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This paper looks at the different approaches and different models that have been developed to value interest rate dependent securities. In a related paper, Strickland [1993a] we examined the approaches to modelling the term structure of interest rates and we take here most of that paper as given and extend it to take account of interest rate derivative securities that are contingent on the level or shape of the yield curve. This paper provides a survey of pricing procedures which are based on mathematical models of the term structure, and can be viewed as a reference for the different interest rate models, with explicit representations, where they exist, for prices of derivative instruments and an analysis of their respective advantages and disadvantages.

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1/ Introduction

This paper looks at the different approaches and different models that have been developed to value interest rate dependent securities. In a related paper, Strickland [1993a] we examined the approaches to modeling the term structure of interest rates and we take here most of that paper as given and extend it to take account of interest rate derivative securities that are contingent on the level or shape of the yield curve. Part of the motivation of this paper is to provide a survey of pricing procedures which are based on mathematical models of the term structure, and as such it can be seen as an extension of an earlier survey paper by Carverhill [1990]. In the same way as for Strickland [1993a] we view this paper as a reference for the different interest rate models, with explicit representations, where they exist, for prices of derivative instruments and an analysis of their respective advantages and disadvantages. We concentrate much of our analysis on models which provide the prices of futures and options on discount bonds, with the justification that many interest rate derivative products can be decomposed into portfolios of these two instruments.

We follow the same division of the term structure literature as our related paper. This can basically be summarised into three main approaches; models in which there is a single source of uncertainty driving the evolution of the yield curve, represented by the papers of Vasicek [1977] and Cox, Ingersoll, and Ross [1985]; models which involve two state variables, represented by the papers of Longstaff and Schwartz [1991], and Fong and Vasicek [1991]; and thirdly, models in which the dynamics of the entire term structure are modeled in a way that is automatically consistent with the initial yield curve, and is represented here by the papers Ho and Lee [1986], Black, Derman and Toy [1990], Hull and White [1990a], and Heath, Jarrow and Morton [1992].

For each of the approaches we review the different models assumptions about the underlying state variables, and explain how interest rate derivatives are priced based upon these assumptions.

The plan of the paper is as follows.

In section 2 we provide a survey of interest rate derivative securities dealing separately with 'money market' and Government debt instruments. In section 3 we show how a number of commonly traded interest rate derivative instruments can be viewed as portfolios of futures and options on pure discount bonds. Based upon this analysis sections 4 and 5 deal with models that provide the prices of derivatives on discount bonds in a single factor and multi-factor setting respectively. Section 6 extends our analysis to cover models that are designed to be consistent with an observed initial term structure. Finally, section 7 presents a summary and our conclusions.

2/ A Survey of Interest Rate Securities

In this section we present a survey of the most important interest rate instruments. The list includes the most popular interest rate based derivatives as well as some institutional detail about their underlying instruments. We deal separately with "money market" debt instruments, and "Government" debt instruments, the main distinction being in the default risk of the two types of instruments.

2.1 Government Securities Debt Instruments

These are securities issued by the government and it is usual to make the assumption that the probability of the issuers defaulting is zero. The basic government security is the government bond. These issues are known as "Gilts" in the United Kingdom, and "treasury issues" in the United States and we can imply riskless interest rates from the prices of these bonds. A government bond with coupon c percent will pay this proportion of the principal or face value of the bond, usually in the form of two six-monthly payments of $\frac{c}{2}$, and at maturity it will pay this together with the principle.

In the UK there are about 80 such bonds with coupons ranging between 3% and 15%, and maturity dates which may be up to 30 years into the future. In addition there are also index linked bonds whose coupon payments are related to a measure of inflation. In the United States there are three types of treasury issues. US treasury bills are issued with maturities upto 1 year, and are discount instruments with the only cashflow occurring at the maturity of the bond. Treasury notes and treasury bonds pay interest semiannually with notes being issued with maturities up to seven years, whilst treasury bonds may be issued with any maturity, but generally with an original maturity of over 10 years

US treasury STRIPS (Separate Trading of Registered Interest and Principal of Securities) have been available since February 1985 and are zero-coupon instruments derived from selected treasury bonds or notes of 10 or more years to maturity. The underlying treasury bonds and notes are separated on the books of the Federal Reserve into their component parts of principle and coupon payments. The resulting "zero-coupon" securities may be separately owned, and pay no interest until maturity.

Both futures and options are traded on government bonds. The LIFFE long gilt futures contract (the right to buy or sell a notional 9% coupon, 20 year maturity bond at a specified date) is the most popular government debt derivative contract in the UK (see Strickland[1993b] for details and an empirical analysis of the contract). Options on the long gilt futures contract are marked-to-market in the same way as the underlying contract. Options are also traded on specific Gilts, which are themselves chosen to be representative Gilts at various terms to maturity.

2.2 Money Market Instruments

In this category we include instruments which are not issued by the government and as such are not considered to be risk-free. The most basic money market instrument is LIBOR (the London Interbank Offer Rate). It is a floating reference rate of interest determined by the trading of deposits between banks on the Eurocurrency market. At any one time it is possible to obtain LIBOR reference rates for 1-month, 3-month, 6-month and 1-year maturities. The rates refer to borrowing in the spot market with the interest payment in arrears.

A Forward Rate Agreement (FRA) is an agreement between two counterparties which guarantees the rate on borrowing a nominal sum over a certain time, beginning at a certain time in the future. The only cashflow occurs at the maturity of the contract as the difference in the interest payments on the nominal sum. The underlying reference rate for these agreements is usually based on one of the LIBOR rates. Exchange traded futures contracts based on 3-month LIBOR are called "short sterling" futures in the UK, and "eurodollar" futures in the US. Options are also traded with the underlying instrument being the futures contract.

One of the most popular interest rate instruments is a cap rate agreement. These agreements are made "over-the-counter" (OTC), privately between two counterparties, rather than via an exchange. An interest rate cap involves restricting the interest rate variations on floating rate loans. Caps guarantee that the rate of interest on the loan will never go above a certain level. The rate charged will be the lesser of the prevailing rate and the cap rate. Caps are usually negotiated on 3 month, 6 month, and 1 year sterling LIBOR rates.

As an example consider an investor who has a floating rate £100m loan based on the 6 month LIBOR rate. The investor buys, from a financial institution, a fiˇve year cap, which caps the interest rate that he has to pay over the 5 year period at 12%. The terms of his original loan dictate that every 6 month period the rate of interest that the investor has to pay, 6 months in arrears, i.e. at the end of the period, is set equal to the 6 month LIBOR rate prevailing at the beginning of the period. The cap ensures that the writer of the cap has to pay the excess, if any, of the 6 month LIBOR rate over 12% to the investor. The payments, by the writer, are again made 6 months in arrears. In this way the investor never pays over 12% on his floating rate loan.

Interest rate floors and interest rate collars can de defined analogously to caps. Floors place a lower limit on the interest rate that will be charged. If interest rates fall below the floor the borrower obtains financing at the floor rate. Collars specify both upper and lower limits for the rate of interest that will be charged.

Interest Rate Swaps are private agreements between two companies to swap interest rate cashflows on a nominal sum according to a prearranged formula. The simplest type of interest rate swap involves the exchange of fixed payments (say annual) for floating payments (say 3-month or 6-month LIBOR). Swaps are quoted on bankers screens for periods between 2 years and 10 years. The quoted rate is the rate of fixed payments that can be swapped for LIBOR over the period. Also increasing in popularity are options on interest rate swaps. These instruments are called swaptions and give the holder the right, but not the obligation, to enter into a certain interest rate swap at a certain time in the future.

3/ Decomposing Interest Rate Derivative Instruments into Portfolios of Derivatives on Pure Discount Bonds

In this section we show how a number of complicated interest rate derivative instruments can, for valuation and hedging purposes, be viewed as portfolios of futures and options contracts on pure discount bonds. This enables us to restrict our attention in later sections to models which provide answers to these problems, still allowing us to price the complicated instruments.

3.1 Pricing European Options on Coupon Bearing Bonds.

When there is only one factor of uncertainty, and an analytic relation between the discount bond prices can be derived, Jamshidian [1989] has suggested a method to value options on coupon bearing bonds which views the problem as a portfolio of options on discount bonds.

Consider a European call option with exercise price X and maturity T on a coupon bearing bond. Let the price of a bond at time T, when the interest rate is r, which matures at time s, and pays off c_i at time s_i, $s_i > T$, ($i = 1,..,n$) be given by $B(r,T,s,c_i,s_i)$. The option will be exercised if:

$$B(r,T,s,c_i,s_i) > X$$

i.e.
$$\sum_{i=1}^{n} c_i P(r,T,s_i) > X \tag{3.1}$$

Bond prices are decreasing functions of the short-term interest rate r. Let r^* be such that;

$$\sum_{i=1}^{n} c_i P(r^*,T,s_i) = X \tag{3.2}$$

If at time T;

$$r < r^* \quad \text{then} \quad \sum_{i=1}^{n} c_i P(r,T,s_i) > X$$

$$r > r^* \quad \text{then} \quad \sum_{i=1}^{n} c_i P(r,T,s_i) < X$$

Let $X_i = P(r^*,T,s_i)$.

The payoff from the option at time T is given by:

$$\max\left[0, \sum_{i=1}^{n} c_i P(r,T,s_i) - X\right]$$

or

$$\sum_{i=1}^{n} c_i \max\left[0, P(r,T,s_i) - X_i\right] \tag{3.3}$$

This implies that an option on a coupon paying bond is equivalent to a portfolio of options on the individual discount bonds with appropriate exercise prices[1].

3.2 Caps, Floors and Collars as Pure Discount Bond Options.

An interest rate cap limits the borrowers floating interest rate to a fixed level, the cap rate, for a given period of time. We will show that an interest rate cap can be interpreted as a series of put options on zero coupon bonds, and so can be valued as such. Analogously an interest rate floor can be valued by summing the premiums on a portfolio of call options on discount bonds, and a collar as the sum of premiums of a portfolio consisting of both put and call options.

Consider an option that caps the interest rate on \$1 at the rate r_C between times t_1 and t_2. Δt is the time difference between t_1 and t_2. r is the interest rate at time t_1 for the period (t_1,t_2).

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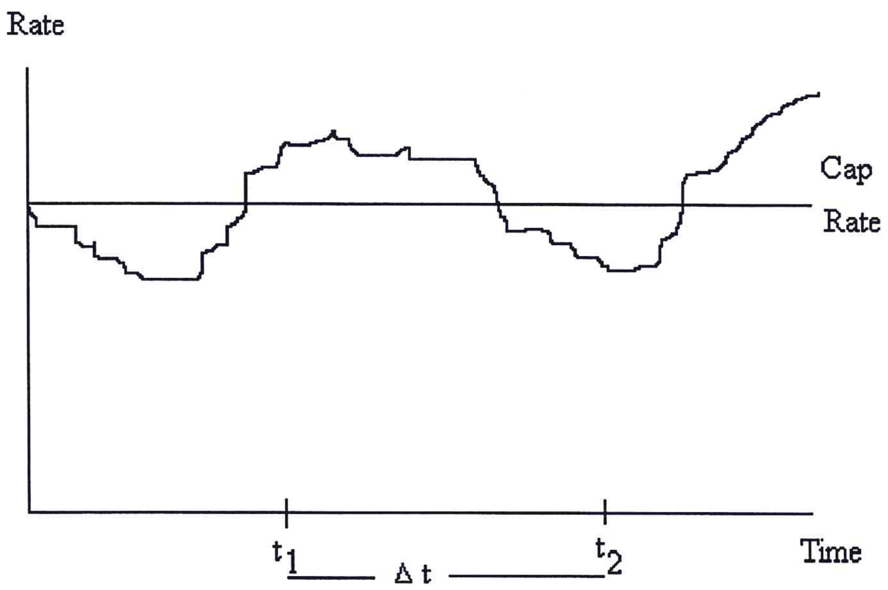
1 Usually a portfolio of options is more complex than an option on a portfolio, but here they all have the same payoff depending on the sign of $(r - r^*)$ making the problem simpler.



The payoff to the option at time t_2 is

$$\Delta t \max(r - r_c, 0)$$

At time t_1 the discounted value of this payoff is given by

$$\frac{\Delta t}{1 + r\Delta t} \max(r - r_c, 0) \tag{3.4}$$

This expression is equivalent to

$$(1 + r_c \Delta t) \max\left(\frac{1}{1 + r_c \Delta t} - \frac{1}{1 + r\Delta t}, 0\right) \tag{3.5}$$

Therefore, an option which caps the interest rate at r_c between t_1 and t_2 is equivalent to $(1+r_c\Delta t)$ European put options with exercise price $X_c = \frac{1}{1 + r_c \Delta t}$ and expiration date t_1 on a $1 face value discount bond maturing at time t_2. More generally an interest rate cap is a portfolio of European puts on a series of discount bonds.

An interest rate floor agreement can be defined analogously. It places a lower limit on the interest rate that will be charged. An interest rate floor can be interpreted as a series of call options on zero coupon bonds. Consider an option that provides a floor on the interest rate on \$1 at the rate r_F between times t_1 and t_2.

The payoff to the option at time t_2 is

$$\Delta t \max(r_F - r, 0)$$

At time t_1 the discounted value of this payoff is given by

t

$$\frac{\Delta t}{1+r\Delta}\max(r_F - r, 0) \tag{3.6}$$

This expression is equivalent to

$$(1+r_c\Delta t)\max\left(\frac{1}{1+r\Delta t} - \frac{1}{1+r_F\Delta t}, 0\right) \tag{3.7}$$

Therefore, an option which provides a floor on the interest rate at r_F between t_1 and t_2 is equivalent to $(1+r_F\Delta t)$ European call options with exercise price $X_F = \frac{1}{1+r_F\Delta t}$ and expiration date t_1 on a \$1 face value discount bond maturing at time t_2. More generally an interest rate floor agreement is a portfolio of European calls on a series of discount bonds.

A collar is just a long position on a cap and a short position on a floor with the same characteristics of settlement dates and reset intervals. This implies that the price of a collar is equal to the difference between the price of the put string with strike price X_C, and the price of the call string with strike X_F.

In the following analysis we will be concentrating on term structure derivatives. For each of the classifications of the introduction, the pricing of pure discount bonds was analysed in Strickland [1993a]. In this article we concentrate on the pricing of futures contracts and options contracts on discount bonds with the justification of the results contained in this section.

4/ Single Factor Models for Pricing Derivatives on Pure Discount Bonds.

In the first part of this section we concentrate on the valuation of discount bond derivative securities based on the explicit modeling of the underlying variable, i.e. the level of the bond price - an approach that we term the direct approach. We then analyse approaches that concentrate upon deriving the values of all interest rate derivative securities as a function of the instantaneous interest rate - the indirect approach. The difference between the approaches is really just a matter of parameterisation, but it allows us to categorise the literature.

Black and Scholes [1973]

The simplest (and reputedly most widely used model) for valuing options on pure discount bonds is the Black and Scholes [1973] model which models the bond price as lognormal. The model was originally developed for the valuation of stock options and assumes that the variance of the rate of return on the underlying security is constant.

The model assumes that the clean (i.e. exclusive of accrued interest) price, at time t of a discount bond which pays off unity at time s, $P(t,s)$, follows the familiar Geometric Brownian Motion with constant drift, μ, and variance, σ.

$$\frac{dP(t,s)}{P(t,s)} = \mu dt + \sigma dz \tag{4.1}$$

dz represents an increment in a Wiener process in a small increment of time dt. If no coupon payments are due to be received during the life of the option (with maturity date T), the Black Scholes model gives analytic solutions for the prices of European call and put options, with exercise price X, as:

$$c(t,T,s) = P(t,s)N(d_1) - e^{-R(t,T)(T-t)}XN(d_2) \tag{4.2}$$

$$p(t,T,s) = e^{-R(t,T)(T-t)}XN(-d_2) - P(t,s)N(-d_1) \tag{4.3}$$

where

$$d_1 = \frac{\log(P(t,s)/X) + (R(t,T) + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma$$

$R(t,T)$ is the current interest rate applicable to a riskfree investment maturing at the same time as the option.

If coupon payments are due to be received during the life of the option, their present value must be subtracted from $P(t,s)$ in the above equations, making the underlying stochastic variable the forward clean price of the bond. If the options are American numerical procedures such as the binomial technique[2] or finite difference method[3], or an analytic approximation[4] must be used.

However, a number of problems exist when using this model to price bond options. Firstly, the cashflows are discounted at a constant interest rate, $R(t,T)$, whilst at the same time the assumption is made that the bond price follows a stochastic process. The interest rate used, however, corresponds to the life of the option, and not to the instantaneous rate.

Secondly, and perhaps more importantly, the model assumes that the volatility of the bond is constant. Although this assumption is plausible for stocks, we know however, that the price volatility of bonds reduces as the bond approaches its maturity date, because at that time the value is known with certainty to be the face value of the bond. This problem increases in importance as the maturity of the option becomes closer to the maturity of the underlying bond. It may not, however, be too unreasonable for short maturity options on long term bonds. In spite of these problems practitioners frequently use the Black-Scholes model to value debt options because of its overwhelming simplicity.

Margrabe [1978]

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2 See for example Cox, Ross and Rubenstein [1979]

3 See for example Hull and White [1990b]

4 See for example Barone-Adesi and Whaley [1987]

In order to overcome the first problem of using the Black-Scholes formula for pricing bond options, we can perform the following. Still keeping within the Black-Scholes framework, i.e. the underlying instrument is a discount bond and the evolution of bond prices can be described by a diffusion of the form (4.1), we can think of an option with maturity date T, and strike X, on an s-maturity discount bond, as an option to exchange a discount bond with face value X and maturity T for the bond underlying the option[5]. Margrabe [1978] has solved this exchange problem and has determined a closed form solution for the pricing of European options. The Margrabe formula for a European option, exercisable at time T adapted to price s-maturity discount bond options is given as:

$$c(t,T,s) = P(t,s)N(d_1) - XP(t,T)N(d_2) \tag{4.4}$$

where

$$d_1 = \frac{\log(P(t,s)/XP(t,T)) + (\sigma_P^2/2)(T-t)}{\sigma_P\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma_P$$

The main difference between equation (4.4) and the earlier (4.2) is the volatility input σ_P. Now, because both assets, $P(t,T)$ and $P(t,s)$, can vary, the variance of both is important to the overall volatility of the option. The input volatility is determined to be:

$$\sigma_P^2 = \sigma_{P(t,s)}^2 - 2\sigma_{P(t,s)}\sigma_{P(t,T)}\rho + \sigma_{P(t,T)}^2 \tag{4.5}$$

where $\sigma_{P(t,T)}$ is the volatility of the T-maturity bond

$\sigma_{P(t,s)}$ is the volatility of the s-maturity bond

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5 See for example Hull and White [1990a].

ρ is the instantaneous correlation between the Wiener processes for the T-maturity and s-maturity bond.

Schaefer and Schwartz [1987]

An interesting approach to retaining a 'priced-based' approach whilst incorporating the systematic reduction in the volatility of spot bond prices has been developed by Schaefer and Schwartz [1987] who use a simple procedure for valuing options on bonds that takes into account the changing characteristics of the underlying bond by assuming that the bond's standard deviation of returns is proportional to it's duration. Their model is general enough to take into account continuously paid coupons and, as with Black-Scholes, has the bond as the state variable with a standard deviation of return which is set to be proportional to a duration measure. This adjustment allows the model to capture the attenuation of a bonds standard deviation of return as it approaches maturity.

The authors assume that the price of a default free bond at time t follows the geometric diffusion process:

$$dP(t,s) = \mu P(t,s)dt + kP(t,s)^{\alpha} D(P,t)dz \tag{4.6}$$

where $\sigma(P,t)$ is the instantaneous standard deviation of return, and μ is the instantaneous expected return, possibly stochastic.

The standard deviation of returns, $\sigma(P,t) = kP^{\alpha-1}D(P,t)$,allows the model to reflect the reduction in price volatility as the bond approaches maturity by assuming that it is proportional to its duration. $D(P,t)$ is specified as the Reddington [1952] duration, a duration measure that depends only on P and t. α and k are constants which control the sensitivity of the bonds return to its price level.

The pricing model developed by Schaefer and Schwartz assumes that the short-term rate of interest is a constant, *r*, which the authors acknowledge to be inconsistent with both the stochastic variation in the prices of long bonds, and with empirical evidence. It also has the consequence that bond prices are not guaranteed to converge to par at maturity. Finally, the resulting partial differential equation subject to the usual terminal and early exercise boundary conditions does not have an analytical solution and so must be solved numerically (e.g. by a finite difference method).

Another, more popular, way of handling the characteristics of the bond price volatility through time is to use the forward price of the bond at the expiry date of the option as the underlying variable in the Black-Scholes formula[6]. This ad hoc adaptation of the formula will solve the volatility problem, but it does not lead to an integrated approach. For example, the approach just described assumes that the forward price is lognormal, caps are often valued using this approach with the forward zero coupon yield assumed to be the lognormal variable, and swaps with the underlying being the forward swap yield. This leads to 'one model - one product', each incompatible with each other, i.e. the deltas cannot be aggregated.

Prices of interest rate dependent securities do not always depend on the price of discount bonds and so in many traditional models of the term structure it is the process followed by the interest rate that is important. Although the difference from the models discussed so far is simply a matter of parameterisation, this approach has the advantage of focusing attention on the dynamics of the instrument that attracts the most attention in bond markets. To be consistent with the paper Strickland [1993a] we will concentrate on the modelling of the state variable under the risk neutral measure.

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6 i.e. apply Black's [1976] formula.

Merton [1973]

The earliest and simplest model of the term structure is the arithmetic Brownian motion assumption for the short rate, r, of Merton [1973].

$$dr = \mu dt + \sigma dz \tag{4.7}$$

where the instantaneous drift, μ, and variance, σ, of the process are both constants. Under this framework, the price at time t of a European call option with strike X, which matures at time T on an s-maturity discount bond ($t \le T \le s$) is given by

$$C(t,T,s) = P(t,s)N(h) - XP(t,T)N(h-\sigma_p) \tag{4.8}$$

with $$h = \frac{\log\left(\frac{P(t,s)}{XP(t,T)}\right)}{\sigma_p(t)} + \frac{\sigma_p(t)}{2}$$

and where $$\sigma_P^2 = \sigma^2(s-T)^2(T-t) \tag{4.9}$$

Prices of European put options on pure discount bonds follow by the put-call parity relationship

$$p(t,T,s) = C(t,T,s) - P(t,s) + P(t,T)X \tag{4.10}$$

which for this process is equal to

$$XP(t,T)N(\sigma_p - h) - P(t,s)N(-h) \tag{4.10a}$$

Prices of the T-maturity and s-maturity discount bonds, $P(t,T)$ and $P(t,s)$, are theoretically calculated prices given by the Merton [1973] formula. The process for the short rate has a number of obvious disadvantages, which are discussed in Strickland [1993a] and which have led a number of researchers to propose other, more realistic, processes for the short rate.

Vasicek [1977]

Two of the best known models of the term structure were proposed by Vasicek [1977] and Cox-Ingersoll-Ross [1985]. In both of these models the risk-adjusted short rate is assumed to be the single source of uncertainty:

$$dr = \alpha(\gamma - r)dt + \sigma(r,t)dz \tag{4.11}$$

The instantaneous drift of the (risk-adjusted) process $\alpha(\gamma - r)$ represents a mean reverting force that keeps pulling the process back towards its long term risk-adjusted mean γ with a force α which is proportional to the deviation of the process from the mean. The instantaneous volatility of the process is represented by a function $\sigma(r,t)$. In the paper of Vasicek the short rate is assumed to follow the Ornstein-Uhlenbeck diffusion process and the volatility of the process $\sigma(r,t)$ is represented by a constant σ.

Although Vasicek does not derive a bond option pricing formula Jamshidian [1989], assuming the same Gaussian process for the instantaneous interest rate, derives a closed form solution for European options on pure discount bonds. The prices at time t, for T-maturity European call and put options with strike X, on a pure s-maturity bond are given by (4.8) and (4.10a) respectively, with:

$$\sigma_p = \frac{v(t,T)\left(1 - e^{-\alpha(s-T)}\right)}{\alpha} \tag{4.12}$$

$$v^2(t,T) = \mathrm{var}_{r,t}[r(T)] = \frac{\sigma^2\left(1 - e^{-2\alpha(T-t)}\right)}{2\alpha}$$

Bond prices $P(t,s)$ $(s \geq t)$ are given by the Vasicek pure discount bond price formula[7]. The parameters that need to be estimated to implement this model are the same as for the Vasicek term structure model. The interested reader is referred to the relevant section of Strickland [1993a].

The above analysis for options does not allow for the early exercise feature of American options. Carverhill [1992] using the techniques developed in Nelson and Ramaswamy [1990] presents a binomial procedure which allows for the pricing of American options in a Vasicek framework. Hull and White [1990b] show how trinomial trees can be used to value American bond options and other interest rate contingent claims in the same model. Because bond prices are known analytically at each node (via the Vasicek bond price formula), when evaluating option prices the bond price lattice need only be extended until the life of the option and not until the life of the bond.

Chen [1992a] focuses on futures prices and European options on futures, on pure discount bonds under the same normal process, deriving a closed-form solution for each. The futures price, at time t, with maturity date T_F, on a pure discount bond that matures at time s, $F(t,T_F,s)$, is given by;

$$F(t,T_F,s) = e^{-rX(t)-Y(t)} \tag{4.13}$$

where

7 See Strickland [1993a] section 2.

$$X(t) = \phi(t,s) - \phi(t,T_F)$$

$$Y(t) = R_{\infty}[s - T_F - X(t)] - \frac{\sigma^2}{2\alpha^2}\left(X(t) - \frac{\alpha}{2}X(t)^2 - \phi(T_F,s)\right)$$

$$\phi(t,s) = \frac{1 - e^{-\alpha(s,t)}}{\alpha}$$

R_{∞} is the yield on an infinity maturity bond and is given by $\gamma - \frac{1}{2}\frac{\rho^2}{\alpha^2}$

The price of a European call futures option at time t, on a futures contact which matures at time T_F, is given by the formula;

$$C(t,T_F,s) = H(r,t)N(h) - XP(t,T)N(h - \sigma_H) \tag{4.14}$$

where $$h = \frac{\log\left(\frac{H(r,t)}{XP(t,T)}\right)}{\sigma_H} + \frac{\sigma_H}{2}$$

$$\sigma_H = \sigma\sqrt{\frac{1 - e^{-2\alpha(T-t)}}{2\alpha}} = [\phi(T,s) - \phi(T,T_F)] \tag{4.15}$$

$$H(r,t) = P(t,T)F(t,T_F,s)e^{\frac{\sigma^2}{2}\phi(t,T)^2 X(T)}$$

In a separate paper, Chen [1992b], the author derives a closed form (but lengthy) solution for the futures price under discrete marking-to-market. Simulations suggest, however, that, under the realistic assumption of daily marking-to-market, unless investors are highly risk adverse discrete futures prices are little different from continuous ones.

Cox, Ingersoll, and Ross [1985]

The underlying process for both the Merton and Vasicek models discussed above allows negative interest rates to occur with positive probability, due to the nature of the assumptions about short rate volatility. Cox, Ingersoll, and Ross [1985] develop a model in which the volatility of the short rate is related to the (square root of the) level of the rate. This allows interest rate volatility to be conditionally heteroskedastic thus permitting the short rate to be more volatile when rates are high and less volatile in time of low rates, and also precluding negative interest rates. Capturing, to a certain extent, the dynamic behaviour of interest rate volatility provides the model with the potential to give more realistic values to interest rate contingent claims such as options since their values are closely related to interest rate volatility. The dynamics of the short term interest rate for the single factor model developed in their 1985 paper are given by equation (4.11) with the volatility of the process, $\sigma(r,t)$, increasing with the square root of the rate itself, $\sigma(r,t) = \sigma\sqrt{r}$, with σ constant.

Under these dynamics of the short rate the price of a European call option at time t, is given by:

$$
\begin{aligned}
C(t,T,s,X) = {} & P(t,s)\chi^2\left(2r^*[\phi+\psi+B(T,s)];\ \frac{4\alpha\gamma}{\sigma^2},\ \frac{2\phi^2 r e^{\theta(T-t)}}{\phi+\psi+B(T,s)}\right) \\
& - XP(t,T)\chi^2\left(2r^*[\phi+\psi];\ \frac{4\alpha\gamma}{\sigma^2},\ \frac{2\phi^2 r e^{\theta(T-t)}}{\phi+\psi}\right)
\end{aligned}
\qquad (4.16)
$$

where

$$\theta \equiv \sqrt{(\alpha + 2\sigma^2)}$$

$$\phi = \frac{2\theta}{\sigma^2\left(e^{-\theta(T-t)} - 1\right)}$$

$$\psi = \frac{\alpha + \theta}{\sigma^2}$$

$$r^* = \ln\left(\frac{A(T,s)}{X}\right) \Big/ B(t,s)$$

and where $\chi^2(.;p,q)$ is the noncentral chi-squared density with p-degrees of freedom and non-centrality parameter q. Prices of pure discount bonds are given by the Cox-Ingersoll-Ross [1985] formula[8]. Put prices are given by an application of (4.10).

This pricing formula (4.16) has a similar interpretation to the Black-Scholes formula. The two terms in the equation correspond to the discounted value of the underlying instrument, and the discounted value of the exercise price multiplied by the probability of the option finishing in-the-money respectively. Although the pricing formula is closed-form in the same sense as Black-Scholes, practitioners have criticised it for its diminished tractability, due to the need to solve for the particular cumulative distribution function.[9].

Sankaran [1963] provides an efficient algorithm for approximating the cumulative non-central chi-squared distribution function using the cumulative standard normal distribution function[10]. Based on this algorithm $\chi^2(z;v,k)$ is approximated by:

$$\chi^2(z;v,k) \approx 1 - N(Q(z;v,k))$$

where

8 See Strickland [1993a] section 2.

9 See Babbs [1991].

10 See also Schroder [1989].

$$Q(z;v,k)=\frac{1-hp[1-h+\frac{1}{2}(2-h)mp-[z/(v+k)]^h}{h\sqrt{2p(1+mp)}}$$

$$h=1-\frac{2}{3}(v+k)(v+3k)(v+2k)^{-2}$$

$$p=\frac{v+2k}{(v+k)^2}$$

$$m=(h-1)(1-3h)$$

Longstaff [1993] extends the Cox-Ingersoll-Ross analysis deriving closed-form expressions for European call and put options on coupon bonds under the same square root process. The formulas that he derives are consistent with (and an explicit representation of) the approach of section 3.1 of viewing the problem as a portfolio of options on discount bonds

Nelson and Ramaswamy [1990] develop a binomial approximation of the Cox-Ingersoll-Ross diffusion that can be used to price American bond options. Tian [1992] also develops a binomial approach which avoids the multiple jumps which may be needed at each node in the Nelson-Ramaswamy model to incorporate mean-reversion in the short rate process, and which increases the numerical complexity of the model.

In an earlier paper, Cox, Ingersoll, and Ross [1981], give the price of a futures contract that matures at time T_F on a s-maturity discount bond as the risk-adjusted expectation of the discount bond price at delivery:

$$F(t,T_F,s)=A^*(t,T_F,s)\exp(-rB(t,T_F,s)) \tag{4.17}$$

where

$$A^{*}(t,T_F,s) = A(T_F,s)\left(\frac{2\alpha}{2\alpha+\sigma^2 B(T_F,s)\left(1-e^{-\alpha(T_F-t)}\right)}\right)^{\frac{2\alpha\gamma}{\sigma^2}}$$

$$B^{*}(t,T_F,s) = \left(\frac{2\alpha e^{-\alpha(T_F-t)} B(T_F,s)}{2\alpha+\sigma^2 B(T_F,s)\left(1-e^{-\alpha(T_F-t)}\right)}\right)$$

Some of the most heavily exchange traded interest rate options are Treasury Bill futures options, Eurodollar futures options, and LIBOR futures options. Feldman [1993] derives a closed form expression for the pricing of European options on bond futures contracts under the square root process. His expression is similar to equation (4.16), reflecting the nature of the underlying contract.

Finally, Chen and Scott [1993] illustrate how the Cox-Ingersoll-Ross one factor interest rate model can be used to price options on bond futures with futures style margining. The partial differential equation that must be satisfied by the option on the futures contract is similar to the original Cox-Ingersoll-Ross equation but without the carry term for the option because of the absence of any initial investment required for this option. The authors note that Jamshidian's trick for pricing options on coupon bonds can be applied under the conditions of their paper. This adaptation of the well-known Cox-Ingersoll-Ross model is important in the London Market due to the futures style margining for options currently used at the London International Financial Futures and Options Exchange (LIFFE). Chen's [1992b] methodology for marking-to-market of the futures price, that we discussed in relation to the Ornstein Uhlenbeck process, is general enough to extend to the square root process.

In this section we have described two, single factor, approaches to pricing term structure derivatives. The first approach is based in the classic 'Black-Scholes' framework which takes the underlying bond as the stochastic variable and derives a differential equation that must be satisfied by any bond derivative. Inconsistent assumptions about the constant nature of the short rate however, have led to the development of models which are applications of models originally designed to explain the term structure. Single factor models for pricing derivative instruments on discount bonds are attractive, especially to practitioners, due to the closed form nature of their solutions, and the observability and relatively low number of parameters that have to be estimated[11]. Disadvantages in this approach include the restrictive nature of the possible term structures that the models allow, the fact that the long rate is a deterministic function of the risk-adjusted parameters, and that the models assumptions about term structure movements and interest rate volatility are not matched by the empirical evidence[12]. Also, this approach is unsuitable for the integrated treatment of portfolios of derivatives especially when the portfolio includes non-standard deals.

Finally, perhaps the biggest disadvantage of this approach for pricing interest rate derivative securities stems from its origin. The derivative pricing models stem from models that were developed to theoretically derive the term structure. As a consequence the yield curve is a product of the model rather than an input into it. In other words derivatives end up being priced in relation to a curve which is different from that implied by market prices.

5/ Two Factor Models for Pricing Pure Discount Bond Derivatives.

The disadvantages associated with single factor models have led a number of researchers to propose models which have more than one factor to describe the evolution of the term

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11 See Strickland [1993b] for a more complete discussion on parameter estimation for models discussed in this section.

12 See Strickland [1993c]

structure. These models permit a greater variety of possible term structures and allow for the empirical evidence that more than one factor is driving the term structure.

In this section we concentrate on a two factor model which is currently popular for pricing interest rate derivatives and results in closed-form solutions for pure discount bonds and discount bond options. Because of the difficulty of evaluating the cumulative distribution functions when we move from single factor to multi-factor models of the term structure, we also discuss the use of monte-carlo simulation for multi-factor processes. This general approach allows us to use other two factor models of the term structure to price term structure derivatives.

Longstaff and Schwartz [1992]

Longstaff and Schwartz [1992] develop a general equilibrium framework for valuing interest-rate-sensitive contingent claims using a two state variable version of the continuous-time economy modeled by Cox-Ingersoll-Ross. The two factors are the short interest rate, r, and the variance of changes in the short term interest rate, v, thus allowing contingent claim prices to reflect both the current level of interest rates and the current level of interest rate volatility[13].

This paper is one of only a number that deal with the same broad class of models, which can be thought of as multi-variate versions of the single factor Cox-Ingersoll-Ross model. Other papers that deal with this class of model include Beaglehole and Scott [1991], Chen and Scott [1992], Constantinides [1992], and Duffie and Kan [1993][14].

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13 In part they are motivated by Dybvig's [1989] conclusion that the second factor of a two factor model should be chosen as the variance of the first factor, especially if we are interested in bond derivative pricing.

14 Duffie and Kan model spot rates at selected fixed maturities as multivariate square root processes. They point out that typically one can perform a change of basis that makes the state variables of the other models equivalent to yields at various maturities as in their model.

The framework that Longstaff-Schwartz develop allow them to obtain closed-form expressions for options on discount bonds which depend on r and v. The inclusion of the latter state variable is an advantage over one-factor models as volatility is a fundamental determinant of option values. The authors first develop the dynamics of two economic factors which evolve independently of one another and are used to characterise the process for realised returns on physical investment:

$$\begin{aligned} dx &= (\gamma - \delta x)dt + \sqrt{x}dz_1 \\ dy &= (\eta - \theta y)dt + \sqrt{y}dz_2 \end{aligned} \tag{5.1}$$

The equilibrium instantaneous interest rate and the variance of changes in this rate are given in this framework as a weighted sum of the state variables, x and y, where the weights relate to parameters of the return process for physical investment.

$$\begin{aligned} r &= \alpha x + \beta y \\ v &= \alpha^2 x + \beta^2 y \end{aligned} \tag{5.2}$$

The form of r and v allows the authors to express their results in terms of r and v as the state variables. The value of a call option, on an s-maturity bond, with exercise price X and time to maturity $\tau = T\text{-}t$, is a complicated expression involving the solution of the bivariate noncentral chi-square distribution function:

$$\begin{aligned} C(t,T,s) \quad &= P(t,s)\Psi(\theta_1,\theta_2;4\gamma,4\eta,\omega_1,\omega_2) \\ &\qquad -XP(t,T)\Psi(\theta_3,\theta_4;4\gamma,4\eta,\omega_3,\omega_4) \end{aligned} \tag{5.3}$$

where

$$\theta_1 = \frac{4\zeta\phi^2}{\alpha(\exp(\phi T)-1)^2 A(s)}$$

$$\theta_2 = \frac{4\zeta\psi^2}{\beta(\exp(\psi T)-1)^2 B(s)} \quad (5.4)$$

$$\theta_3 = \frac{4\zeta\phi^2}{\alpha(\exp(\phi T)-1)^2 A(T)}$$

$$\theta_4 = \frac{4\zeta\psi^2}{\beta(\exp(\psi T)-1)^2 B(T)}$$

and

$$\omega_1 = \frac{4\phi\exp(\phi T)A(s)(\beta r - V)}{\alpha(\beta-\alpha)(\exp(\phi T)-1)A(T-s)}$$

$$\omega_2 = \frac{4\psi\exp(\psi T)B(s)(V-\alpha r)}{\beta(\beta-\alpha)(\exp(\psi T)-1)B(T-s)}$$

$$\omega_3 = \frac{4\phi\exp(\phi T)A(T)(\beta r - V)}{\alpha(\beta-\alpha)(\exp(\phi T)-1)}$$

$$\omega_4 = \frac{4\psi\exp(\psi T)B(T)(V-\alpha r)}{\beta(\beta-\alpha)(\exp(\psi T)-1)}$$

$$\zeta = \kappa(s-T) + 2\gamma\ln A(s-T) + 2\eta\ln B(s-T) - \ln X$$

The s-maturity and T-maturity discount bond prices, $P(t,s)$ and $P(t,T)$, are given by the Longstaff and Schwartz discount bond pricing formula[15]. Estimation of the parameters required to derive the term structure of bond prices are discussed in Clewlow and Strickland [1994]. The cumulative distribution function, $\Psi(\theta_1,\theta_2;4\gamma,4\eta,\omega_1,\omega_2)$, is the bivariate noncentral chi-squared. The Longstaff-Schwartz solution involves a double integral across the product of two univariate non-central χ^2 distribution functions.

15 The functions $A(\tau)$ and $B(\tau)$ are defined in the Longstaff and Schwartz paper. See also Strickland [1993a].

$$\int_{0}^{\theta_1}\int_{0}^{\theta_2-\theta_2 u/\theta_1}\chi^2(u;4\gamma,\omega_1)\chi^2(v;4\eta,\omega_2)\,dv\,du \tag{5.5}$$

As with most multi-factor models, solving for all but a few types of derivative security prices is computational expensive. Longstaff-Schwartz note that the computation of the cumulative distribution function can be cumbersome enough to warrant numerically solving the partial differential equation for the discount bond option price, or integrating the option payoff across the resulting joint density. A number of the papers referenced earlier for the multi-variate class of square root models propose ways of reducing this computational burden. Chen and Scott [1993] for example show that the multi-variate integrals of the bond option pricing formula can be reduced to univariate numerical integrations, reducing substantially the computation time required by the model. Duffie and Kan [1993] present a practical finite difference algorithm for the same purpose.

In our related paper, Strickland [1993a], we discussed a number of other two-factor models of the term structure (including Cox-Ingersoll-Ross [1985], Brennan and Schwartz [1979], Schaefer and Schwartz [1987] and Fong and Vasicek [1991]). The Longstaff-Schwartz paper that we have just discussed is the only paper amongst those listed that deals explicitly with the problem of pricing options on discount bonds. In order to obtain European option prices for models which assume different stochastic processes from the one discussed here we can make use of Monte-Carlo simulation in conjunction with the results from these papers concerning the pricing of bonds. We will use as an illustration the two-factor stochastic volatility model of Fong and Vasicek [1991, 1992a, 1992b].

Fong and Vasicek [1991]

Fong-Vasicek derive the prices of pure discount bonds under the standard no arbitrage condition, starting from the direct characterisation of two stochastic factors; the short term interest rate and the volatility of this rate:

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1$$
$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2 \tag{5.4}$$

$\bar{r}$ and $\bar{v}$ are the risk-adjusted long-term means of the short rate and the short rate volatility respectively. The two processes have an instantaneous correlation coefficient of ρ.

In order to price, say, an option with maturity date T on a pure discount bond maturing at time s, we simulate, a number of times, the joint process of r and v from their initial values, (r_0, v_0), until the maturity of the option, leading to a vector of paired values $\{(r_0, v_0), (r_1, v_1), \ldots\ldots, (r_m, v_m)\}$ for an m time-step simulation. The analytic solution for the price of a discount bond with time to maturity s-T is then calculated via the Fong and Vasicek formula using the level of the state variables obtained at the end of the simulation, and the payoff to the option is discounted using the path of the short term interest rate realised during the simulation[16].

Although the use of Monte-carlo simulation for models with a single diffusion process is a (relatively) time consuming procedure it becomes more attractive when we consider more than one state variable. Analytic solutions, such as Longstaff-Schwartz, require a computationally expensive procedure for the cumulative distribution function, whilst numerically solving the partial differential equation is also expensive in a two state world.

Two factor stochastic volatility models have the advantage that they can usually be expressed so as to nest GARCH-like variation in interest rates. A major problem with the approach discussed in this section, however, as far as practitioners are concerned is that, in common with the single factor models discussed in the previous section, the models are used to price derivatives on discount bonds with assumptions about the interest rate

16 See Strickland [1993c] for an analysis of the Fong-Vasicek pure discount bond pricing formula.

process that lead to incorrectly priced bonds when compared with market prices. Although we have discussed elsewhere the Longstaff-Schwartz model, parameter estimation and computation time remains a problem[17] for models which allow for more than one source of uncertainty.

6/ 'Term Structure Consistent' Models

As we have mentioned, the approach outlined in the previous two sections has the severe disadvantage that the underlying term structures that the models produce, provide a limited family which are not consistent with the term structure of interest rates available in the market. By valuing interest rate derivatives with reference to a theoretical yield curve rather than the actually observed curve, these models produce contingent claims prices that disregard key market information affecting the valuation of any interest rate derivative security. The most obvious market data is that of the term structure of (spot or forward) interest rates and the term structure of rate volatilities. In this section we discuss models which belong to what we term the 'term structure consistent' approach; models of interest rate derivatives that are designed to be exactly consistent with the observed term structure.

Ho and Lee [1986]

Ho and Lee [1986] were the first authors to build a model that set out to model the dynamics of the entire term structure given the prices of all zero coupon bonds on the valuation date. The risk-neutral continuous time limit can be characterized by the short rate process:

$$dr = \theta(t)dt + \sigma dz \tag{6.1}$$

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17 See Clewlow and Strickland [1993].

where $\theta(t)$, the time-dependent drift term, reflects the slope of the initial forward rate curve, $f(0,t)$, and the volatility parameter of the short rate process:

$$\theta(t) = f_t(0,t) + \sigma^2 t \tag{6.2}$$

It can be shown that, in this framework, analytical solutions for the prices of pure discount bonds and options on discount bonds are obtainable (see Hull and White [1991]) as a function of the short rate and the initial yield curve. The price at time t, of European call and put options maturing at time T, on a pure discount bond with maturity s, is given by the modified Black-Scholes formulas:

$$C(t,T,s) = P(t,s)\, N(d_1) - XP(t,T)\, N(d_2) \tag{6.3}$$

$$p(t,T,s) = XP(t,T)\, N(-d_2) - P(t,s)\, N(-d_1) \tag{6.4}$$

where

$$d_1 = \frac{\log\left(P(t,s) \big/ XP(t,T)\right)}{\sigma_P} + \frac{\sigma_P}{2}$$
$$d_2 = d_1 - \sigma_P$$
$$\sigma_P = \sigma(s-T)\sqrt{T-t}$$

The parameter $\sigma(s - T)$ is equivalent to the variance of the instantaneous return on the forward price (at time T) of a s-maturity bond. An advantage of the Ho-Lee set up is that option valuations are preference independent as in the Black-Scholes analysis. In order to price options in this framework, therefore, we need only to be able to observe pure discount bond prices and σ, the volatility of the short rate process. These option formulas are similar to the ones developed by Merton [1973] and which we analysed in section 4, the difference being in the values of the bonds maturing at times s and T; Merton's bond prices are theoretically derived under assumptions about the risk-neutral process for the short rate, Ho and Lee prices are observed market prices.

The problems associated with the Ho-Lee model are analysed in Strickland [1993b] but can be summarised as follows; all possible yield curves at a future time are parallel to each other, there is a positive probability that future interest rates will become negative, and the volatility structure, determined mathematically within the model, implies that spot rates and forward rates are all equally variable.

Heath, Jarrow and Morton [1990a], [1990b], and [1992] seek to extend the earlier work of Ho-Lee by constructing a family of continuous time stochastic processes for the term structure, consistent with the observed initial term structure data incorporated into the Ho-Lee model, but also with term structure data describing interest rate volatility. In order to model the dynamics of the term structure one can chose between equivalent formulations in terms of bond prices, short term interest rates or forward rates. Heath-Jarrow-Morton chose to model forward instantaneous interest rates, due to volatility considerations concerned with the maturity of zero coupon bonds, and so take as given the initial forward rate curve $\{f(0,s): s \geq 0\}$.

The forward rate curve's dynamics are exogenously given by the equation:

$$f(t,s) = f(0,s) + \int_0^t \alpha(v,s)dv + \sum_{i=1}^{n} \int_0^t \sigma_i(v,t)dZ_i(v) \tag{6.5}$$

where $\alpha(v,T)$ is the instantaneous forward rate's drift, σ_i are the volatilities of the forward rates, and Z_i denotes the i'th Wiener process (for $i = 1, ..., n$).

Equation (6.5) is expressed in its most general form, with n independent Brownian motions determining the stochastic fluctuation of the forward rate curve. Heath-Jarrow-

Morton [1992] show that prices of pure discount bonds satisfy a stochastic differential equation that states that the instantaneous return on the *T-th* maturity bond has a drift rate equal to the spot rate plus a term premium which is a function of the forward rate's drift and volatility, and volatilities which are also a function of the forward rate volatilities. It can be shown that the Heath-Jarrow-Morton model is consistent with bond prices converging to par at maturity, and the behaviour of the instantaneous rate. By applying the insights of Harrison and Kreps [1979] the process is shown to be arbitrage free, and contingent claim values are obtained via an application of Harrison and Pliska [1981].

The class of interest rate models given by Heath-Jarrow-Morton is very general and includes a very wide range of term structure consistent models. As with the Black-Scholes analysis the drift term serves only a technical role in the pricing of options. The volatility function(s) are chosen at the discretion of the user. Choosing the function to be constant, or as a function of time and the forward rate, sometimes leads to path-independent models with analytical solutions such as the continuous time Ho-Lee, Vasicek or Cox-Ingersoll-Ross models. Choosing the volatility functions to best fit historical term structure movements or market options data, generally leads to the evolution of the term structure being path-dependent, with a considerable increase in computation times due to the exponential growth in the tree of the discrete time approximation[18].

Heath-Jarrow-Morton present two examples to illustrate their principles of contingent claim valuation. Both turn out to be computationally simple. The first involves a single source of uncertainty and is equivalent to the continuous-time Ho-Lee model we have just seen. The second example involves two sources of uncertainty and can be represented by a stochastic differential equation (equivalent to equation (6.5)) of the form :

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18 The issues of which volatility functions lead to analytical solutions and how to choose the volatility functions are discussed in Strickland [1993d].

$$df(t,s) = \alpha(t,s)dt + \sigma_1 dZ_1(t) + \sigma_2 e^{-(\lambda/2)(s-t)} dZ_2(t) \tag{6.6}$$

Here σ_1, σ_2, and λ are positive constants which allow the volatility functions to impact on the forward rate curve in different ways, allowing for 'tilts' in the yield curve as well as parallel shifts. The value of European call and put options, with maturity T, are given by equations (6.3) and (6.4) with:

$$\sigma_P^2 = \sigma_1^2 (s-T)^2 (T-t) + \left(\frac{4\sigma_2^2}{\lambda^3}\right)\left(e^{-(\lambda/2)s} - e^{-(\lambda/2)T}\right)^2 \left(e^{\lambda T} - e^{\lambda t}\right) \tag{6.7}$$

In general, depending on the choice of the volatility function the model of Heath-Jarrow-Morton will be non-Markov, requiring the binomial tree representing the path of the short rate to be non-recombining. The choice and specification of the volatility functions will depend on the type and derivative being priced; an increase in the number of factors can lead to monte-carlo simulation being feasible[19].

There are two main approaches to achieving the 'fit' between interest rate contingent claim pricing models and the observed term structure data. The work of Heath, Jarrow, and Morton just described develops an arbitrage pricing model for valuing interest rate contingent claims where the stochastic structure is imposed directly on the term structure of (forward) interest rates. Alternatively, the approach of Hull and White [1990], Jamshidian [1991], Dybvig [1989], and others, involves writing down a general process for the short rate as in the traditional approach described in section 4 and then effectively expanding the parameterisation set by replacing the constant parameters of the approach by deterministic functions of time. The time dependent functions are then chosen to match the model solution of the term structure to that given by market data[20].

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19 See Strickland [1993d]

20 The difference between the two approaches is really only technical. However, they do lead to rather different forms.

Hull and White [1990]

Motivated by practical considerations Hull and White derive two one-state variable models for the short rate. They seek to reconcile the tractability of the Vasicek and Cox-Ingersoll-Ross models with the consistency of a model that fits the observed term structure data:

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)dz \tag{6.8}$$

$$dr = [\theta(t) - \phi(t)r]dt + \sigma(t)\sqrt{r}\, dz \tag{6.9}$$

Hull-White propose that the three functions of time $\theta(t)$, $\phi(t)$ and $\sigma(t)$ are chosen so that the models, determined by equations (6.8) and (6.9), fit the initial term structure of interest rates, the term structure of spot rate volatilities, and, to overcome one of the problems with the Black-Derman-Toy model, the anticipated variability across time of the instantaneous spot rate.

For the 'extended Vasicek' model represented by equation (6.8), the prices of European call and put options with maturity T, on a s-maturity pure discount bond are given by the modified Black-Scholes formulas in equations (6.3) and (6.4) with:

$$\sigma_P^2 = [B(0,s) - B(0,T)]^2 \int_0^t \left[\frac{\sigma(\tau)}{\partial B(0,\tau)/\partial \tau} \right]^2 d\tau \tag{6.10}$$

The characterisation of this model leads to normally distributed interest rates and lognormally distributed bond prices with the resulting disadvantage that interest rates can become negative with positive probability. The advantage of the model is the analytical tractability of prices for European options. To solve for option prices the user fits the function $\{B(0,s); s \geq 0\}$ to empirical data, a relatively simple procedure once the term structures of spot rates and spot rate volatilities are established, and performs a numerical differentiation followed by a numerical integration[21].

The second model, represented by equation (6.9) - the 'extended Cox-Ingersoll-Ross' model, allows the variability of the short rate to be related to the its level. This has the effect of eliminating the possibility of negative interest rates but is not as analytically tractable as the extension to Vasicek. The solution to the option pricing problem involves using numerical procedures to solve the partial differential equation governing the instrument's evolution. For a special case of (6.9) which Jamshidian [1993] refers to as the simple class of square root models[22] he is able to obtain closed form solutions for the prices of options on discount bonds.

Both of the 'extended' models discussed here accept the observed term structure of (spot) rate volatility and future expectations for the short rate volatility - although the latter is often assumed to be constant and equal to it's observed level. It can be shown that the observed generalised duration measure $\{B(0,s); s \geq 0\}$ is the solution to a partial differential equation, and as such is constrained to evolve through time in a deterministic way. This implies that once the initial spot rate volatility function has been specified its subsequent evolution is deterministic, and may evolve in a way that was not originally intended by the user (see Strickland [1993c]).

The special case of equation (6.8) discussed by Strickland [1993a] in the context of term structure modeling, allows us to price options on discount bonds without the need for the numerical procedures just discussed. This model, known as the 'Hull-White' model is the

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21 See Strickland [1993e] for an analysis of the Hull and White approach.

22 The case where $\theta(t)/\sigma^2(t)$ is a constant.

extended Vasicek model with both the revision rate and the instantaneous standard deviation constant. The short rate process is given by the diffusion:

$$dr = [\theta(t) - ar]dt + \sigma dz \tag{6.11}$$

This model can be thought of as the Vasicek model with time dependent drift or the Ho-Lee model with mean-reversion, retaining the property of consistency with the term structure, whilst allowing for mean reversion and being analytically tractable. The prices of European call and put options on a pure discount bond are again given by equations (6.3) and (6.4), with:

$$\sigma_P^2 = \frac{\sigma^2}{4a^3}\left(1 - e^{-a(s-T)}\right)^2\left(1 - e^{-2a(T-t)}\right) \tag{6.12}$$

i.e. the Jamshidian model discussed in section 4, with different (i.e. market) bond prices.

Black, Derman and Toy [1990]

Black, Derman and Toy [1990] develop a Markov model to match the observed term structure of spot interest rate volatilities as well as the term structure of interest rates, and which conversations with practitioners suggest is currently popular[23]. As with the original Ho-Lee, the model is developed algorithmically describing the evolution of the entire term structure in a discrete-time binomial lattice framework. Although this means that the model is rather opaque with regard to its assumptions about the evolution of the short rate, it has been shown that the continuous time limit of the model is given by the stochastic differential equation:

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23 A similar model has been proposed by Black and Karasinski [1991]

$$d \log r(t) = \left[\theta(t) - \frac{\sigma'(t)}{\sigma(t)} \log r(t) \right] dt + \sigma(t) dz \tag{6.13}$$

This representation of the model allows us to better understand the assumptions implicit in the model. The Black-Derman-Toy model incorporates two functions of time; $\theta(t)$ chosen so that the model fits the term structure of spot interest rates, and $\sigma(t)$ so that it fits the term structure of spot rate volatilities. In this model changes in the short rate are lognormally distributed, with the resulting advantage that interest rates cannot become negative. Once $\theta(t)$ and $\sigma(t)$ are chosen, the future short rate volatility is entirely determined, and an unfortunate consequence of the model is that for certain specifications of the volatility function $\sigma(t)$, i.e. if the future short rate volatility declines over time, the short rate can be mean-fleeing rather than mean-reverting. The model also has the advantage that the volatility unit is a percentage, conforming with the market convention. Unfortunately due to its lognormality, neither analytic solutions for the prices of bonds or the prices of bond options are available, and a 'trial and error ' procedure is required to derive the short rate tree that correctly yields the market term structures. However, Jamshidian [1991] provides an elegant and computationally fast procedure to calculate the short rate tree making its real time implementation possible.

Attractive though the class of models just described may be, they are not without their dangers. Although there exists a large amount of freedom created by the time dependency of coefficients, this can lead to estimation problems; at any date we must estimate functions for the term structure of interest rates, the term structure of volatility and the time path of volatility. However, just as with implied volatility in a conventional options model, when we look at market prices at a later date, there is no guarantee that they will be consistent with the previously estimated functions. In particular, choosing volatility functions to some extent independently of fitting to the term structure of interest rates poses the danger that we may ignore the information that the shape of the term structure contains about the anticipated volatility of interest rates. Convexity considerations or other more formal analysis lead very easily to relationships between the concavity of spot rates with respect to maturity and the prospective volatility of interest rates[24]. Finally, the extension from models where both the term structures of interest rates and interest rate volatilities are implied by the parameters of the model, to models where both term structures are implied from market data, does not mean that the latter models describe the actual stochastic evolution of interest rates in an improved way. This remains an open empirical question.

7/ Summary and Conclusions

This paper has focused on analysing a number of different approaches to pricing derivatives on pure discount bonds with the justification that many more complicated interest rate derivatives can be considered as portfolios of these instruments. Single factor models that we study can be split into two groups; those that explicitly model the discount bond price as the underlying variable, and those that consider that it is the process followed by interest rates that is important. This class of models have the advantage of being relatively easy to compute but with the disadvantage that unrealistic assumptions are made about the underlying processes in order to achieve this.

Equilibrium models that appeal to the empirical evidence that more that one factor of uncertainty is driving the term structure suffer from the fact that, although a richer variety of term structures are possible, the models value discount bond derivatives with reference to a theoretical yield curve rather than the actually observed curve. Once the parameters of the stochastic processes are determined the yield curve of spot rate volatilities are also determined within the model.

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24 See Strickland, Carverhill and Hodges [1993] for an analysis of the shape of the term structure and interest rate volatility.

Recent research has concentrated on building models that are designed to be exactly consistent with the observed term structure as given by the prices of traded instruments. Two main approaches are currently popular to achieve this. The first involves imposing the stochastic structure of term structure evolution directly on the interest rate curve, and is the approach favoured by Heath-Jarrow-Morton. The second approach involves allowing the constant parameters of the traditional models to be time dependent, effectively increasing the parameterisation of the models. Models of this kind can be made to fit the term structure of interest rates by allowing the drift term of the stochastic process to be time dependent. If the short rate volatility is also made time dependent, the term structure of interest rate volatilities can also be made to be a fitted function.

A quantitative comparison of each of the above described approaches is left to a further paper.

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