMM). HK established a link between viability and the existence of one or more EMMs, under the restrictive assumption that agents can trade only at a finite number of pre-specified fixed times[12]. We adapt HK' framework to a term structure context and make use of results in Babbs which link EMMs to viability under a square integrability condition on trading strategies, of the kind widely accepted as a means of eliminating arbitrage (see eg Dybvig and Huang [1988]).[13]

A by-product of this second task of the paper is that the equivalent martingale measure is in fact unique. It then follows (Section 4) that all contingent claims are priced by arbitrage.

In Section 5, we discuss a number of sub-families of our models. We shed fresh light on the question of under what term structure dynamics conventional duration is the correct bond portfolio risk measure, generalising the setting used in Cox, Ingersoll and Ross [1979]. We also derive a number of existing models as special cases of our family. In particular, the "extended Vasicek" model

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[11] All the results here are contained in chapter 5, pp129-52 of Babbs.

[12] HK made no attempt to defend this restriction on economic grounds, and pointed out its undesirability, eg in excluding the option replication strategy in the Black and Scholes [1973] model.

[13] The results in Babbs are of independent interest, and we hope to present them fully elsewhere.

of Hull and White [1990] is shown to be a special case; the continuous time limit of the "state-time-independent" model of Ho and Lee [1986b] represents a further specialisation. One effect of this is that our results can be applied to show that these models are viable.

Section 6 provides brief concluding remarks.

2 CONSTRUCTION OF ITO PROCESSES

A common assumption is that the state of the world at any time can be described by the values of some fixed finite set of state variables. We have no need to impose this restriction, and therefore omit it. Instead, we shall focus directly on the term structure itself. We shall see, however, (see Section 5 below) that certain subclasses of our family of Ito processes can be described using state variables.

In setting out to construct stochastic processes for the term structure, one immediate question is whether to base the construction on asset prices - those of pure discount bonds are the natural choice - or with interest rates. Inspection of the stochastic processes of asset prices in existing literature (eg equation (30) in Section 8 of Merton [1973]) reveals that the instantaneous riskless interest rate can be a prominent element of the "drift" or "trend" component. If therefore, having eschewed the use of state variables, we base the construction on the prices of pure discount bonds, we may find it difficult to specify their stochastic processes fully without prior analysis of the instantaneous riskless rate, which is itself a function[14] of bond prices. To avoid this kind of vicious circle, we base our construction on interest rates.

To limit the burdens of notation, we will present our results in single-factor form, ie using only one Brownian motion to drive the dynamics of the term structure. We would emphasise that this is a purely expositional device, since the multi-factor version of our results can be obtained by simply adding additional terms identical in form to those arising from the single-factor case, a procedure whose details we regard as self-evident. We believe that our approach to this issue best serves expositional clarity.

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[14] See equations (8)-(9).

2.1 CONSTRUCTION

By means of Proposition 2.1 and Theorem 2.1 below, we construct a collection of Ito processes, one for each pure discount bond, that is consistent with the initial term structure and which ensures that each bond price converges to par at maturity.

We then proceed (in subsection 2.1.1) to employ an argument, due to Vasicek [1977], to obtain a restriction on the drift terms that is a necessary condition for the absence of arbitrage opportunities. Imposing that restriction enables the stochastic processes for the term structure to be re-expressed in an intuitively appealing way:

$$\frac{dB(M,t)}{B(M,t)} = \{r(t,\omega)+\theta(t,\omega)\sigma(M-t,t,\omega)\}dt + \sigma(M-t,t,\omega)dZ(t) \qquad (1)$$

where: $B(M,t)$ denotes the price at time t of unit nominal of a pure discount bond maturing at time $M \geq t$; $r(t,\omega)$ denotes the instantaneous spot interest rate; and $Z(\,)$ is a standard Brownian motion[15].

Equation (1) bears the straightforward interpretation that each pure discount bond has an expected instantaneous rate of return which differs from the instantaneous spot interest rate by an amount proportional to the bond's instantaneous price volatility. The factor of proportionality represents the market price of interest rate risk, and is the same across all bonds. The presence, as a functional argument, of the representative element $\omega \in \Omega$ of the set of all possible paths for the evolution of the term structure, indicates that the instantaneous spot rate, the price of risk, and the volatility of bond prices are allowed to be state-dependent in a very general fashion, as well as time-dependent.

Unfortunately, (1) is not a satisfactory starting point since, as discussed above, the instantaneous spot rate is itself a function of the prices of pure discount bonds. To avoid this kind of vicious circle, we base our construction upon instantaneous forward interest rates for all dates. An heuristic preview may be helpful:

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15 To be precise, $Z(\;)$ denotes a standard Brownian motion starting at zero, defined on some probability space $(\Omega,\mathcal{F},P)$ equipped with an increasing family of sub-sigma-algebras of $\mathcal{F}$, $\{\mathcal{F}_t \,:\, t \in [0,T]\}$ for some fixed T > 0, satisfying the "usual conditions" (see eg Karatzas and Shreve [1987] 1.2.25 Definition p10), and where, without loss of generality, $\mathcal{F}_0$ is almost trivial, and $\mathcal{F}_T = \mathcal{F}$.

Having no particular intuitions about the dynamics of instantaneous forward interest rates, we suppose (Proposition 2.1) that they follow very general Ito processes, subject only to conditions on the processes of the different rates sufficient to ensure that the prices at all dates of pure discount bonds can be recovered via the appropriate integration (see (7)). It follows from that integration that bond prices attain par at maturity (see (5)), and that future instantaneous spot interest rates are well-defined (see (6), (8) and (9)).

To turn the expression (10) for future bond prices, obtained in Proposition 2.1, into an Ito process, we must reverse the order of some repeated integrals, one of them involving an Ito stochastic integral. We achieve this in Theorem 2.1, subject to additional regularity conditions upon the instantaneous standard deviations of the forward rate processes.

In subsection 2.1.1, we make the final step to reach our conjectured bond price dynamics (ie (1)), by noting that an argument due to Vasicek [1977], concerning arbitrage between bonds of different maturities, requires a relationship (see (14)) between the drifts and instantaneous standard deviations of instantaneous forward rates.

Proposition 2.1 *Let the initial term structure* $B(.,0) : [0,\infty) \rightarrow \Re$ *be strictly positive and differentiable on* $[0,\infty)$

Let $a,b : [0,T]\times[0,\infty)\times\Omega \rightarrow \Re$ *satisfy appropriate*[16] *measurability conditions and the following regularity conditions.*

(i) $a(.,.,\omega) : [0,T]\times[0,\infty) \rightarrow \Re$ *is integrable over bounded rectangles almost surely.*

(ii) $\mathrm{E}\left[\int_0^t b^2(s,m,\omega)ds\right] < \infty; \ \forall t \leq m$

(iii) $\int_0^t b(s,m,\omega)dZ(s)$ *is integrable in* $m \geq t$ *over bounded intervals.*

16 The measurability conditions are omitted for the sake of brevity. They are simply the kind of conditions customary in stochastic calculus, see eg Ikeda and Watanabe [1981].

Then we may construct Ito processes, consistent with the initial term structure, for the instantaneous forward interest rate $f(m,.)$ for each date $m \geq 0$:

$$f(m,t) = f(m,0) + \int_0^t a(s,m,\omega)ds + \int_0^t b(s,m,\omega)dZ(s); \qquad t \leq m \tag{2}$$

where

$$f(m,0) = -\frac{d}{dm}\ln B(m,0) \tag{3}$$

Moreover, we can derive from these processes an expression for the price of any pure discount bond at any future date:

$$\ln B(M,t) = \ln B(M,0) + \int_0^t f(s,0)ds - \int_t^M \int_0^t a(s,m,\omega)dsdm - \int_t^M \int_0^t b(s,m,\omega)dZ(s)dm \tag{4}$$

with

$$B(M,M) = 1 \qquad \text{almost surely}, \quad \forall M > 0 \tag{5}$$

Furthermore, the instantaneous spot interest rate $r(t)$ follows a well-defined process:

$$r(t) = f(t,0) + \int_0^t a(s,t,\omega)ds + \int_0^t b(s,t,\omega)dZ(s) \tag{6}$$

Proof By the assumed differentiability of the initial term structure, (3) well-defines an initial curve $f(.,0) : [0,\infty) \rightarrow \Re$ of instantaneous forward interest rates.

By assumptions (i) and (ii), we may now define a family of Ito processes, indexed by $m \geq 0$, as given by (2).

By assumptions (i) and (iii), each summand on the RHS of (2) is integrable in m over bounded intervals. Therefore, we have a well-defined stochastic process

$$\ln B(M,t) = -\int_t^M f(m,t)dm; \qquad \forall t \in [0,M] \tag{7}$$

for the price of each pure discount bond.

Equation (5) follows immediately from (7), while substituting (2) and performing elementary manipulations gives (4).

Also from (7), we have that

$$\frac{-\ln B(M,t)}{M-t} \rightarrow f(t,t) \quad \text{as} \quad M \downarrow t \tag{8}$$

whence we may well-define the spot instantaneous interest rate, *r(t)* and establish (6), by setting:

$$r(t) = f(t,t); \qquad \forall t \geq 0 \tag{9}$$

and substituting (2) on the RHS. ▮

To turn (4) into an Ito process, we need to reverse the order of integration of the repeated integration. This requires some additional regularity conditions which, as we shall see in later sections, are satisfied in a number of significant cases.

Theorem 2.1 *If the conditions of Proposition 2.1 hold and, in addition, b( ) satisfies the following regularity conditions:*

(iv) $\int_0^t b(s,t,\omega)dZ(s)$ *is almost surely pathwise integrable in* t .

(v) $b(s,.,\omega) : \Re \rightarrow \Re$ *is integrable over finite intervals* $\forall s$ *almost surely.*

(vi) $E\left[\int_0^t \left(\int_s^M b(s,m,\omega)dm\right)^2 ds\right] < \infty; \qquad \forall t \leq M$

(vii) $\left(E\left[\int_0^t b^2(s,m,\omega)ds\right]\right)^{\frac{1}{2}}$ *is integrable with respect to* m *over finite intervals* $(m \geq t)$

Then we may construct Ito processes for the price of each pure discount bond, consistent with the initial term structure, with the stochastic differential form:

$$\frac{dB(M,t)}{B(M,t)} = \left\{r(t) - \int_t^M a(t,m,\omega)dm + \frac{1}{2}\left(\int_t^M b(t,m,\omega)dm\right)^2\right\}dt - \left\{\int_t^M b(t,m,\omega)dm\right\}dZ(t) \tag{10}$$

where $r(\,)$ *is the spot instantaneous interest rate process given by (6), and where each bond price converges to par at maturity, as described by (5).*

Proof By assumptions (i) and (iv), equation (6) is almost surely pathwise integrable over finite intervals, giving:

$$\int_0^t r(m)dm = \int_0^t f(s,0)ds + \int_0^t\int_0^m a(s,m)dsdm + \int_0^t\int_0^m b(s,m)dZ(s)dm \qquad (11)$$

which, being pathwise an ordinary integral, is absolutely continuous (cf eg Weir [1973] p67) and thus, *a fortiori*, a continuous VF process and a semi-martingale.

Subtracting (11) from (4), and combining the ranges of integration, now yields:

$$\ln B(M,t) - \ln B(M,0) = \int_0^t r(m)dm - \int_0^M\int_0^{\min\{m,t\}} a(s,m)dsdm$$
$$- \int_0^M\int_0^{\min\{m,t\}} b(s,m)dZ(s)dm \qquad (12)$$

Applying Fubini's theorem to

$$\int_0^M\int_0^{\min\{m,t\}} a(s,m)dsdm$$

and using assumptions (ii), (v), (vi) and (vii) to enable us to apply a Fubini-type theorem for stochastic integrals[17] to

$$\int_0^M\int_0^{\min\{m,t\}} b(s,m)dZ(s)dm$$

we may rewrite (12) as:

$$\ln B(M,t) - \ln B(M,0) = \int_0^t r(m)dm - \int_0^t\int_s^M a(s,m)dmds$$
$$- \int_0^t\int_s^M b(s,m)dmdZ(s) \qquad (13)$$

The final term on the RHS of (13) is a continuous martingale by assumption (vi). The preceding terms are absolutely continuous in t and thus VF. Hence, $\ln B(M,.)$ is an Ito process. By Ito's lemma, we deduce that *B(M,.)* is itself an Ito process, whose differential form is (10) as required. ▮

[17] The required theorem is an adaptation of Lemma 4.1 on pp116-9 of Ikeda and Watanabe [1981] to the special case of martingales based on Brownian motion. See Babbs (chapter 28, pp379-83) for details.

2.1.1 Necessary restriction for absence of arbitrage opportunities

We may apply an argument identical to that used in Section 3 of Vasicek [1977], concerning arbitrage between bonds, to deduce from (10) that we require:

$$-\int_t^M a(t,m,\omega)dm + \frac{1}{2}\left\{\int_t^M b(t,m,\omega)dm\right\}^2 = -\theta(t,\omega) \int_t^M b(t,m,\omega)dm \qquad (14)$$

where

$$\theta(t,\omega) = \text{price of interest rate risk at } (t,\omega); \qquad (15)$$

If we now define

$$\sigma(M-t,t,\omega) \equiv -\int_t^M b(t,m,\omega)dm \qquad (16)$$

we may use (15) to re-express (10) in the form of (1), completing the construction.

3 VIABILITY

While we took steps, in the preceding subsection, to eliminate arbitrage opportunities, we have not established that none remain. A strictly wider question is whether members of our family of models are "viable", ie capable of being supported in a general equilibrium in an economy populated by rational agents.

To address these matters, we must embed our term structure dynamics in an economy, and decide what class of trading strategies are available to economic agents. For simplicity, we will utilise the pure exchange economy employed in HK; in the interests of brevity, we refer the reader to Sections 1 and 2 of HK for details.

We need however to make some various adaptations to accommodate term structure models. In particular, we will suppose that the set of traded securities consists of pure discount bonds[18]; we do not require the collection of available securities to be finite, and we allow for maturing bonds by prescribing that bonds may not be held beyond their maturity dates.

HK demonstrated a link between viability and equivalent martingale measures (EMMs). An EMM is a reassignment of probabilities under which appropriately normalised security price prices are martingales; we give a formal definition below. The first stage of the link was to show that a price system for the traded subset of contingent claims (here understood as state-contingent claims to

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18 A far richer setting, involving coupon bonds and other assets, is readily constructed, but would obscure the essentials of the exposition.

consumption at the terminal date of the economy) is viable if and only if it can be extended[19] to all claims. The second was to show that each such extension could be used to define an EMM, and *vice versa*.

The natural numeraire in this economy is the security which has a certain unit payoff at T, ie the pure discount bond maturing at that date. With this numeraire, HK's definition of an EMM is modified to become:

Definition *A probability measure P^* on $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure if and only if the following three conditions hold:*

(i) *P^* and P are equivalent, ie have the same null sets. A necessary and sufficient condition for this is that the Radon-Nikodym derivative*

$$\frac{dP^*}{dP}$$

be strictly positive[20].

(ii) *bond price processes, after normalisation by dividing through by $B(T,.)$, are martingales under P^**

(iii) $$\frac{dP^*}{dP} \in L^2(\Omega, \mathcal{F}, P) \qquad (17)$$

The second stage of the link demonstrated by HK was achieved under the assumption that each agent can follow only "simple" trading strategies, involving trading only at a finite set of fixed dates selected at the outset of the economy. As HK observed, this is undesirably restrictive. At this stage, therefore, we make instead the:

Provisional definition *A trading strategy is provisionally defined as a non-negative integer n, a selection of bond maturity dates, $M_1, \ldots, M_n$, together with an n-dimensional real valued stochastic process $\psi : [0,T] \times \Omega \to \Re^n$, whose jth component ψ_j represents time-state-dependent holdings of bond j, satisfying:*

(i) *The stochastic integral in (18) is well-defined;*

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19 by a continuous and strictly positive linear functional

20 Note that since P, P^* have the same null sets, we may use the term "almost surely" without qualification as to which probability measure is intended.

(ii) $\psi_j(t,\omega) = 0$ *if bond* j *matures strictly before* t;

(iii) *self-financing:* $\forall t, \omega$:

$$V_\psi(t,\omega) = \sum_{j=1}^{n} \psi_j(t,\omega) B(M_j,t) = V_\psi(0,\omega) + \sum_{j=1}^{n} \int_0^t \psi_j(u,\omega) dB(M_j,u) \tag{18}$$

This requirement says that changes in the value of the strategy must be attributable precisely to capital gains on bond holdings, ie with no net injections or extractions of funds.

(iv) *terminal value in the terminal consumption space:*

$$V_\psi(T,.) \in L^2(\Omega,\mathcal{F},P) \tag{19}$$

We now avail ourselves of a result in Babbs (chapter 5, pp134-9) which extends HK' work to admit a wider class of trading strategies.

Theorem 3.1 *Suppose* P^* *is an EMM and that we further restrict strategies by requiring that their discounted*[21] *value processes are* P^* *-martingales; then the model is viable, with* $\kappa : L^2(\Omega,\mathcal{F},P) \rightarrow \Re$ *defined by:*

$$\kappa(x) = B(T,0)\, E^*[x] \tag{20a}$$

extending the price system to all contingent claims.

Conversely, if we suppose that the model is viable, with κ *extending the price system to all claims, and make the alternative restriction that the discounted price of each traded bond is P-square integrable, then there exists an EMM,* P^* *given by:*

$$P^*(A) = \frac{\kappa(1_A)}{B(T,0)} \tag{20b}$$

where 1_A *is the indicator function of* A .

Proof See Babbs, (*loc. cit.*) ▮

We are now ready to address the viability of the Ito process models of the term structure constructed in Section 2.1. We begin (Theorem 3.2) by showing that there is at most one EMM, and specifying its Radon-Nikodym derivative with respect to P - a step which lends itself to obtaining sufficient regularity conditions for existence. By imposing these conditions, we establish (Theorem 3.3) the existence of an EMM and hence viability.

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21 ie normalised by dividing by the price of the numeraire security

Theorem 3.2 *If the bond price processes are described by (1), then the economy just specified has at most one EMM.*

If this EMM, P^*, *exists, then*[22]

$$\xi(t) \equiv \exp\left\{-\int_0^t \{\theta(u)-\sigma(T-u,u)\}dZ(u) \;-\; \frac{1}{2}\int_0^t \{\theta(u)-\sigma(T-u,u)\}^2 du\right\}$$

$$\text{is a } P\text{-martingale, with } \xi(T) \in L^2(P) \tag{21}$$

and P^* *has Radon-Nikodym derivative*

$$\frac{dP^*}{dP} = \xi(T) \tag{22}$$

Conversely, if (21) holds, define a probability measure, P^*, *by (22). Then a sufficient condition for* P^* *to be an EMM is:*

$$\exp\left\{\int_0^t \{\sigma(M-u,u)-\sigma(T-u,u)\}dZ^*(u) \;-\; \frac{1}{2}\int_0^t \{\sigma(M-u,u)-\sigma(T-u,u)\}^2 du\right\}$$

$$\text{is a } P^*\text{-martingale } \forall M \in [0,T] \tag{23}$$

where $Z^*(\;)$ *is the standard Brownian motion under* P^* *given by (35).*

Proof Define the discounted bond price processes:

$$B^*(M,t) \equiv \frac{B(M,t)}{B(T,t)} \qquad \forall M,t \tag{24}$$

(The argument which now follows is largely due to Pages [1987].)

Suppose that there exists an EMM, P^*. By the definition of an EMM,

$$\frac{dP^*}{dP} \in L^2(P) \tag{25}$$

Hence $\eta(\;)$, defined by:

$$\eta(t) \equiv \mathrm{E}\left[\left.\frac{dP^*}{dP}\right| \mathcal{F}_t\right] \tag{26}$$

is a square-integrable P-martingale with

$$\eta(0) = \mathrm{E}\left[\frac{dP^*}{dP}\right] = 1 \tag{27}$$

22 From this point on, we usually drop notational dependence on ω in the interests of brevity.

We can therefore apply martingale representation theory[23] to obtain that[24]

$$\eta(t) \;=\; 1 \;+\; \int_0^t \alpha(u,\omega)dZ(u) \tag{28}$$

for some P-square-integrable process $\alpha(\;)$.

By (1),

$$\frac{dB^*(M,t)}{B^*(M,t)} \;=\; \{\sigma(M-t,t)-\sigma(T-t,t)\}\;[\;\{\theta(t)-\sigma(T-t,t)\}dt \;+\; dZ(t)\;] \tag{29}$$

Since P^* is an EMM, $B^*(M,.)$ is a P^*-martingale; implying[25] that $B^*(M,.)\eta(\;)$ is a P-martingale. Applying Ito's lemma,

$$\frac{d(B^*(M,t)\;\;\eta(t))}{B^*(M,t)} \;=\; \{\sigma(M-t,t)-\sigma(T-t,t)\}\;[\alpha(t) \;+\; \{\theta(t)-\sigma(T-t,t)\}\;\;\eta(t)]\;\;dt$$

$$+\;[\alpha(t) \;+\; \{\sigma(M-t,t)-\sigma(T-t,t)\}\;\;\eta(t)]\;\;dZ(t) \tag{30}$$

Now an Ito process is a martingale only if[26] it has zero drift. Thus we require

$$\alpha(t) \;+\; \{\theta(t)-\sigma(T-t,t)\}\;\;\eta(t) \;=\; 0 \tag{31}$$

Multiplying this through by $dZ(t)$, substituting for $\alpha(t)dZ(t)$ by means of (28), and rearranging:

$$\frac{d\eta(t)}{\eta(t)} \;=\; -\{\theta(t)-\sigma(T-t,t)\}\;\;dZ(t) \tag{32}$$

whose solution is

$$\eta(t) \;=\; \xi(t) \qquad \forall t \tag{33}$$

---

23 see eg Liptser and Shiryayev [1977] Theorem 5.5 p162 - essentially the Kunita-Watanabe Representation Theorem.

24 We have generally not felt the need to make explicit the various measurability conditions involved in the stochastic calculus we have been undertaking. The validity of (28) however depends on the assumption that the increasing family $\{\mathcal{F}_t : t \in [0,T]\}$ with which our probability space is equipped is that generated by $Z(\,)$. Restrictions of this kind, introduced by HK (Section 5) and subsequently commonplace, restrict agents' information to the past price history of the traded securities.

25 see eg Liptser and Shiryayev [1977] Lemma 6.6 p226; Pages [1987] cites Dellacherie and Meyer [1982] Lemma VII.48

26 It is well known that this condition is necessary but not sufficient. Its necessity flows from the result that, on a finite interval, a continuous local martingale of finite variation is constant (see eg M U Dothan [1990] Theorem 10.28 p249).

Thus, if P^* is an EMM, then $\xi(\ )$ is a P-martingale with

$$\xi(T) = \eta(T) = \frac{dP^*}{dP} \in L^2(P) \tag{34}$$

Conversely, if the condition (21) in the theorem is fulfilled, then, by the Girsanov Theorem[27], (22) defines a probability measure P^* equivalent to P, under which

$$Z^*(t) \equiv Z(t) + \int_0^t \{\theta(u) - \sigma(T-u,u)\}\, du \tag{35}$$

is a standard Brownian motion, and we may re-express the discounted security price process (29) as

$$\frac{dB^*(M,t)}{B^*(M,t)} = \{\sigma(M-t,t) - \sigma(T-t,t)\}\ dZ^*(t) \tag{36}$$

whose unique solution is the P^*-supermartingale

$$B^*(M,t) = B^*(M,0)\ \exp\left\{\int_0^t \{\sigma(M-u,u) - \sigma(T-u,u)\}\, dZ^*(u) - \frac{1}{2}\int_0^t \{\sigma(M-u,u) - \sigma(T-u,u)\}^2\, du\right\} \tag{37}$$

If (23) holds, the RHS of (37) is, in fact, a P^*-martingale; thus P^* is an EMM as required. ▮

To utilise the above Theorem in the most straightforward way, we now impose the following:

Regularity conditions *The conditions (21) and (23) of Theorem 3.2 are fulfilled.*

Remark These regularity conditions are satisfied in a range of interesting cases. For example, by the Novikov condition[28], (23) will always be fulfilled if $\sigma(\ )$ is globally bounded (eg when it is deterministic); similarly (21) will be satisfied if $\theta(\ )$ is also globally bounded. We shall see examples of this in Section 5.

Since Theorem 3.2 tells us that there is at most one EMM, and mindful of Theorem 3.1, we elect to finalise our definition of trading strategies as follows:

Definition *Define a trading strategy by the Provisional Definition above, supplemented by the following additional requirement:*

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[27] see eg Karatzas and Shreve [1987] Theorem 3.5.1 p191

[28] see eg Karatzas and Shreve [1987] Corollary 3.5.13 p199

(v) $V_\psi(\cdot,\cdot) \ / \ B(T,\cdot)$ *is a* P^**-martingale*

We can now conclude:

Theorem 3.3 *Our term structure model is viable, with a unique EMM. In particular, arbitrage opportunities and "suicide" strategies are precluded.*

Proof The unique EMM follows immediately from the above regularity conditions and Theorem 3.2. Viability follows by Theorem 3.1. Arbitrage and suicide strategies are readily shown to be precluded by requirement (v) on trading strategies. ▮

4 PRICING OF CONTINGENT CLAIMS

As HK (Section 3, Corollary to Theorem 2) pointed out, the existence of a unique EMM implies that there is a unique extension of market prices to all contingent claims. Thus every claim has a unique price consistent with equilibrium; HK termed this price "determined by arbitrage".

Theorem 4.1 *Let the conditions of Theorem 3.3 be fulfilled, Then any* $x \in L^2(\Omega, \mathcal{F}, P)$ *is priced by arbitrage, at a value:*

$$V^{(x)} = B(T,0) \ \mathrm{E}^*[x] \tag{38}$$

Proof Let P^* be the unique EMM. Let κ be the corresponding extension of the price system. Then, by Theorem 3.1, κ is given by:

$$\kappa(x) = B(T,0) \ \mathrm{E}^*[x] \tag{39}$$

The result follows. ▮

The valuation equation (38) applies across the whole family of models constructed in this paper, and is thus inevitably abstract. For particular models, formulae capable of explicit computation can be derived from it.

For example, for a broad class of models within the subfamily discussed in Section 5.3 below, Babbs has obtained closed-form pricing formulae for a wide range of contingent claims (Babbs chapters

6-17 pp153-272), including: european bond options[29]; interest rate caps and floors; futures contracts on bonds[30] and on short-term interest rates; and for options, margined in a manner akin to futures, on those futures contracts, such are traded on the London International Financial Futures Exchange (LIFFE). A computationally efficient binomial approximation scheme, using the general framework of Ho and Lee [1986b], extends the coverage (Babbs chapters 18-25 pp273-370) to include, *inter alia*: american bond options; and non-margined options on futures. We hope to present a number of these results in future papers.

5 PROPERTIES OF SUB-FAMILIES

In this Section we consider various sub-families of the models constructed in Section 2. In each case we are able to shed light on topics discussed in earlier literature in more restricted settings, or to obtain existing models as special cases. One effect of this is that our results on viability (Theorem 3.3) and the valuation of contingent claims (Theorem 4.1) can be applied to a range of models of theoretical and practical interest.

We shall see that in some cases - though not in others - the evolution of the term structure can be described in terms of a finite set of state variables. This vindicates our decision, at the outset of Section 2, to eschew basing our construction upon state variables. At the same time, the existence of a state variable representation for some classes of our models may be useful to other researchers seeking to incorporate our models in wider endeavours.

5.1 b( ) independent of M

If we write

$$b(t,M,\omega) \quad = \quad b(t,\omega) \tag{40}$$

in (16), we see that the volatility of $B(M,t)$ is

$$\sigma(M-t,t,\omega) \quad = \quad (M-t)\ b(t,\omega) \tag{41}$$

Moreover, from (14) we find that

$$a(s,M,\omega) \quad = \quad b(s,M,\omega)\left\{\theta(s,\omega) \;+\; \int_s^M b(s,m,\omega)dm\right\} \qquad \forall\ s,M \tag{42}$$

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29 The formula in question yields that of Jamshidian [1990] as a special case, when applied to the Vasicek [1977] term structure model.

30 The analysis sets aside the various delivery options often embedded in such contracts.

From (41) it is easy to verify[31] that the volatility of the value of any bond portfolio, not just that of a single pure discount bond, is proportional to conventional duration. Thus the sub-family discussed here is significant as the sub-family for which conventional duration is the correct tool for portfolio immunisation.

However, if we substitute (41) and (42) into (2), it is readily shown that

$$f(M_2,t) - f(M_1,t) = f(M_2,0) - f(M_1,0) + (M_2 - M_1)\int_0^t b^2(s,\omega)ds \tag{43}$$

The interpretation of (43) is that, with probability one, the forward instantaneous rate curve, with respect to forward date, steepens as time elapses. Thus the sub-family for which conventional duration is the correct immunisation tool has implausible properties.

Cox, Ingersoll and Ross [1979] suggested (p55) that the only dynamics for which conventional duration is a valid measure of risk is that which corresponds in our framework to the case *b( ) = constant.* This stemmed from their restricting their attention to term structure dynamics for which their equation (6) holds. Our findings thus reinforce, from the vantage point of a more general framework, the conclusion of Cox *et al.* that using conventional duration as a measure of risk is consistent only with implausible term structure dynamics.

---

31 Let the promised cash flows from the portfolio be $c_1, \ldots, c_n$ at dates $M_1 < \ldots < M_n$ respectively, with M_1 greater than current time, *t*. Then the value of the portfolio is:

$$V(t) = \sum_{j=1}^{n} c_j B(M_j,t)$$

Applying Ito's lemma, and substituting (1),

$$dV(t) = \sum_{j=1}^{n} c_j B(M_j,t)\{r(t)+\theta(t)\sigma(M_j-t,t)\}\,dt + \sum_{j=1}^{n} c_j B(M_j,t)\sigma(M_j-t,t)\,dZ(t)$$

whence

$$\frac{dV(t)}{V(t)} = \{r(t)+\theta(t)D(t)\}dt + D(t)dZ(t)$$

where

$$D(t) = \left\{\sum_{j=1}^{n} c_j B(M_j,t)\sigma(M_j-t,t)\right\} \div \left\{\sum_{j=1}^{n} c_j B(M_j,t)\right\}$$

Thus *D(t)* is the volatility of the value of the portfolio, and the appropriate measure of risk. Comparison with conventional duration:

$$C(t) = \left\{\sum_{j=1}^{n} c_j B(M_j,t)(M_j-t)\right\} \div \left\{\sum_{j=1}^{n} c_j B(M_j,t)\right\}$$

quickly reveals that volatility will be proportional to conventional duration, for all portfolios, if and only if $\sigma(\ )$ is of the form given by (41).

Substituting (40) and (42) into (2) also yields:

$$f(m,t) = f(m,0) + m\int_0^t b^2(s,\omega)ds - \int_0^t \{sb^2(s,\omega) - \theta(s,\omega)b(s,\omega)\}ds + \int_0^t b(s,\omega)dZ(s) \tag{44}$$

Hence

$$\int_0^t b^2(s,\omega)ds \tag{45a}$$

and

$$\int_0^t \{sb^2(s,\omega) - \theta(s,\omega)b(s,\omega)\}ds - \int_0^t b(s,\omega)dZ(s) \tag{45b}$$

constitute the state variables for the entire forward instantaneous rate curve and hence, via (7), of the entire term structure itself.

5.2 b( ) constant

Setting

$$b(t,M,\omega) \equiv b \tag{46}$$

obviously represents an extreme case of the sub-family just considered above, with the proportionality factor between volatility and conventional duration being state and time independent. Various authors[32] have shown that the Ho and Lee [1986b] state-time-independent model represents a discrete time approximation to precisely this extreme case.

For this sub-family, it is possible to derive a closed-form expression for future term structures, as follows.

Substituting (46) into (41) and (42) and thence both into (6) and (1) yields:

$$r(t) = f(t,0) + b\int_0^t \theta(s,\omega)ds + \frac{1}{2}b^2t^2 + bZ(t) \tag{47}$$

and

$$\frac{dB(M,t)}{B(M,t)} = \{r(t) + \theta(t,\omega)(M-t)b\}dt + (M-t)b\,dZ(t) \tag{48}$$

32 Babbs (chapter 20, pp308-15) and Heath, Jarrow and Morton [1990] provide independent, and indeed quite different, treatments. We are unclear to what extent Carverhill [1989] and Hull and White [1990] are indebted to Heath *et al.*

whence straightforward manipulations give

$$B(M,t) = \frac{B(M,0)}{B(t,0)} \exp\left[(M-t)\{f(t,0)-r(t)\} - \frac{1}{2}t(M-t)^2 b^2\right] \tag{49}$$

(Note that if, in (48), we set $\theta(\ ) = 0$ and $b = -g$, we obtain the model described in footnote 43 on p163 of Merton [1973].)

It may be seen from (49) that, for this sub-family, the instantaneous interest rate *r(t)* is a state variable and that no other such variable is required.

5.3 b( ) state-independent

By removing state-dependence, ie dependence on ω, from *b( )*, the volatility of pure discount bond prices becomes state-independent, as can be seen from (16), but may nevertheless be a function of time as well as of term to maturity. Models in this family are readily shown to be viable, if (say) the market price of risk is globally bounded (see Theorem 3.2 and subsequent Remark).

For this sub-family the impact of the innovations of the Brownian motion *Z( )* upon any particular part of the term structure depends on the maturity date in question[33]. In general, therefore, no finite set of state variables for the entire term structure exists for this sub-family.

In pricing contingent claims, Babbs focusses much attention upon a portion[34] of this sub-family in which *b( )* has the functional form:

$$b(s,m) = -G'(m)\lambda(s) \tag{50}$$

where $G, \lambda : [0,\infty) \rightarrow [0,\infty)$ with $G(0) = 0$ and where $G'(\ )$ denotes the first derivative of *G( )*.

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33 For example, it is easy to verify that for this sub-family, (2) becomes:

$$f(m,t) = f(m,0) + \int_0^t b(s,m)\left\{\int_s^m b(s,n)dn - \theta(s,\omega)\right\}ds + \int_0^t b(s,m)dZ(s)$$

34 Babbs shows that, after adjusting the probability assignment to the "equivalent martingale measure", future term structures can be described by a single Gaussian state variable. This facilitates the derivation of concrete valuation techniques - in many cases closed-form expressions - for a very wide range of contingent claims of commercial interest (see the end of Section 4 above).

For this portion, it is easy to verify that (2) becomes

$$f(m,t) = f(m,0) + G'(m)\int_0^t \{G(m)-G(s)\}\lambda^2(s)ds$$

$$- G'(m)\left[\int_0^t \theta(s,\omega)\lambda(s)ds + \int_0^t \lambda(s)dZ(s)\right] \qquad (51)$$

so that

$$Y(t,\omega) \equiv \int_0^t \theta(s,\omega)\lambda(s)ds + \int_0^t \lambda(s)dZ(s) \qquad (52)$$

constitutes the single state variable for the whole term structure.

It is worth exploring the properties of the instantaneous spot rate of interest for this portion of the wider subfamily.

Recalling from (9) that

$$r(t) = f(t,t) \;; \quad \forall t$$

we put $m = t$ in (51) and use (52) to obtain

$$r(t) = f(t,0) + G'(t)\int_0^t \{G(t)-G(s)\}\lambda^2(s)ds - G'(t)Y(t) \qquad (53)$$

If $G'(\ )$ is differentiable[35], we may re-express (53) in differential format:

$$dr(t) = \mu'(t)dt - \frac{G''(t)}{G'(t)}\{\mu(t)-r(t)\}dt - G'(t)dY(t) \qquad (54)$$

where $\mu'(t)$ is the derivative of

$$\mu(t) \equiv f(t,0) + G'(t)\int_0^t \{G(t)-G(s)\}\lambda^2(s)ds \qquad (55)$$

(54) can be interpreted as expressing a mean reversion process involving a moving mean, $\mu(t)$, and a generalised innovations process $G'(t)dY(t)$. Analysing contingent claims under this model, Babbs has obtained (chapters 6-25, pp153-370) an extensive range of continuous time closed-form results and a binomial approximation scheme (see also the end of Section 4 above).

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[35] Assumption (iii) of Proposition 2.1 implicitly imposes the requirement that $G'(\ )$ exist, but it will not necessarily be differentiable.

5.3.1 The "extended Vasicek" model of Hull and White [1990] (HW)[36]

HW propose an "extended Vasicek" model, in which the dynamics of the instantaneous spot instantaneous rate can be written in the form:[37]

$$dr(t) \;=\; \{\alpha(t)+\beta(t)r(t)\}dt \;-\; \gamma(t)dZ(t) \tag{56}$$

and the market price of risk process is a function of time alone, to be determined from the other parameters and the initial term structure.

HW propose that $\alpha(\;)$, $\beta(\;)$ and $\gamma(\;)$ be chosen so that the model fits: the initial term structure, the initial variabilities of spot interest rates of all maturities, and the prospective variability across time of the instantaneous spot rate.

It is readily shown, by comparing (56) with (54), that any "extended Vasicek" model is a special case of our "*b( )* state-independent" sub-family, with:

$$G(t) \;=\; \int_0^t \exp\left\{\int_0^s \beta(u)du\right\}ds \tag{57a}$$

$$\lambda(t) \;=\; \gamma(t)\exp\left\{-\int_0^t \beta(u)du\right\} \tag{57b}$$

and

$$\theta(t) \;=\; -\alpha(t) \;-\; \beta(t)f(t,0) \;+\; f_1(t,0) \;+\; \int_0^t \gamma^2(s)\exp\left\{2\int_s^t \beta(u)du\right\}ds \tag{57c}$$

The significance of this representation is threefold:

Firstly, we can apply the results in this paper to establish that HW's "extended Vasicek" is viable - an issue HW overlook.

Secondly, HW achieve the "fit" of their model by imposing an artificial structure (57c) on the market price of risk, $\theta(\;)$. By contrast, the greater generality of (54), over that of (56), enables us to achieve the same "fit" while leaving the price of risk, $\theta(\;)$, free to take any form.

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36 HW also put forward an "extended CIR" model, building on that in Cox, Ingersoll and Ross [1985]. A similar analysis to that given here for the "extended Vasicek" model can be brought to bear (see Babbs p96), but no gain in tractability accrues.

37 We have changed the notation, to avoid confusion below when HW's models are compared with ours.

Under our approach, all we need to parametrize[38] the model are $G(\,)$ and $\lambda(\;)$. The former, together with $\lambda(0)$, is determined from the initial absolute variabilities of spot rates of all maturities:

$$\sigma_{spot}(M,0) \begin{cases} = \dfrac{G(M)}{M}\lambda(0), & M > 0 \\ = G'(0)\lambda(0), & M = 0 \end{cases} \tag{58}$$

where, without loss of generality, we set $G'(0) = 1$. We can now determine $\lambda(\;)$ from the anticipated absolute variability of instantaneous spot rates:

$$\sigma_{spot}(0,t) = G'(t)\lambda(t) \tag{59}$$

Thus, not only is our approach more general but also, as comparison of (58)-(59) with equations (15)-(16) in HW's paper reveals, we are able to parametrize our model using substantially simpler - and more computationally tractable - expressions.

Thirdly, in pricing contingent claims, we can apply the results in Babbs (chapters 6-25, pp153-370) referred to at the end of Section 4 above.

6 CONCLUSIONS

In this paper, we have identified a need for continuous time models of the dynamics of the term structure of interest rates, consistent with the initial term structure actually observed. We have addressed that need by constructing a general family of Ito process models, and by extending the work of Harrison and Kreps [1979] in such a way as to enable us to identify modest sufficient conditions under which members of that family are viable, ie can be supported in general equilibrium. As a by-product of our analysis of viability, we have established that all contingent claims are priced by arbitrage, and exhibited a general pricing equation.

We have examined some subfamilies of our models. In so doing, we shed fresh light on the implausibility of the term structure dynamics under which conventional duration is the correct measure of bond portfolio risk, extending a critique advanced by Cox, Ingersoll and Ross [1981]. We also pointed out that various existing models, including the continuous time limit of the state and time independent model of Ho and Lee [1986b], and the "extended Vasicek" model of Hull and

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38 The "price of risk" process is irrelevant to the pricing of contingent claims, and so can be left unspecified.

White [1990], are special cases of our framework. In particular, we showed that we could generalise the "extended Vasicek" model in such a way as to remove the artificial form of the market price of risk, imposed by Hull and White.

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