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This paper looks at a number of different continuous-time approaches that have been developed to model the term structure of interest rates. These techniques span the interest rate literature over the last 20 years or so, and are the most commonly used among both academics and practitioners. We view this paper as a reference for the different term structure models, aiming to bring together the three most commonly used approaches, emphasising their differences. analysing their respective advantages and disadvantages, and with explicit representations, where they exist, for prices of discount bonds.

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1/ Introduction

This paper looks at a number of different continuous-time approaches that have been developed to model the term structure of interest rates. These techniques span the interest rate literature over the last 20 years or so, and are the most commonly used among both academics and practitioners. In some way we view this paper as a reference for the different term structure models, aiming to bring together the three most commonly used approaches, emphasising their differences, analysing their respective advantages and disadvantages, and with explicit representations, where they exist, for prices of discount bonds (i.e. bonds which have a payoff of unity at maturity with no intermediate cashflows).

We can represent the yield curve in three different but equivalent ways. The first representation is by the prices of discount bonds that differ in their time to maturity. We define the function $P(t,s)$ to be the price at time t of a discount bond which matures at time s, with $t \leq s$. We can also represent the term structure in terms of spot rates by looking at the yields on discount bonds:

$$R(t,s) = \frac{1}{s-t} \ln P(t,s) \tag{1.1}$$

The third formulation is in terms of the forward rate curve. This represents at time t, the instantaneously maturing interest rate at time s and is derived from the discount bond function by applying the following transformation:

$$f(t,s) = \frac{\partial}{\partial t} \ln P(t,s) \tag{1.2}$$

The results that we present will normally be in terms of the first two representations, although at any time we can apply equation (1.2) to derive expressions for the forward rate.

Many traded bonds are not pure discount bonds. The majority of government and corporate bonds pay an annual, or semi-annual, coupon and often come with options, either to the seller or to the buyer of the bond, attached. We are interested in discount bonds because they form the building blocks for these other types of bonds; coupon bonds can be regarded as a portfolio of discount bonds with payoffs and maturities that match the coupon payments, and callable or puttable bonds as portfolios of a straight (or coupon) bond and an option.

We divide the literature into essentially three main approaches to modeling the term structure. The first describes models in which there is a single source of uncertainty driving the evolution of the yield curve[1]. The state-variable in these models is assumed to be the short-term rate of interest r. These models are represented by the papers of Merton [1973], Vasicek [1977] and Cox, Ingersoll, and Ross [1985]. The second approach results in a group consisting of models which involve two state variables. These models are represented by the papers of Brennan and Schwartz [1979], Schaefer and Schwartz [1984], Longstaff and Schwartz [1991], and Fong and Vasicek [1991]. The third approach models the dynamics of the entire term structure in a way that is automatically consistent with the initial (observed) yield curve, and is represented here by the papers Ho and Lee [1986] and Hull and White [1990].

Although we have chosen to categorise the literature in this way, we could have chosen to attribute the models in the first two sections in a different way, based upon the way in which the state variables are defined. The equilibrium approach starts from a description of the underlying economy and from assumptions about the stochastic evolution of one or more state variables in the economy and about the preferences of a representative investor. General equilibrium conditions are used to endogenise the interest rate and the price of all contingent claims. The equilibrium approach was pioneered by Cox, Ingersoll, and Ross [1985], and has been used more recently in two-factor models by Longstaff and Schwartz [1991], and Chen and Scott [1993].

The other approach is based upon no-arbitrage considerations. The arbitrage approach starts from assumptions about the stochastic evolution of one or more interest rates and derives the prices of all contingent claims by imposing the condition that there be no arbitrage opportunities in the economy[2]. Papers that adopt this approach include Vasicek

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1 The perfect correlation of rates implied by a one factor model, means that these models could be re-parameterised in terms of any one, or linear combination, of the other spot rates.

2 No arbitrage models can often be embedded in an equilibrium approach.

[1977], in the one factor case, and Brennan and Schwartz [1979], and Fong and Vasicek [1991] who use two factors.

Although Cox-Ingersoll-Ross discuss the comparison between the equilibrium approach and the no-arbitrage approach, the two are essentially equivalent. Dybvig and Ross [1989] apply to the 'Fundamental Theorem of Asset Pricing', concluding that the two approaches are simply different ways of making the same assumption. Indeed, no-arbitrage models can always be embedded in an equilibrium, although the utility function may in some cases be state dependent.

Our comparison of the term structure models takes the following form. For each alternative approach we analyse the processes assumed for the state variables and the assumptions, where appropriate, about the market price of interest rate risk. We write down the resulting fundamental differential equation that must be satisfied by all interest rate contingent claims based on the processes. Under the boundary condition that the pure discount bond must equal its (known) face value at maturity, we write down the expressions for the pure discount bond price and the pure discount bond price yield if they exist.

We then analyse the possible shapes of the yield curves that each of the models permit, and comment on the implications for the spot rate volatility term structure. Finally, we discuss the number of parameters that need to be estimated for the model and analyse the difficulties in estimating these parameters.

The plan of the paper is as follows.

In section 2 we analyse single factor models of the term structure where the sole source of uncertainty is assumed to be the instantaneous rate of interest. The restrictive nature of the resulting possible term structures and the empirical evidence of more than one factor 'driving' the evolution of the term structure have led to the development of a number of models which assume two sources of uncertainty, and which we analyse in section 3. Section 4 contains an analysis of models which describe the evolution of the yield curve from an initial (observed) state and, finally, section 5 contains a summary and our concluding discussion.

2/ Single Factor Models of the Term Structure

In this section we analyse models of the term structure where the prices of all default-free bonds can be expressed in terms of a single state variable. In all of the models that we study the single source of uncertainty is assumed to be the instantaneous rate of interest, r[3]. A general representation for this class of models is given by the following:

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dz \tag{2.1}$$

where $\mu(r,t)$ and $\sigma(r,t)$ represent the instantaneous (risk-adjusted) drift and variance of the short rate process respectively. dz represents an increment in a Wiener process in a small increment of time dt. Under the assumption of the stochastic process for r, arbitrage pricing theory gives us that the price of a default-free discount bond is the expectation of the payoff discounted at future levels of the short rate;

$$P(t,s) = \hat{E}_t\left[\exp\left(-\int_t^s r_\tau d\tau\right)\right] \tag{2.2}$$

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3 Although any rate could be chosen for this single variable, for convenience it is usually chosen to be the short term interest rate.

where the expectation is taken with respect to the (risk-adjusted) process of the short rate[4]. The process for the instantaneous rate is sufficient, in the absence of arbitrage opportunities, to determine the evolution of the term structure of interest rates. In the following analysis, we will concentrate on the modelling of *r* under the risk-neutral measure, with the resulting advantage that the market price of interest rate risk does not appear in any of the subsequent formulas[5].

Merton [1973]

Perhaps the first and simplest model of the term structure is the random walk model of Merton [1973], which assumes that the short rate follows an arithmetic Brownian motion.

$$dr = \mu\, dt + \sigma\, dz \tag{2.3}$$

Following the methodology of Cox, Ingersoll, and Ross [1985], and applying Ito's lemma to the price of the bond and making use of the no arbitrage condition, it can be shown that under the above assumptions the value of all default-free discount bonds must satisfy the following fundamental differential equation:

$$\frac{1}{2}\sigma^2 P_{rr} + \mu P_r - P_\tau - rP = 0 \tag{2.4}$$

where subscripts denote partial derivatives and $\tau = s - t$. This equation is satisfied not only by discount bonds but also by any interest rate contingent claims which are dependent on at most *r* and *t*. Imposing the maturity boundary condition that the bond pays off its face

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4 See Ingersoll [1987]

5 We can transform the short rate process into the risk neutral measure, from the objective measure, using the analysis of Harrison and Kreps [1979] and Harrison and Pliska [1981].

value, leads to the following solutions for the price of a pure discount bond and its corresponding yield:

$$P(t,s) = \exp\left(-r\tau - \frac{1}{2}\mu\tau^2 + \frac{1}{6}\sigma^2\tau^3\right) \tag{2.5}$$

$$R(t,s) = r + \frac{1}{2}\mu\tau - \frac{1}{6}\sigma^2\tau^2 \tag{2.6}$$

There are clearly undesirable properties associated with the random walk assumption[6]. The possibility of very large (negative and positive) interest rates in the future, becomes more and more likely, as the variance of the distribution for r increases without limit. Equation (2.6) tells us that discount bond yields first increase and then decrease (if the variance of the short rate > 0). In the limit as $\tau \rightarrow \infty$ the spot rate tends to $-\infty$ and the discount bond price to $+\infty$. One of the methods used to overcome the problem of the properties of the distribution for r is to incorporate the concept of mean-reversion into the short rate process.

Vasicek [1977]

Two of the best known models of the term structure were proposed by Vasicek [1977] and Cox, Ingersoll, and Ross [1985]. Vasicek develops an explicit characterisation of the term structure in an efficient market. In a specific case of the general model that he develops the short rate is assumed to change over time consistent with the following process:

$$dr = \alpha(\gamma - r)dt + \sigma dz \tag{2.7}$$

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6 Merton uses the model, as we do here, purely for expositional purposes.

The instantaneous drift of the process $\alpha(\gamma - r)$ represents a force that keeps pulling the process back towards its risk-adjusted long term mean γ with a force α which is proportional to the deviation of the process from the mean. The volatility of the process is represented by a constant σ.

From a computational point of view, the major advantage of using this process is that the conditional distribution of the short rate at any point in time in the future, given the current level of interest rates, is normal. Unfortunately, under this specification for the short rate process, the interest rate can become negative with positive probability[7].

Under the Vasicek process, also known as the Ornstein-Uhlenbeck process, the drift term is adjusted by the market price of risk, λ, which is similar to the risk premium for beta in the Capital Asset Pricing Model and, as a consequence of assuming no arbitrage, is constant across different interest rate securities in this model.

The fundamental equation for interest rate dependent securities is given by:

$$\tfrac{1}{2}\sigma^2 P_{rr} + \alpha(\gamma - r)P_r - P_\tau - rP = 0 \tag{2.8}$$

and the solution to the bond pricing problem is given by:

$$P(t,s) = \exp\left[\tfrac{1}{\alpha}\left(1 - e^{-\alpha(s-t)}\right)(R_\infty - r) - (s-t)R_\infty - \frac{\sigma^2}{4\alpha^3}\left(1 - e^{-\alpha(s-t)}\right)^2\right] \tag{2.9}$$

By applying equation (1.1) we obtain the term structure of spot interest rates.

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7 See Carverhill [1992] or Babbs [1990] for probabilities under realistic parameter values.

$$R(t,s) = R_\infty + (r - R_\infty)\frac{1}{\alpha(s-t)}\left(1 - e^{-\alpha(s-t)}\right) + \frac{\sigma^2}{4\alpha^3(s-t)}\left(1 - e^{-\alpha(s-t)}\right)^2 \tag{2.10}$$

where

$$R_\infty = \gamma - \frac{1}{2}\frac{\sigma^2}{\alpha^2}$$

determines the yield on the long (infinite maturity) bond.

The possible shapes permissible under equation (2.10) allow the term structure of discount bond yields to be uniformly rising, uniformly declining, or slightly humped.

Once the level of the state variables and the parameters of the risk-adjusted process are specified, the mathematical relationships within the model determine the volatilities of the spot rates that differ in their time to maturity. Applying Ito's lemma to equation (2.7) and using (2.10) we find the volatility structure of spot interest rates, $\sigma_R(t,s)$, to be determined by both the volatility and the rate of mean reversion of the short rate[8]:

$$\sigma_R(t,s) = \frac{\sigma}{\alpha(s-t)}\left(1 - e^{-\alpha(s-t)}\right) \tag{2.11}$$

This implies that long rates are less volatile than short rates (the curve is negative exponential declining to zero at the long end - expression (2.9) shows the long end to be fixed and not dependent on the level of the short rate). The rate of mean reversion, i.e. the

8 We focus on $\sigma_R(r,s)$ instead of the perhaps more familiar generalised duration measure due to the widespread usage of the term 'term structure of spot rate volatility' in connection with term structure consistent models.

rate at which the standard deviations of spot (and hence forward rates) decline, is dictated by the parameter α.

Not all of the parameters of the Vasicek model can be identified from time series linear regression. The identifiable structural parameters are α and σ^2, with (the non-risk-adjusted) γ and market price of risk not identifiable separately. To estimate the parameters of the model cross-sectionally, a procedure similar to that used by Brown and Dybvig [1986] can be implemented. The instantaneous interest rate is not normally known as it is unusual for a bond with the correct maturity (i.e. instantaneously) to be trading. Most tests of interest rate models therefore try to recover the short rate parameter and so the full set of, jointly estimated, parameters is r, α, γ, and σ^2. The market price of risk is not individually observable, but is incorporated in the risk-adjusted drift term.

Cox, Ingersoll, and Ross [1985]

The Gaussian interest rate process of the Vasicek paper ignores the empirical evidence of the positive relation between interest rates and the level of interest rate uncertainty[9]. Cox, Ingersoll, and Ross [1985] develop a model in which the term structure is determined within a general equilibrium framework, and which relates the variability of the short rate process to the level of the rate itself, hence allowing interest rate volatility to be conditionally heteroskedastic. The general equilibrium conditions allow the term structure, its dynamics, and the form of the market price of interest rate risk to be endogenously determined as part of the equilibrium (in contrast to the Vasicek model which is imposed on the model in order to obtain analytical tractability). In this paper Cox-Ingersoll-Ross describe a number of models, the simplest of which is based on a special case of a single factor model of interest rates. The risk-neutral dynamics of the short term interest rate for this model are given by the process:

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9 See, for example, Chan, Karolyi, Longstaff, and Sanders [1991]

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\, dz \qquad (2.12)$$

where $\kappa(\theta\text{-}r)$ is the, mean-reverting, instantaneous rate of drift of the process (as in the Vasicek model), and *dz* again represents the uncertainty of the process. In this model the volatility of the short rate increases with the square root of the level of the rate, precluding the existence of negative interest rates, and allowing more variability at times of high interest rates, and less variability when rates are low. This dependence has led the model to be widely known as the 'square-root process'.

The fundamental equation for the price of a pure discount bond is given by:

$$\tfrac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - rP = 0 \qquad (2.13)$$

If we impose the pure discount bond boundary condition that $P(s,s) = 1$, the solution of the differential equation is found to be:

$$P(t,s) = A(t,s)e^{-B(t,s)r} \qquad (2.14)$$

where

$$A(t,s) = \left(\frac{\phi_1 e^{(\phi_2(s-t))}}{\phi_2\left(e^{\phi_1(s-t)} - 1\right) + \phi_1} \right)^{\phi_3}$$

$$B(t,s) = \left(\frac{e^{(\phi_2(s-t))} - 1}{\phi_2\left(e^{\phi_1(s-t)} - 1\right) + \phi_1} \right)$$

and where

$$\phi_1 \equiv \sqrt{\kappa^2 + 2\sigma^2}$$
$$\phi_2 \equiv \frac{(\kappa + \phi_1)}{2}$$
$$\phi_3 \equiv \frac{2\kappa\theta}{\sigma^2}$$

Discount bond prices, under this process, are therefore a function of r, the time to maturity, and the parameters ϕ_1, ϕ_2, and ϕ_3, which are themselves, related to the risk-adjusted parameters of the interest rate process.

For Cox-Ingersoll-Ross, pure discount bond yields are given by applying equation (1.1).

$$R(t,s) = \frac{1}{s-t}[rB(t,s) - \ln A(t,s)] \tag{2.15}$$

As with the Vasicek model there are only three possible shapes obtainable for discount bond yields; uniformly rising, uniformly declining, or slightly humped.

For long times to maturity, the price of a pure discount bond tends to;

$$P \approx e^{-(\phi_1 - \phi_2)\phi_3(s-t)} \tag{2.16}$$

with the corresponding discount rate on such long bonds tending to the value given by:

$$R_\infty = (\phi_1 - \phi_2)\phi_3 = \frac{2\kappa\theta}{\kappa + \theta} \tag{2.17}$$

The constant term in the volatility of the short rate is related to the parameters of the discount bond price by the equation:

$$\sigma^2 = 2[\phi_1\phi_2 - \phi_2^2] \tag{2.18}$$

The volatility structure of spot rates is again determined by the state variable and the parameters of the process, via Ito's lemma, to be:

$$\sigma_R(t,s) = \frac{\sigma\sqrt{r}}{s-t} B(t,s) \tag{2.19}$$

Equation (2.14) defines the Cox-Ingersoll-Ross model in terms of discount bond prices. The model contains four parameters, 3 involved in the objective process for the short rate, plus one to represent the market price of risk. Estimating cross-sectionally the parameters from prices of bonds reduces the parameter space to three, $\phi_1, \phi_2,$ and ϕ_3, as these depend only on the risk-adjusted process. Time series estimation using a two-step ordinary least squares procedure in principle enables the separate identification of the short rate and the structural parameters, α, γ, σ, and the risk parameter.

Other Models

Other models of the term structure involving a single source of uncertainty include the Dothan [1977] model. The short rate is determined to follow a mean-reverting process with the volatility increasing with the level of the rate. The solution for bond prices involves the evaluation of modified Bessel functions. The basic drawback of this model and others of its kind, i.e. models with the form :

$$dr = \alpha(\gamma - r)dt + \sigma r^{\beta} dz$$

where $\beta > 0$ and not equal to 1/2, is the absence of closed-form solutions and hence the need for the application of numerical methods.

Nearly all empirical tests of single factor models have been carried out using the Cox-Ingersoll-Ross model. Almost all of the studies (including Brown and Dybvig [1986] using US T-bills, notes and bonds, Barone, Cuoco, and Zautik [1989] using Italian Treasury bonds, and Brown and Schaefer [1991] who employ a database of British Government index-linked bonds) find that the implied volatility of the short rate corresponds closely to time series estimates, but that the results concerning the fixed nature of the long rate and the stability of the parameter estimates suggest that this single factor model of the term structure is mis-specified[10]. One exception to this is the results of Brown-Schaefer who find that the estimated real long term zero-coupon yield is quite stable.

The models that we have concentrated on in this section have been characterised by their tractability and ease of use. We have discussed the stochastic process assumed by each of the models and outlined the parameters that need to estimated to implement the models. The number of parameters are relatively few but with the resulting disadvantage of unrealism in the assumed stochastic process for the interest rate. All single factor models suffer from the assumption that all that is known about future interest rates is impounded in the current short rate, implying that apart from deterministic shifts over time, the whole term structure is inferred from the current short rate.

3/ Two-State Variable Models

There are obvious disadvantages to models that assume a single source of uncertainty, the majority of which we have just discussed. A number of studies[11] and casual empiricism have concluded that the variability across rates of different maturities can better be explained by incorporating more than one source of uncertainty. This has led a number of

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10 See also Gibbons [], Heston [], and Pearson and Sun [].

11 Notably Dybvig [1989] on US data and Steeley [1991] on UK data.

authors to propose models of the term structure which incorporate two state variables or stochastic factors. Although all of the models take the short rate (in one form or another) as one of the driving forces of the term structure, they chose the second state variable to be one of a number.

Richard [1978] and Cox-Ingersoll-Ross [1985]

Two of the earliest models, Richard [1978] and another model developed in the 1985 Cox-Ingersoll-Ross paper, take as their state variables the instantaneous real rate of interest and the rate of inflation. The analysis of both of these papers (based upon the exclusion of arbitrage opportunities) results in the partial differential equation governing the evolution of bond prices containing two preference functions that can be regarded as the market prices of (state variable) risk . In both models the authors assume a representative investor economy with the (very stong) assumption of logarithmic utility, with Richard also considering the risk-neutral case. One of the advantages of these models is that the long term rate of interest is endogenously derived within the model and as such is not constrained to be a constant dictated by the parameters of the model, although the assumptions about, and estimation of, the utility functions are a severe disadvantage for practical implementation.

Brennan and Schwartz [1979]

Brennan and Schwartz [1979] and Schaefer and Schwartz [1984] develop models of the term structure which are based on the specification of stochastic processes for two interest rates. Brennan-Schwartz chose as their two factors the short (r) and the long (l) rates allowing the model to reflect the assumption that the current long term interest rate contains information about the future value of the short rate. This approach, because it takes the observed values for the rates at the extremes of the term structure, is only attempting to explain the intermediate points of the term structure. An advantage of the choice of the long term rate as the second state variable is that by taking its value as the yield on a traded instrument (a consol bond) the risk associated with it can be hedged away.

After choosing specific forms for the drift of the short rate and volatility functions for both of the processes that preclude the existence of negative interest rates, the authors show that pure discount bonds must satisfy the following partial differential equation:

$$\tfrac{1}{2}P_{rr}r^2\sigma_1^2 + P_{rl}\rho r l\sigma_1\sigma_2 + \tfrac{1}{2}P_{ll}l^2\sigma_2^2 + P_r\beta_r + P_l\beta_l - P_\tau - rP = 0 \qquad (3.1)$$

σ_1 and σ_2 dictate the variance of the short and long rates respectively, whilst the drift parameters β_r and β_l are influenced by the volatility parameters, parameters relating the level of the two interest rates, and the utility dependent function for the market price of short term interest rate risk.

Unfortunately, from the perspective of practitioners there is no known closed-form solution to the differential equation (3.1) and so it has to be solved numerically or by simulation.

Schaefer and Schwartz [1984]

Schaefer-Schwartz use the same information about interest rates as Brennan-Schwartz, but express their model in terms of the long rate and the spread ($s = r - l$) between the long rate and the short rate. This change is really just a redefinition of variables which allows the authors to obtain an analytic solution to the valuation problem. The change of variables trick, used to exploit the empirical evidence that the two state variables l and s are orthogonal, gives an exact solution to the problem that is an approximation to that studied by Brennan-Schwartz. The analytical approximation is solved as the product of two expressions which are essentially the Vasicek and Cox-Ingersoll-Ross single factor solutions. This is not surprising given the specification of the stochastic processes for the state variables:

$$ds = m(\mu - s)dt + \gamma dz_1$$
$$dl = \beta_l dt + \sigma\sqrt{l}dz_2 \tag{3.2}$$

The process for the spread is the mean-reverting Ornstein-Uhlenbeck process of the Vasicek model discussed earlier, where the market price of spread risk is assumed to be a constant. As we discussed in section 2 this process admits negative values, which is empirically reasonable for the spread between two interest rates. The drift term for the process for the consol (long) rate does not enter the valuation equation as the rate is again defined as the yield on an infinite maturity bond. The variability of the long rate is allowed to depend on the level of the rate itself in the same way as in the Cox-Ingersoll-Ross model. The estimation of the models parameters is similar to those discussed for the single factor models of the previous section, although to make the solution tractable, an extra parameter has to be estimated as the solution that equates two related partial differential equations.

Two recent papers cause us to question the approach jof both Schaefer & Schwartz, and Brennan & Schwartz. Dybvig, Ingersoll, and Ross [1992] prove that the long rate is non-decreasing (in the above its proxy - the consol rate - is a stochastic process). Hogan [1993] proves that there are no real-valued solutions to the diffusion equations written down for the long and short rate by Brennan and Schwartz, thus allowing for the existence of arbitrage.

The models of Brennan and Schwartz, and Schaefer and Schwartz are consistent with the results of principle components analysis of the term structure[12] that the first two factors explaining the variability across rates of different maturities, should reflect the level and slope of the yield curve. Dybvig [1989] proposes that the second factor of a two factor

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12 The interested reader is refereed to the papers of footnote 8.

model should be chosen as the variance of the first factor, especially if we are interested in bond derivative pricing. In the next part of this section we discuss, in detail, two of the most recent multi-factor papers in the term structure literature[13]. Both of these papers use as the two sources of uncertainty the two factors Dybvig suggests and which anecdotal evidence suggests are the most intuitively appealing to practitioners; the short rate and the volatility of the short rate.

Longstaff and Schwartz [1992]

Longstaff and Schwartz [1992] develop a general equilibrium framework for valuing interest-rate-sensitive contingent claims using a two state variable version of the continuous-time economy modeled by Cox-Ingersoll-Ross[14]. The two factors are the short interest rate, r, and the variance of changes in the short term interest rate, v, thus allowing contingent claim prices to reflect both the current level of interest rates and the current level of interest rate volatility. The authors first develop the dynamics of two economic factors which evolve independently of one another and are used to characterise the process for realised returns on physical investment:

$$\begin{aligned} dx &= (\gamma - \delta x)dt + \sqrt{x}dz_1 \\ dy &= (\eta - \theta y)dt + \sqrt{y}dz_2 \end{aligned} \tag{3.3}$$

γ, δ, η, and θ are parameters of the risk adjusted process for the uncorrelated state variables. The fundamental partial differential equation for all default-free interest rate contingent claims, H, is shown to be:

$$\frac{1}{2}xH_{xx} + \frac{1}{2}yH_{yy} + (\gamma - \delta x)H_x + (\eta - \theta y)H_y - (\alpha x + \beta y)H = H_\tau \tag{3.4}$$

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13 An analysis of both of these models in relation to their assumptions and implications for term structure volatility is contained in Strickland [1993b]

14 See also Langetieg [1980].

The equilibrium instantaneous interest rate and the variance of changes in this rate are given in this framework as a weighted sum of the state variables, x and y, where the weights relate to parameters of the return process for physical investment.

$$\begin{aligned} r &= \alpha x + \beta y \\ v &= \alpha^2 x + \beta^2 y \end{aligned} \tag{3.5}$$

The form of r and v allows the authors to express their results in terms of r and v as the state variables, although expressing the results in terms of the original x and y is computationally easier. The transformation between the two sets of state variables, given by equation (3.5) implies a high degree of correlation between r and v. Also, in order for x and y to be non-negative we need v to satisfy the following bounds:

$$\alpha r \leq v \leq \beta r \tag{3.6}$$

The resulting joint process for the dynamics of the short rate and the volatility of the short rate (or of the original state variables) allow us to write down closed form solutions for the prices of pure discount bonds. Bond prices, when expressed in terms of r and v, take the form,

$$P(t,s) = \exp(G(\tau) + C(\tau)r + D(\tau)v) \tag{3.7}$$

where $\tau = s - t$. The price is a function of the state variables and their risk adjusted parameters. The functions of time G, C, and D are tractable and easy to compute:

$$G(\tau) = \kappa\tau + 2\gamma \ln A(\tau) + 2\eta \ln B(\tau)$$

$$A(\tau) = \frac{2\phi}{(\delta + \phi)(\exp(\phi\tau) - 1) + 2\phi}$$

$$B(\tau)=\frac{2\psi}{(\theta+\psi)(\exp(\psi\tau)-1)+2\psi}$$

$$C(\tau)=\frac{\alpha\phi(\exp(\psi\tau)-1)B(\tau)-\beta\psi(\exp(\phi\tau)-1)A(\tau)}{\phi\psi(\beta-\alpha)}$$

$$D(\tau)=\frac{\psi(\exp(\phi\tau)-1)A(\tau)-\phi(\exp(\psi\tau)-1)B(\tau)}{\phi\psi(\beta-\alpha)}$$

where

$$\phi=\sqrt{2\alpha+\delta^2}$$
$$\psi=\sqrt{2\beta+\theta^2}$$
$$\kappa=\gamma(\delta+\phi)+\eta(\theta+\psi)$$

The discount bond price, therefore, depends on the six parameters α, β, γ, δ, η, θ. Yields on the s-maturity bond are obtained via equation (1.1) as:

$$R(t,s)=\frac{-(\kappa\tau+2\gamma\ln A(\tau)+2\eta B(\tau)+C(\tau)r+D(\tau)v)}{\tau} \qquad (3.8)$$

As τ approaches infinity the yield to maturity on the infinitely maturing bond converges to a constant independent of the current values of r and v, implying that the long end of the curve is fixed once the parameters of the process have been specified.

$$R_\infty=\gamma(\phi-\delta)+\eta(\psi-\theta) \qquad (3.9)$$

At the very short end of the term structure, the prices of very short maturity bonds are unaffected by changes in the volatility of the short rate.

The reliance of the yield curve on both the short rate of interest and the volatility of the short rate allows the term structure to assume a greater variety of shapes than equivalent single factor models. The curves permissible under the equation for the yield curve can be monotone increasing or decreasing, to have a hump or a trough, or a combination of the two. Changes in both state variables with the other held fixed can have significant effects on the slope and shape of the yield curve.

The expression for the term structure of spot yield volatility implied by the model depends on the two factors, allowing a greater variety of permissible volatility structures, and is given by:

$$\sigma_R(t,s) = \frac{1}{s-t}\sqrt{\left(\frac{\alpha\beta\psi^2(\exp(\phi\tau)-1)^2A^2(\tau)-\alpha\beta\phi^2(\exp(\psi\tau)-1)^2B^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right)r + \left(\frac{\beta\phi^2(\exp(\psi\tau)-1)^2B^2(\tau)-\alpha\psi^2(\exp(\phi\tau)-1)^2A^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right)v} \tag{3.10}$$

To price discount bonds under the formulation of equation (3.7) requires, as inputs, the level of the instantaneous rate, the volatility of the rate and the 6 additional parameters. Longstaff and Schwartz [1993] outline a parameter estimation method that uses the historical time series of interest rates and estimated time series of interest rate volatilities. By looking at these time series, and appealing to Equation (3.6), α and β can be approximated as the minimum and the maximum of the ratio $\frac{v}{r}$ respectively. The long run stationary distributions for r and v, and the yield of the infinite maturity discount bond are all expressions dependent only on the parameters of the risk adjusted process and α and β. Rearranging these expressions we obtain the following:

$$\delta = \frac{\alpha(\alpha+\beta)(\beta E[r]-E[v])}{2\left(\beta^2 Var[r]-Var[v]\right)}$$

$$\gamma = \frac{\delta(\beta E[r]-E[V])}{\alpha(\beta-\alpha)}$$

$$\xi = \frac{\beta(\alpha+\beta)(E[v]-\alpha E[r])}{2\left(Var[v]-\alpha^2 Var[v]\right)}$$

$$\eta = \frac{\xi(E[v] - \alpha E[r])}{\beta(\beta - \alpha)}$$

The one remaining parameter to be estimated θ can then be estimated cross-sectionally as the value that minimizes some error term for a cross-section of discount bonds. In a recent paper Clewlow and Strickland [1994] use simulation techniques to show that although, in theory, the estimation methodology works, the nature of financial time series available to financial researchers and market participants make its practical implementation extremely difficult.

Fong and Vasicek [1991]

In a series of related papers, Fong and Vasicek (1991, 1992a, 1992b) present a two-factor model of the term structure of interest rates, with the same two factors driving the prices of discount bonds as the Longstaff-Schwartz paper we have just discussed. The authors derive the prices of pure discount bonds under the equilibrium condition of no arbitrage. Fong-Vasicek start from the direct characterisation of the evolution of their stochastic factors. The behaviour of the short rate is determined by the diffusion process:

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1 \tag{3.11}$$

$\bar{r}$ is the long-term mean of the short rate, which has instantaneous volatility v.

Fong-Vasicek assume that the variance, v, is stochastic following the process:

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2 \tag{3.12}$$

$\bar{v}$ is the long-term average of the volatility. The random component of the volatility process has a variance proportional to the current level of the volatility. Both processes are described in the risk-neutral measure. The random element, dz_1, of the short rate, and the random element, dz_2, of the volatility process are correlated with a coefficient of ρ.

One of the problems with the joint specification of the state variables given in equations (3.11) and (3.12) is that interest rates can become negative. The process for the short rate is essentially Vasicek [1977]. The extra uncertainty added by allowing the variance of the short rate itself to follow a stochastic process implies that the probability of observing negative rates is higher in the two factor model than its one-factor equivalent.

The above assumptions describe the model. The resulting partial differential equation governing the price of a pure discount bond is found to be:

$$\begin{aligned} &P_t + \alpha(\bar{r} - r)P_r + \gamma(\bar{v} - v)P_v \\ &+ \frac{1}{2}vP_{rr} + p\xi vP_{rv} + \frac{1}{2}\xi^2 vP_{vv} - rP = 0 \end{aligned} \tag{3.13}$$

subject to the usual boundary condition.

The price of a pure discount bond, which is the solution to the partial differential equation is given as:

$$P(t,s) = \exp(-rD(t) + vF(t) + G(t)) \tag{3.15}$$

where $D(t) = \frac{1}{\alpha}\left(1 - e^{-\alpha(s-t)}\right)$

is the duration measure of the Vasicek [1977] paper. $F(t)$ and $G(t)$ are computed to be complicated expressions involving the confluent hypergeometric function. Although these functions are difficult to evaluate - the solution proposed by Fong and Vasicek requires complex (as opposed to real) algebra - we have developed extremely efficient series solutions which allow us very easily to compute pure discount bond prices, see Selby and Strickland [1993].

Yields on discount bonds, after an application of equation (1.1), are given by:

$$R(t,s) = \frac{rD(t)}{\tau} - \frac{vF(t)}{\tau} - \frac{G(t)}{\tau} \tag{3.16}$$

allowing the shape of the yield curve to take the same possible forms as the Longstaff-Schwartz model. The shape of the spot rate volatility function is found by applying Ito's lemma to equation (3.16) to be:

$$\sigma_R(t,s) = \frac{1}{s-t}\sqrt{\left(D(\tau)^2 - 2\rho\xi F(\tau)D(\tau) + \xi^2 F(\tau)^2\right)v} \tag{3.17}$$

In total eight parameters need to be estimated; the parameters of the risk-adjusted process, the correlation between the state variables and the coefficients of the factor risk premiums.

In summary, two factor models of the term structure allow a more realistic representation of the yield curve than their single factor equivalents. They appeal to the empirical evidence that more than one factor is working in real-world markets. The models of our analysis allow a greater variety of permissible yield curves and term structures of volatility, and are fairly easy to compute, but with the disadvantage of their practical implementation (unobservability and number of parameters).

4/ 'Term Structure Consistent' Models

We have discussed approaches to the term structure of interest rates which start from a specification of state variable(s) and the processes followed by those variables(s), and which then attempt to construct a term structure of bonds which differ only in their time to maturity. Although the parameters of the model are chosen so as to provide a reasonable fit to the term structure actually observed, the model will probably not do so perfectly. The approach that we now discuss is mainly used to price term structure derivative securities such as options on bonds, due to the fact that prices of discount bonds are an input to the model rather than an output as in the approaches we have seen so far. We use the approach here to derive an interest rate term structure at some time in the future consistent with a given, initially observed, state.

Ho and Lee [1986]

This approach is what we term the 'term structure consistent' approach and it originated with Ho and Lee [1986] who were the first authors to build a model that set out to model the dynamics of the entire term structure in a way that was automatically consistent with the initial (observed) term structure of interest rates. The Ho and Lee model is developed in the form of a binomial tree relating future movements of the yield curve explicitly to it's initial state but it has been shown that the continuous time limit can be characterized by the short rate process:

$$dr = \theta(t)dt + \sigma dz \tag{4.1}$$

where $\theta(t)$, the drift of the stochastic process during the short time interval dt, is a function of time in order to make the model consistent with the initial term structure of interest rates. The drift term reflects the slope of the initial forward curve and the volatility of the short rate. The risk preferences of investors are imbedded in the market prices of discount bonds which determines the drift term.

Within this framework discount bond prices as a function of the level of the short rate, it's uncertainty, and the initial exogenously defined term structure $\{P(0,s): s \geq 0\}$ can be shown to be:

$$P(t,s) = A(t,s)e^{-B(t,s)r} \tag{4.2}$$

where

$$B(t,s) = (s-t)$$
$$A(t,s) = \ln\frac{P(0,s)}{P(0,t)} - B(t,s)\frac{\partial \ln P(0,t)}{\partial t} - \tfrac{1}{2}\sigma^2 t B(t,s)^2$$

The single function of time in the short rate process allows the model to fit only the term structure of interest rates, and so the term structure of volatility is determined within the model. As can be seen from equation (4.1) the model describes the short rate volatility by a single parameter σ, leading to a term structure of interest rate volatilities that implies spot rates and forward rates that differ in their maturity are all equally variable, all future spot rates are normally distributed and all possible yield curves at a future time are parallel to each other. i.e.,

$$\sigma_R(t,s) = \sigma \tag{4.3}$$

A further difficulty of the model is that it incorporates no mean reversion, and as a result there is a positive probability that future interest rates will become negative[15]. Dybvig

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15 See Strickland [1993c] for drawbacks of the Ho and Lee model for pricing bond options.

[1989] argues that although the model can obtain a sensible initial yield curve it is only after making unreasonable assumptions about the variance process of the short rate and expected interest rates, thus we should not rely on the model for pricing options on interest rates except at very short maturities.

Hull and White [1990]

Hull and White [1990] seek to reconcile the tractability of the single factor Vasicek and Cox-Ingersoll-Ross models with the consistency of a model that fits the initial yield curve. Their proposed models can be seen, and are presented as, extensions to the Vasicek and Cox-Ingersoll-Ross models, due to the similarity of the nature of the short rate processes to those presented in the original papers:

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)dz \tag{4.4}$$

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)\sqrt{r}\,dz \tag{4.5}$$

Hull-White propose that the three functions of time $\theta(t)$, $\phi(t)$ and $\sigma(t)$ are chosen so that the models, determined by equations (4.4) and (4.5), fit the initial term structure of interest rates, the term structure of spot rate volatilities, and the anticipated variability across time of the instantaneous spot rate.

For equation (4.4), the 'extended Vasicek' model, the bond price as a function of the short rate, and maturity, is given by equation (4.2) with;

$$B(t,s) = \frac{B(0,s) - B(0,t)}{\partial B(0,t)/\partial t}$$

$$\hat{A}(t,s) = \hat{A}(0,s) - \hat{A}(0,t) - B(t,s)\frac{\partial \hat{A}(0,t)}{\partial t} - \frac{1}{2}\left[B(t,s)\frac{\partial B(0,t)}{\partial t}\right]^2 \int_0^t \left[\frac{\sigma(\tau)}{\partial B(0,\tau)/\partial \tau}\right]^2 d\tau$$

where $\hat{A}(t,s) = \ln[A(t,s)]$

$B(0,s)$ and $A(0,s)$ are defined by the initial term structure of spot (or forward) rates and rate volatilities. Fitting these functions to term structure data is a relatively simple operation once the term structures and the volatility of the short rate have been determined[16]. The only other function to specify is $\sigma(\tau), t \leq \tau \leq s$, the future variability of the short rate.

The second model, represented by equation (4.5) - the 'extended Cox-Ingersoll-Ross' model, allows the variability of the short rate to be related to the its level. This has the effect of eliminating the possibility of negative interest rates but the solution is not as analytically tractable as in the extension to Vasicek. The solution involves using numerical procedures to solve the partial differential equation governing the evolution of the bond price.

Independently, Jamshidian [1993] studying the same class of models, finds that if one restricts $\theta(t)$, $\phi(t)$, and $\sigma(t)$ to the case that $\theta(t)/\sigma^2(t)$ is a constant then results concerning the bond pricing formula are tractable.

In a recent paper Strickland [1993b] discusses the constraints on the volatility functions for the extended models analysed above. He finds that, for the extended Vasicek model, the yield volatilities at time t are constrained to be related to the initial , time 0, curve in the following way[17].

$$\sigma_R(t,s) = \frac{\sigma(t)}{(s-t)} \frac{\left[s\sigma_R(0,s) - t\sigma_R(0,t)\right]}{\left[t\sigma_R'(0,t) + \sigma_R(0,t)\right]} \tag{4.6}$$

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16 See Strickland [1993d]

17 This result was developed in discussions with Stewart Hodges. See also Carverhill [1992].

A similar relationship (in closed form only for the Jamshidian restriction) exists for the extended Cox-Ingersoll-Ross model. Equation (4.6) implies that once an initial spot rate volatility function, $\sigma_R(0,s)$, has been specified, its subsequent evolution is deterministic, and may evolve in a way not originally intended by the user.

One of the special cases of equation (4.4) is the extended Vasicek model with both the mean revision and the instantaneous standard deviation constant. This model can be thought of as the Vasicek model with time dependent drift or the Ho-Lee model with mean-reversion, retaining the property of consistency with the term structure, whilst allowing for mean reversion. The short rate process becomes:

$$dr = [\theta(t) - ar]dt + \sigma dz \tag{4.7}$$

The authors describe this model as the Hull and White "benchmark" model. The bond price as a function of the short rate is given by equation (4.2) with:

$$B(t,s) = \frac{1}{a}\left(1 - e^{-a(s-t)}\right)$$
$$\ln A(t,s) = \ln \frac{P(0,s)}{P(0,t)} - B(t,s)\frac{\partial \ln P(0,t)}{\partial t} - \frac{1}{4a^3}\sigma^2\left(e^{-as} - e^{-at}\right)^2\left(e^{2at} - 1\right) \tag{4.8}$$

The volatility structure, of spot interest rates, is determined by both the volatility and the rate of mean reversion of the short rate and is the same as for the Vasicek model, i.e.

$$\sigma_R(t,s) = \frac{\sigma}{a}\left(1 - e^{-a(s-t)}\right)$$

Other Term Structure Consistent Models

Other models which are constructed to be consistent with initial term structure data include that of Black, Derman and Toy [1990] and the approach of Heath Jarrow and Morton [1992]. Neither of these approaches result in analytical solutions to the bond pricing problem. Their real value lies in pricing bond derivatives and so a discussion of these models is left to our sister paper.

The term structure consistent approach has the advantage over the models from the previous two sections, of pricing pure discount bonds (at some valuation date in the future) in a way which allows the model to be consistent with the observed bond prices[18]. There is however, no published evidence that although these models fit the market data, the process that they imply for the short rate account for the stochastic evolution of the term structure any better than the approaches contained in section 3 or section 4. As we have already mentioned the major advantage of this approach lies in its ability to price pure discount bond derivatives.[19]

5/ Summary and Conclusions

We have analysed a number of different approaches to modeling the term structure of interest rates. Each approach has advantages as well as disadvantages when compared on the basis of the tractability of the model solution, the number and estimatability of the parameters, and the amount of market information used. The first approach involves models that are based on equilibrium characteristics of the term structure and a single source of uncertainty, and seeks to determine the whole term structure in terms of key

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18 See Strickland [1993d] for an analysis of the dangers of 'overfitting' with a resulting loss of stability in estimating the parameters.

19 See Strickland [1993c] for a thorough analysis of the advantages and disadvantages of this approach.

parameters such as the short rate. Models that belong to this category can be characterised by their tractability and ease of use, but with the resulting disadvantages of unrealistic assumptions about the stochastic process for the short rate, and the limitation of possible shapes that the term structures can take.

Models which involve two stochastic factors allow for a more realistic representation of the yield curve and are relatively efficient in terms of computation. However, they suffer from the empirical observability of their parameter values and their failure to fit the observed term structure of interest rates as implied from the prices of traded bonds.

The last approach that we analyse sets out to build models that model the dynamics of the whole term structure from an initially input state. These models can be fitted to term structure volatility data as well as interest rate term structure data, and their main use is for pricing derivative instruments written on discount bonds. Although they use the maximum amount of data available, there may be estimation problems as, on each valuation day, a number of different term structures have to be estimated. It is also an open empirical question as to whether or not models that adopt this approach describe the stochastic evolution of the term structure an better than the traditional approach.

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