Equilibrium and the Role of Options in an Economy with Stochastic Volatility

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Abstract

The paper develops a model of market equilibrium in an economy where a single risky asset evolves with stochastic volatility. Our work is in the spirit of Bick (1987). In our economy, unlike Bick's which is complete and supports Black-Scholes option values, the market is incomplete unless some kind of volatility sensitive (ie. option-like) asset is traded. The model we present enables us to examine the role options play in enabling investors (other than the representative investor) to optimize their investments. In steady state equilibria involving investors with the same horizons but different degrees of risk aversion the gains from options appear to be relatively minor.

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Equilibrium and the Role of Options in an Economy with Stochastic Volatility

1. Introduction

The purpose of our paper is to explore the role which options play in completing markets, and enabling investors to hedge changes in volatility. To do this we develop a model of market equilibrium in an economy where a single risky asset evolves with stochastic volatility. Our work is therefore related to and builds on that of Bick (1987). In our economy, (unlike Bick's which is complete and supports Black-Scholes option values), the market is incomplete unless some kind of volatility sensitive (ie. option like) asset is traded. We introduce an equilibrium model of stochastic volatility which is very closely related to the stochastic volatility models of Hull and White (1988) and Scott (1987) which use a Cox, Ingersoll and Ross (1985) square root process for the asset's variance. Other related papers by He (1993) and by Pham and Touzi (1994) have characterized conditions for a stochastic volatility model to be consistent with this form of equilibrium.

Instead of defining the asset process entirely exogenously, we follow Bick and choose as exogenous the process (but now with stochastic volatility) for the amount of a consumption good available to be consumed at a single future date. Prices are derived endogenously under the assumption of a representative investor equilibrium characterized by a constant proportional risk averse (CPRA) investor. Our model enables us to examine the role options play in enabling investors (other than the representative investor) to optimize their investments. The representative investor is, of course, happy to retain the market portfolio throughout and has a zero demand for options. Other investors with different levels of risk aversion require non-linear payoffs, and will benefit (to extents that we can quantify) if the market is made complete. The role of options in completing markets was first analysed by Ross (1976).

The paper has the following structure. We first describe the assumptions that we will make and introduce some of our notation. Next we characterize how securities are priced in this economy. We then consider the difference in expected utility that non-representative investors can obtain depending on whether or not the market is complete. We quantify these differences for the special case where all investors have very long horizons.

2. The Model

In this section we will describe the assumptions we make in our market equilibrium model, and discuss their relationship to those made in other related papers.

Assumptions

We make the following assumptions:

1. Consumption Good Process.

All consumption takes place at time T. A random amount D_T of the consumption good is available at time T, and at any earlier time t, we use D_t to denote the expected value of D_T.
D_t evolves under the diffusion process

$$\frac{dD}{D} = \sqrt{v}\, dz_1$$
$$dv = \kappa(\theta - v)\, dt + \sigma\sqrt{v}\, dz_2, \tag{1}$$

where dz_1 and dz_2 are independent standard Brownian processes.

2. Representative Investor

Prices are set in an economy characterized by a representative investor who maximizes

$$E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \text{ of terminal consumption } W_T. \tag{2}$$

Thus, this agent only consumes at date T, and maximizes expected utility of a constant proportional risk aversion (CPRA) utility function with relative risk aversion of $\gamma > 0$.

3. Financial Market

There is a frictionless forward market in which in which fractions of the total market supply of D_T can be traded for fixed units of the consumption good at date T.

Discussion

The process we have assumed for D_t has two features worth commenting on. Since no consumption takes place until we reach time T, we do not need to specify a process for the amount of the consumption good through time, only for the amount expected at date T. Indeed, in some sense it may not even exist until time T, eg. a physical crop which is unripe. The process for such an expectation is necessarily a Martingale (ie. its drift is zero). This contrasts slightly with Bick's assumption of a particular growth rate for the physical commodity. We would need such an assumption to consider intermediate or continuous consumption, but for now our expectations process is consistent with quite general kinds of time dependent drift in the quantity of the consumption good. The other feature is the incorporation of stochastic volatility. Our stochastic volatility model is the same mean reverting square root model which Cox Ingersoll and Ross (1985) first introduced to describe the term structure of interest rates, and which has been used by Hull and White (1988) and by Scott (1987) as a stochastic volatility model. However these papers took the whole of the price process rather than just its horizon value as exogenous.

The adoption of a representative equilibrium framework and the choice of a CPRA investor is fairly conventional and represents a natural choice. However, there is one slight problem with this formulation. We want to make comparisons between a market structure which is incomplete and one which is complete. In general, if we start from a collection of heterogeneous agents with initial endowments, and characterize the resulting equilibrium in terms of a representative investor, we shall find that the nature of the equilibrium and of the representative agent depend on whether or not the market was complete. Thus, in terms of making such comparisons, we need to make the stronger assumption that almost all agents share the preferences of the representative investor.

Finally, it is worth noting that by only trading forward contracts we sidestep all issues to do with the behaviour of interest rates and discount bond prices. Bick also finesses this in a way which is closely related. He takes the price of a bond which pays one unit of the consumption good at date T and which is in zero net supply, as numeraire for his economy.

3. The Pricing of Contingent Claims

We now consider the mathematics of pricing forward contracts which pay some function $C_T(D_T)$ of the value of D_T at date T. The marginal utility of the representative investor is $D_T^{-\gamma}$ (where $\gamma > 0$ is the investor's relative risk aversion). Hence the date t forward price of the claim is given as

$$C_t = \frac{E\left[C_T(D_T)\,D_T^{-\gamma}\right]}{E\left[D_T^{-\gamma}\right]}. \tag{3}$$

We shall now see what this implies for pricing the market asset and some other simple claims.

Expected Values of Powers of D_T

We begin by deriving formulae for the expectation of the powers of D_T which we require for pricing. Since our process for the proportional changes in the consumption good only depends on v and time, it is clear that

$$E_t\left[D_T^b\right] = D_t^b\,\phi(b;v,t) \tag{4}$$

for some function $\phi(b\,;v,t)$. We shall make considerable use of this homogeneity property with respect to the expected level of the consumption good throughout the paper. We will now characterize the behaviour of the ϕ function and obtain its solution. Applying Ito's Lemma to the function $D^b\,\phi$ (for any fixed power b), we find that the expression for its rate of drift is

$$D^b\left(\phi_t + \kappa(\theta - v)\,\phi_v + \tfrac{1}{2}\sigma^2 v\,\phi_{vv} + \tfrac{1}{2}b(b-1)v\,\phi\right) \tag{5}$$

(where the subscripts now denote partial derivatives rather than a particular function). This drift must be zero, since the expectation is a Martingale. Thus the expression in brackets provides a partial differential equation for $\phi(b\,;v,t)$, subject to the boundary condition that $\phi(b\,;v,T) = 1$.

The solution to this partial differential equation must take the form

$$\begin{aligned}&\phi(b;v,t) = \exp\{A(b;t)v + B(b;t)\}\\ &\text{with } A(b;T)=0,\ B(b;T)=0, \text{to satisfy the boundary condition.}\end{aligned} \tag{6}$$

We obtain the following ordinary differential equations for the functions A and B

$$\begin{aligned}&A' - \kappa A + \tfrac{1}{2}\sigma^2 A^2 = \tfrac{1}{2}b(1-b),\\ &B' + \kappa\theta A = 0.\end{aligned} \tag{7}$$

Two cases arise, depending on the sign of

$$\kappa^2 + \sigma^2 b(1-b).$$

In the case where this is negative, the form of the A function is essentially a trigonometric tangent function of time, and it becomes unbounded within a finite time span. As we shall see, this form can arise when we come to value the risky asset in situations with low mean reversion, and high risk aversion (since the second term is always negative and its magnitude increases with the risk aversion γ). We shall instead focus on the positive case. Here the function is essentially a hyperbolic tangent and we obtain solutions for all t and T, including an asymptotic steady state solution.

The solution functions are as follows:

$$A(b;t) = -\frac{b(1-b)(e^{\rho(T-t)}-1)}{(\kappa+\rho)(e^{\rho(T-t)}-1)+2\rho}$$

$$B(b;t) = \frac{2\kappa\theta}{\sigma^2}\left[\frac{(\kappa+\rho)(T-t)}{2} - \log\left\{\frac{\kappa+\rho}{2\rho}(e^{\rho(T-t)}-1)+1\right\}\right] \tag{8}$$

where $\rho = \rho(b) = \sqrt{\kappa^2 + b(1-b)\sigma^2}$.

In the case where T is large, the solution to the partial differential equation approaches asymptotically to:

$$\phi(v,t) \to \left(\frac{2\rho}{\kappa+\rho}\right)^{2\kappa\theta/\sigma^2} e^{\frac{\kappa-\rho}{\sigma^2}(v+\kappa\theta(T-t))}. \tag{9}$$

Forward Contracts

We will next consider the pricing of a forward contract which pays $D_T^{\,c}$ at future date T, for some power c. In some cases we may need to restrict what powers we consider in order to obtain bounded values. We will consider first the value of such a forward contract in both the finite and infinite horizon cases. We will look at their price processes and derive the risk neutral processes.

From our earlier exposition, it is clear that a claim which pays $D_T^{\,c}$ at future date T has forward price $P_T^{(c)}$ given by

$$P_t^{(c)} = D_t^{\,c} \exp\{v\,\alpha(c,\gamma;t) + \beta(c,\gamma;t)\} \tag{10}$$

where $\alpha(c,\gamma;t) = A(-\gamma;t) - A(c-\gamma;t)$, and

$$\beta(c,\gamma;t) = B(-\gamma;t) - B(c-\gamma;t).$$

From Ito's Lemma we find that the price process for $P_t^{(c)}$ is:

$$\frac{dP^{(c)}}{P^{(c)}} = [-\alpha' v - \beta' - \alpha\kappa(\theta - v) + \tfrac{1}{2}\sigma^2\alpha^2 v + \tfrac{1}{2}c(c-1)v]\,dt + c\sqrt{v}dz_1 - \alpha\sigma\sqrt{v}dz_2. \tag{11}$$

Note that the alpha prime and beta prime terms denote derivatives with respect to time. In general the expressions in this equation are time dependent and fairly complicated. However, we can simplify the last equation in order to show how the drift is attributable to each of the two shocks without too much difficulty. First, we note that the drift is directly proportional to v, as the non v terms disappear since they are related by the differential equation for the B's. Second, we expand the α terms, and make use of the differential equation for the A's. We obtain:

$$\frac{dP^{(c)}}{P^{(c)}}=c\sqrt{v}\left[\gamma\sqrt{v}\,dt+dz_1\right]+\alpha(c,\gamma;t)\;\sigma\sqrt{v}\left[\sigma A(-\gamma;t)\sqrt{v}\,dt-dz_2\right], \qquad (12)$$

and it is clear how the risk premium is related to each of the two shocks through their magnitude and the risk aversion of the representative investor. It is worth noting that even though the increments in v and D are independent, both of these shocks come into the process for $P_t^{(c)}$, and both command a risk premium. In a complete market, it is now obvious how the unique martingale measure would be constructed by transforming so that each of these two processes is a martingale. Contingent claims could then be valued simply as their expectations under these new measures. In an incomplete market, such as we are about to explore, the martingale measure is not unique and there will in general be ambiguity about the prices of some claims. Unsurprisingly, the risk adjustment on the evolution equation for D_t is simply the relative risk aversion of the representative investor times the instantaneous variance.

In the special case of large T for a long-lived representative investor the equation becomes:

$$\frac{dP^{(c)}}{P^{(c)}}=c\sqrt{v}\left[\gamma\sqrt{v}\,dt+dz_1\right]+\overline{\alpha}(.)\sigma\sqrt{v}\left[\frac{\kappa-\rho(-\gamma)}{\sigma}\sqrt{v}\,dt-dz_2\right],$$

$$\text{where } \overline{\alpha}(.)=\overline{\alpha}(c-\gamma,-\gamma)=\frac{\rho(c-\gamma)-\rho(-\gamma)}{\sigma^2}, \qquad (13)$$

and $\rho(.)$ is a defined earlier in equation (8).

4. Behaviour of Non-Representative Investors

In modelling the investment decisions of investors with different risk aversion to the representative investor, but still with CPRA utility, and with the same horizon, the homogeneity properties are still very helpful to us. We will consider first the case of a non-representative investor in a complete market, and then the case where the only traded contract is a forward contract on the market asset.

We introduce the following new notation. Our non representative investor maximizes the expectation of $W_T^{1-g}/(1-g)$ (ie. the non-representative investor has relative risk aversion of g, instead of γ for the representative investor).

The homogeneity property implies that the indirect utility function J of our investor whose endowment at time t has forward value F_t, is given by:

$$J(F_t,v,t)=\frac{F_t^{1-g}}{1-g}\,\psi(g;v,t). \tag{14}$$

We shall now apply this to derive solutions.

The Complete Market Case

Our market will be complete provided that there exists a single European option contract (or a power contract with $c \neq 1$) in addition to the forward contract already mentioned. (See, for instance, Barles, Romano and Touzi (1993)). If the market is complete and our investor has the same horizon as the representative investor, then it is easy to see that the optimal choice for terminal wealth is simply

$$W_T^* = k\,D_T^{\gamma/g} = k\,D_T^{\eta}, \text{for some constant } k,$$

and where $\eta=\gamma/g$ measures the ratio of the relative risk aversions of the two investors.

The investor gets the above payoffs subject to the constraint:

$$k\,P_t^{(\eta)} = k\,D_t^{\eta}\frac{\phi(\eta-\gamma;v,t)}{\phi(-\gamma;v,t)} = F,$$

where F is the forward value of the endowment.

We therefore obtain the following closed expressions for the implied utility function:

$$\begin{aligned} J(F_t,v,t) &= \frac{E[(k\,D_T^{\eta})^{1-g}]}{1-g} \\ &= \frac{k^{1-g}}{1-g}D_t^{\eta(1-g)}\,\phi(\eta(1-g);v,t) \\ &= \frac{F^{1-g}}{1-g}\,\phi(\eta(1-g);v,t)\left(\frac{\phi(-\gamma;v,t)}{\phi(\eta-\gamma;v,t)}\right)^{1-g}. \end{aligned} \tag{15}$$

In the case of long-lived investors, the rate of time appreciation in this function is

$$\left[g\{\rho(\eta-\gamma)-\rho(-\gamma)\}-\rho(\eta(1-g))+\kappa\right]\frac{\kappa\theta}{\sigma^2} \tag{16}$$

We shall use this equation later in our numerical comparisons.

The Incomplete Market Case

We consider the situation in which the only traded asset is the forward contract on the market itself, P. The market is incomplete and the investor has to solve an optimal control problem. The investor must choose the fraction x of the forward value of his wealth to invest in the market at any time in order to maximise expected utility. The investor will choose x to maximise the expected value of dJ, and with that choice, J is a martingale. The optimal strategy is obtained from solving the following:

$$\underset{x}{Max}\ E\left[\left(1+x\frac{dP}{P}\right)^{1-g}\left(1+\frac{d\psi}{\psi}\right)-1\right]\Big/(1-g)=0. \tag{17}$$

The maximand can be expanded as a quadratic in x:

$$\frac{1}{1-g}E\left[\frac{d\psi}{\psi}\right]+E\left[\frac{dP}{P}+\frac{dP}{P}\frac{d\psi}{\psi}\right]x-\frac{g}{2}E\left[\left(\frac{dP}{P}\right)^2\right]x^2 \tag{18}$$

The second order conditions are clearly satisfied since $g > 0$ (the investor is risk averse). Setting the maximum equal to zero then gives:

$$\left\{E\left[\frac{dP}{P}+\frac{dP}{P}\frac{d\psi}{\psi}\right]\right\}^2+\frac{2g}{1-g}E\left[\frac{d\psi}{\psi}\right]E\left[\left(\frac{dP}{P}\right)^2\right]=0. \tag{19}$$

This is a partial differential equation for ψ:

$$\left[\mu-\alpha\sigma^2\frac{\psi_v}{\psi}\right]^2+\frac{2g}{1-g}\left[\frac{\psi_t}{\psi}+\kappa(\theta-v)\frac{\psi_v}{\psi}+\tfrac{1}{2}\sigma^2 v\frac{\psi_{vv}}{\psi}\right]\left[1+\alpha^2\sigma^2\right]=0 \tag{20}$$

where $\mu=\gamma+\alpha(\,;t)\,\sigma^2 A(-\gamma\,;t)$, and
subject to the boundary condition: $\psi(v,T)=1$.

For the special case where $g=\gamma$ we know from the economics that the solution to this optimal control problem is identical to the complete market case of the previous section. We can prove this algebraically for some cases, and we have also confirmed numerically that it is correct.

The solution can be written in the form

$$\psi(v,t) = \exp\{G(t)v + H(t)\} \text{ with } H' + \kappa\theta G = 0.$$

$G(t)$ satisfies the ordinary differential equation:

$$\left[\mu - \alpha\sigma^2 G\right]^2 + m\left[G' - \kappa G + \tfrac{1}{2}\sigma^2 G^2\right] = 0,$$
$$\text{where } m = \frac{2g}{1-g}\left(1+\alpha^2\sigma^2\right), \tag{21}$$

and subject to the boundary condition $G(T) = 0$.

This rearranges to:

$$mG' + \tfrac{1}{2}\sigma^2\left[m + 2\alpha^2\sigma^2\right]G^2 - \left[2\mu\alpha\sigma^2 + \kappa m\right]G + \mu^2 = 0. \tag{22}$$

The equation for $G(t)$ is not very tractable analytically; the coefficients of G and its derivatives are themselves time dependent. However, our principal interest is in the investor's utility function a long way from T. The dynamics of the market price process become time independent, and so too do the coefficients of the differential equation. The rate of change in G becomes a quadratic fuinction of the level of G. If the quadratic has no real roots, G becomes unbounded. Otherwise there is a solution where G tends asymptotically to a constant as t goes to minus infinity. This means that H is linear in time and the investor's implied utility is exponentially increasing in time to maturity.

This asymptotic growth rate is given by:

$$\left[\kappa m + 2\bar{\mu}\bar{\alpha}\sigma^2 - \sqrt{m^2\kappa^2 + 4m\kappa\bar{\mu}\bar{\alpha}\sigma^2 - 2m\bar{\mu}^2\sigma^2}\right]\frac{\kappa\theta}{\sigma^2\left(m + 2\bar{\alpha}^2\sigma^2\right)}, \tag{23}$$

where $\bar{\mu} = \gamma + \bar{\alpha}\left(\kappa - \rho(-\gamma)\right)$.

Note that this corresponds to solving the quadratic for G in equation (22), after setting $G' = 0$, and replacing α by $\bar{\alpha}$.

The choice of the quadratic root is determined by the fact that if the quadratic has a maximum (ie. the quadratic term is negative) the lower root is stable and the higher root is unstable. For more general t, the equations could be solved numerically without any real difficulty.

In the next section we will investigate numerically the asymptotic benefits of a complete market to a non-representative investor whose risk aversion differs from that of the representative investor.

5. Numerical Results

We have attempted to choose parameters for our model to be consistent with the observable behaviour of stock prices.

Our model involves four parameters: we have three variables for volatility (sigma, the vol of vol; theta, the mean vol; and kappa, the speed of mean reversion), and we have one preference parameter, gamma.

We decided to fit the market risk premium (6%) and the volatility of the market (15%). We also want to fit the volatility of volatility in some suitable way - there is not much point in creating a stochastic volatility model where uncertainty about volatility is small. If the long horizon distribution of variance of returns has mean θ and standard deviation S, then the time integral of variance has mean θT and standard deviation $S\sqrt{T}$. Even in our stochastic volatility model we can predict out-turn annualised volatility for long dated options accurately - though volatility times sqrt(time) is subject to error. To avoid all stochasticity vanishing we have have rather arbitrarily decided to set the unconditional standard deviation of volatility equal to the mean (theta). That gives three conditions, and one dimension of freedom.

We found sets of σ, κ, θ, γ which satisfied the following equations:

$$\theta\mu = 6\% \quad \text{(expected return on forward price)}$$

$$\sqrt{\theta(1+\overline{\alpha}^2\sigma^2)} = 15\% \quad \text{(expected volatility of forward price)}$$

$$\frac{\sigma^2}{2\kappa\theta} = 1 \quad \text{(unconditional variance of v)}$$

We solved numerically for the parameters consistent with these three equalities. There is a limited range of parameters for which the problem is soluble. Table 1 lists a number of sets of parameters, with increasing levels of risk aversion of the representative investor which we have used for our numerical work.

Finally, for each assumed level of the risk aversion of the representative investor, ie. γ, we have calculated the asymptotic benefit to non-representative investors with varying degrees of relative risk aversion from 1 to 9.

Table 1 **Parameters Used**

γ	σ	κ	θ
3	0.1738	0.9563	0.01579
4	0.0742	0.3675	0.00750
5	0.0508	0.2885	0.00448
6	0.0394	0.2590	0.00300
7	0.0325	0.2441	0.00216
8	0.0277	0.2354	0.00163

These are reported in Table 2 as a deterministic equivalent interest rate in annual basis points, which if applied to the initial endowment would equate the expected utility obtained in the complete market and incomplete market cases. This is calculated very simply as the difference between the growth rates of the ψ functions divided by (1−g). The justification for this is that if

$$J(F_t,t)=\frac{F_t^{1-g}}{1-g}e^{r(T-t)} \text{ for some growth rate } r\text{, then}$$

$$J(F_t,t)=\frac{G_t^{1-g}}{1-g}\text{, where } G_t=F_t\,e^{\frac{r}{1-g}(T-t)}. \tag{24}$$

Note that 50.0 would mean an annual benefit of 0.50%, which compares to the general risk premium of 6.00%.

Table 2 **Benefits of Options (D.E. Interest Rate in Basis Points)**

γ	$g=1$	2	3	4	5	6	7	8	9
3	28.4	1.3	0.0	0.2	0.4	0.6	0.7	0.7	0.7
4	97.7	12.7	1.4	0.0	0.5	0.8	2.0	2.6	3.0
5	149.3	31.1	7.3	1.1	0.0	0.6	1.6	2.7	3.7
6	189.1	51.1	16.6	4.8	0.9	0.0	0.5	1.6	2.8
7	220.8	70.2	27.6	10.8	3.5	0.6	0.0	0.4	1.4
8	246.5	87.5	39.0	17.9	7.6	2.7	0.6	0.0	0.4

The table shows that the advantage to options is highly sensitive to the other parameters, and the advantages are related to γ and g in a highly non-linear way. First, the table (and other calculations we have done) confirms that when γ equals g the benefits of options are always zero as they should be. For g equal to 1 we have the case of the logarithmic utility function. In this case the growth rates are equal but the limit of their difference divided by (1−g) as g tends to one is a positive amount, as we have tabulated. The results show that for all values of g different from γ the benefits are positive, but often rather small. The benefits of completing the market appear to be greatest to an investor who is not very risk averse, but who lives in an economy where the representative investor is very risk averse. In this case the annual benefit can represent a significant percentage of the normal market risk premium. The deterministic equivalent benefits are small to risk averse investors partly because these investors only hold a small proportion of their wealth in the risky market asset.As noted earlier, this interpretation depends on non-representative investors only being present in minute quantities compared to the representative investor.

In our economy, all investors have power utility and defend a minimum consumption level of zero. Investors who are more risk averse than the representative investor seek concave payoffs, and in a complete market are the sellers of option type contracts. Conversely, investors who are less risk averse will buy option contracts. The mathematics of our model remains essentially unaltered if we generalize the preferences of non-representative investors to the form

$$\text{Maximize } E\left[\frac{(W_T - h)^{1-g}}{1-g}\right].$$

For these preferences, too, concave payoffs are sought whenever $g > \gamma$. See Leland (1980) for an earlier related analysis.

6. Conclusions

We have derived a model of market equilibrium in an economy where a single risky asset evolves with stochastic volatility and applied it to examine the role of options in completing the market. Our work is in the spirit of Bick (1987). In our economy, (unlike Bick's which is complete and supports Black-Scholes option values), the market is incomplete unless some kind of volatility sensitive (ie. option-like) asset is traded.

We have calculated the benefits to completing the market in steady state equilibria involving investors with the same horizons but different degrees of risk aversion. The numbers we have calculated show a number of interesting features, and confirm the main regularities which we have prior insights about. In general the gains from introducing options (or completing the market in any other way) appear to be relatively minor. However, the more risk averse the representative investor becomes, the more a less risk averse investor will benefit from market completeness. The relationships are also surprisingly non-linear.

The gains may become larger in a richer model where investors' horizons differ. In this case we will also find effects stemming from the non-optimality of strategies which are path independent in the level of the market asset. In order to model the behaviour of investors with differing horizons we would need to introduce the possibility of consumption at other dates, or even continuous consumption. The model would need to represent the supply available at more than one date, and we then also need a risk free interest rate (which may be determined either exogenously, as in Bick's paper, or derived endogenously from other assumptions). The extensions required are surprisingly complicated and are left as a subject for further work.

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