The Efficiency of the Single and Multivariate Binomial Technique

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ABSTRACT

We discuss the theoretical efficiency of the binomial technique in its standard and Breen accelerated forms for American style options. The choice of the early exercise opportunity set for the Breen method is considered. Firstly we consider the case of a single underlying instrument, demonstrating the theoretical improvement in efficiency it yields. We then consider the application of Breen's acceleration technique to the multiple asset case and show that here it only more efficient if the early exercise test is relatively computationally costly.

I Introduction

The Binomial option valuation procedure was introduced by Sharpe (1978), Cox, Ross and Rubinstein (1979) and Rendleman and Bartter (1979). It can be used to value options where no analytical solution exists, in particular American put options. It can also be used where the underlying instrument pays continuous or discrete dividends. Geske and Johnson (1984) introduced a method for pricing

American put options based on the compound option model of Geske (1979) which is more efficient than the standard binomial technique. Breen (1991) has recently described a more efficient binomial procedure for American options based on the Geske and Johnson (1984) model.

In this paper we firstly consider the theoretical efficiency of these techniques for a single underlying instrument. In this case we can demonstrate the theoretical improvement in efficiency. We then consider the application of Breen's acceleration technique to the multiple asset case and show how the efficiency depends on the computational cost of the early exercise test.

In section 2 we briefly review the Binomial procedure. Section 3 contains an analysis of the efficiency of the standard Binomial procedure. In section 4 we restate the Binomial acceleration technique of Breen (1991) for American style options and analyse its efficiency. Section 5 considers the application of Breen's acceleration technique and its efficiency in the multivariate case. Finally, a summary and conclusions are in section 6.

II The Binomial Option Pricing Model

In order to help our exposition, we will briefly review the Binomial procedure in its basic form. The idea is simply to replace the random process followed by the underlying instrument by a Binomial random walk in which the steps have the same expectations and standard deviations as the instrument itself. It is appropriate to work with risk-neutral probabilities, and to assume that the price S_t of the underlying instrument obeys the stochastic equation

$$dS_t = rS_t dt + \sigma S_t dB_t \tag{1}$$

in which r is the interest rate and σ is the volatility of the instrument (both assumed to be constant), and dB_t is the increment of the Standard Brownian Motion. To implement the procedure, one must then imagine a Binomial lattice or 'tree', over which the Binomial random walk is to move. Then the underlying process is replaced with a Binomial random walk on the lattice, in which the time steps have length dt. Equation (1) implies that the proportional increments of the underlying over each time step are iid, and so it is possible either to think of the lattice as referring to proportional changes in the underlying, or to work with logarithms of the underlying; and we prefer the latter. So, put $\log(S_t) = s_t$. Then the stochastic equation for s_t is

$$ds_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dB_t \tag{2}$$

From each node on the lattice, the random walk can either step up by a distance

h^+ with probability p^+ , or it can step down by a distance h^- with probability p^-. These parameters must be chosen so that the expectation and standard deviation of the movement of the random walk over each time step match those of the logarithm of the underlying, i.e. the coefficients in Equation (2). There is a degree of freedom in making this choice, because this subjects these 4 parameters to 2 constraints, and there is also the constraint $p^+ + p^- = 1$. Two convenient choices are those of Cox, Ross, and Rubinstein (CRR) (1979), which also imposes $h^+ = -h^-$, and Jarrow and Rudd (JR) (1983), which imposes $p^+ = p^- = \frac{1}{2}$. In full, these choices are as follows:

CRR:

$$h^{\pm} = \pm\sigma\sqrt{dt}$$

$$p^{\pm} = \frac{1}{2} \pm \frac{\mu}{2\sigma}\sqrt{dt}$$

JR:

$$h^{\pm} = \mu dt \pm \sigma\sqrt{dt}$$

$$p^{\pm} = \frac{1}{2}$$

To implement the procedure, one must develop the lattice until the maturity time of the option. At the lattice nodes corresponding to this maturity time, the option values are then known. From these values, one can then develop the option values backwards through the lattice, by using the rule that at each node, the option value is the discounted expectation of its value after the next time step, i.e. (if $\phi_{i,j}$ represents the option value after i time steps, if there have been j up-steps by time i)

$$\phi_{i-1,j} = \exp(-rdt)(p^{-}\phi_{i,j} + p^{+}\phi_{i,j+1}) \tag{3}$$

To value American options, just test for early exercise at each time step, by comparing the value given by Equation (3) with the value the option would yield if it were exercised at that time. If exercising makes the option worth more, then do so, and to proceed further with the calculation, replace the result of Equation (3) with the exercise value. We will call this the 'step- back' method.

For European options, it is possible to jump in one action over all the steps of the Binomial lattice. Suppose the random walk has N steps in all before the maturity of the option. Then to do this, note that the lattice has just $N+1$ nodes corresponding to the option maturity time, and that all paths leading to say the kth node (which corresponds to k 'ups') have the same probability, namely $(p^{+})^{k}(p^{-})^{N-k}$. Therefore the probability of the Binomial walk ending up at say the kth node at maturity is $\binom{N}{k}(p^{+})^{k}(p^{-})^{N-k}$. The present value of the option is therefore,

$$\phi_{0,0} = \sum_{k=0}^{N} \binom{N}{k} (p^+)^k (p^-)^{N-k} \phi_{N,k} \tag{4}$$

We will call this the 'jump-back' method.

III The Computational Efficiency of the Binomial Technique

We can calculate the relative efficiency of the jump-back and step- back methods by computing the number of basic computational operations involved. For both methods we must compute the $N+1$ node values of the option at maturity (terminal nodes), so we may ignore this in the comparison. For the jump-back method we must sum over the $N+1$ terminal nodes the product of the probability of that node and the value of the option at that node,

$$\phi_{0,0} = \sum_{k=0}^{N} \binom{N}{k} (p^+)^k (p^-)^{N-k} \phi_{N,k} \tag{5}$$

Now this can be reduced to three multiplications and an addition for each term in the summation (assume the binomial coefficients are pre-computed and stored, pre-compute $P_k = (p^+)^k (p^-)^{N-k}, k = 0$ then for each subsequent term simply multiply P_k by p^+/p^-). So the computation time is given by,

$$(N+1)(3\tau_m + \tau_a) \quad (6)$$

where τ_m is the time required for a floating point multiplication and τ_a is the time required for a floating point addition.

The step-back method requires two multiplications and an addition for each non-terminal node (see equation 3, the discount factor can be combined with the up and down probabilities). At each step $i, 1 <= i <= N$ there are i nodes to evaluate, therefore the computation time is,

$$\sum_{i=1}^{N} i(2\tau_m + \tau_a) = \frac{1}{2}N(N+1)(2\tau_m + \tau_a) \quad (7)$$

Therefore for $N > 3$ the jump-back method becomes more efficient relative to the step-back method as N increases.

IV The Breen Acceleration Technique

In a recent paper Breen (1991) shows how to use this idea of jumping over the steps of the lattice together with 'Richardson Extrapolation' to speed up the Binomial procedure when valuing American options. The Breen acceleration technique is as follows:

First, calculate the European option value ϕ_1, i.e. assume there is just 1 exercise opportunity, at the maturity time T of the option, by jumping over all the steps, as described above. Then, assume that there are 2 exercise opportunities, at times $T/2$ and T, and calculate the value ϕ_2 of the option by jumping first over the time interval $[T/2, T]$, and then over $[0, T/2]$. Last, calculate the value ϕ_3, using 3 exercise opportunities, at times $T/3, 2T/3, T$. Now, the sequence of option values $\{\phi_1, \phi_2, \phi_3, ..., \phi_n\}$ converges to the American option value ϕ; the Breen technique is to 'accelerate' the convergence to this limit by using Richardson Extrapolation,

$$\phi = \phi_3 + \frac{7}{2}(\phi_3 - \phi_2) - \frac{1}{2}(\phi_2 - \phi_1) \tag{8}$$

The acceleration of Equation (8) can be justified by adapting the approach of Geske and Johnson (1984).

However, there is a problem with this choice of exercise opportunities. The option values will not necessarily be monotonically increasing since the exercise opportunity sets are not nested (Omberg (1987)). An alternative choice is {T}, {T, T/2}, {T, 3T/4, T/2, T/4}, but this increases the computational cost of the technique. One choice which solves the problem and also increases the efficiency slightly over Breen's choice is {T}, {T, 2T/3}, {T, 2T/3, T/3}.

We can calculate the computation time required to value an American Option by both the step-back and Breen acceleration methods in a similar way as before.

For the step-back method we simply need to add the time taken to perform the early exercise test at each node. We assume that the computational cost of the early exercise test (CCEET) can be reduced to simply a floating point comparison and assignment and is therefore at least as fast as a floating point multiplication. The computation time is therefore approximately,

$$\frac{1}{2}N(N+1)(3\tau_m+\tau_a) \tag{9}$$

For the Breen method we must jump-back firstly over N nodes for the final node. Secondly over $N/2$ for each of the $N/2$ mid-point nodes and the final node. Finally over $N/3$ nodes for the $2N/3$ 2/3-point nodes, the $N/3$ 1/3-point nodes and the final node. The most efficient way to perform these multiple jump- backs is to pre-compute the 1-dimensional array of probabilities which multiply the option values. This requires two multiplications (one for the binomial coefficient and one for the probability factor) for each probability. The computational time for a size $(M+1)$ array is therefore,

$$2(M+1)\tau_m \tag{10}$$

The total computation time for the Breen method is therefore,

$$(N+1)(\tau_m+\tau_a)+2(N+1)\tau_m+$$

$$((N/2+1)(\tau_m+\tau_a)+\tau_m)((N/2+1)+1)+2(N/2+1)\tau_m+ \tag{11}$$

$$((N/3+1)(\tau_m+\tau_a)+\tau_m)((2N/3+1)+(N/3+1)+1)+2(N/3+1)\tau_m$$

which simplifies to,

$$(\frac{7}{12}N^2+\frac{29}{3}N+17)\tau_m+(\frac{7}{12}N^2+\frac{9}{2}N+6)\tau_a \tag{12}$$

In this case N must be greater than approximately 10 before the Breen method becomes more efficient than the standard method. However N will normally be much greater than 10 in order to give reasonable accuracy. Asymptotically, as $N \to \infty$ the Breen method is 55% faster than the standard method for a CCEET of one floating point multiplication and becomes increasingly more efficient as the CCEET increases (see Table 1).

V The Multivariate Case

Boyle, Evnine and Gibbs (1989) (BEG) show how the Binomial technique can be generalised to the case of n underlying assets involved in the option valuation. BEG and Stultz (1982) describe examples of options to which this technique will apply, these include instruments in the EuroMarkets such as option bonds (which are corporate bonds with a choice of payoff currencies) and multi- currency options such as the option to convert the ECU into any one of its constituents, at predetermined rates.

The Breen acceleration idea can be applied in a straightforward way to the multivariate case by using the step-back method. We can analyse the efficiency of the standard BEG and Breen accelerated BEG methods in a similar way to the one dimensional case. For the standard method each node depends on 2^n previous nodes and at step i there are i^n nodes. The computation time is therefore,

$$\sum_{i=1}^{N} i^n(2^n(\tau_m + \tau_a) + \tau_m) \tag{13}$$

For the Breen method we must step-back over N steps, $N/2$ steps and $2N/3$ steps and perform the early exercise test at each exercise opportunity. The total computation time is therefore,

$$\begin{aligned} &\sum_{i=1}^{N} i^n(2^n(\tau_m + \tau_a) + \tau_m) + \\ &\sum_{i=1}^{N/2} i^n(2^n(\tau_m + \tau_a) + \tau_m) + (N/2)^n \tau_m + \\ &\sum_{i=1}^{2N/3} i^n(2^n(\tau_m + \tau_a) + \tau_m) + ((2N/3)^n + (N/3)^n)\tau_m \end{aligned} \tag{14}$$

The efficiency of the Breen method depends on the trade-off between the reduction in the number of early exercise tests and the increase in the number of nodes which must be computed.

We can obtain a quantitative comparison of the computational times in units of τ_m if we ignore the computation time for floating point additions (floating point additions are typically much faster than floating point multiplications). Table 1 gives the ratio of the computational times for the Breen and standard methods for one, two and three underlying assets, for a typical range of values of N and for a CCEET of one and five floating point multiplications.

As we can see from Table 1 the Breen method is in fact less efficient than the step-back method for a CCEET of one floating point multiplication. This is because although the Breen method considerably reduces the number of early exercise tests we must perform their computational cost is small compared with the computational cost of the extra nodes that must be computed in more than one dimension. It is possible to obtain Breen's results in the two dimensional case by assuming the early exercise test requires around 20 floating point multiplications.

VI Conclusions

We have considered the theoretical efficiency of the Binomial procedure for a single underlying instrument and the increase in efficiency obtained by applying Breen's (1991) acceleration technique for American style options. We have noted that there is a better choice of the exercise opportunity set than that proposed by

Breen. We then considered the application of Breen's acceleration technique to the multiple asset case and showed that the increase in efficiency depends critically on the computational cost of the early exercise test.

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Table 1

Ratio of Computational Times for

Breen and Standard Binomial Methods

	Early Exercise Computational Cost							
	1 floating point multiplication				5 floating point multiplications			
N	$n = 1^{\dagger}$	$n = 1^{*}$	$n = 2^{*}$	$n = 3^{*}$	$n = 1^{\dagger}$	$n = 1^{*}$	$n = 2^{*}$	$n = 3^{*}$
10	1.048	1.279	1.223	1.168	0.668	0.756	0.801	0.868
20	0.706	1.208	1.182	1.146	0.400	0.627	0.722	0.826
30	0.597	1.183	1.168	1.138	0.319	0.581	0.693	0.810
40	0.544	1.170	1.160	1.133	0.279	0.557	0.679	0.802
50	0.512	1.162	1.156	1.131	0.256	0.543	0.669	0.797
60	0.491	1.157	1.153	1.129	0.241	0.533	0.663	0.793
70	0.477	1.153	1.151	1.128	0.230	0.526	0.659	0.791
80	0.465	1.150	1.149	1.127	0.222	0.521	0.656	0.789
90	0.457	1.148	1.148	1.126	0.215	0.517	0.653	0.787
100	0.450	1.146	1.147	1.125	0.211	0.514	0.651	0.786

N is the number of time steps

n is the dimensionality of the lattice

† Breen method with jump-back

* Breen method with step-back