Option Pricing and Smile Effect when Underlying Stock Prices are Driven By a Jump Process

Luca Pappalardo

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Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry
CV4 7AL
Phone: (01203) 524118

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Abstract

Black and Scholes values of volatility, implied from market prices, show a strong dependence on both the strike price and the maturity of a given European call option. This dependence is called the *Smile effect*. If we believe that stock prices are driven by a diffusion process, in which volatility is allowed to be dependent in either the stock price or time, Dupire showed how it is possible to recover the process from option prices. We assume, instead, a model in which stock prices undergo a diffusion plus a jump term (which is driven by a Poisson process). Option pricing is possible in this model (Merton 1973), but since the model is no more complete some hypothesis on the structure of the market is necessary – such as the CAPM. The aim of this work is to provide a method to recover a jump process from the prices observed in the market. It can be done modifying in a suitable way the procedure followed by Dupire. Some care is needed, since the spot process is not continuous. Hence, deriving the Kolmogorov Equation for the process, Ito lemma has to be applied in its general form. The equation for the *Smile surface* is derived in some particular cases of jump distribution and the problem of the parameter estimation - from which difficulties may arise - is discussed.

[*]Istituto di Metodi Quantitativi per le Scienze Economiche e Aziendali, Università degli Studi di Milano, Via Sigieri 6, 20135 Milano, Italy – phone +(39)2/54170201 – fax +(39)2/55180536 – e-mail PAPPA@IMIUCCA.CSI.UNIMI.IT

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The work is organized as follows: in section 2 there is a brief explanation of the Dupire procedure to recover a diffusion from the option prices observed in the market is explained ; in section 3 – following Merton [15] – the option pricing with a model which includes jumps is discussed ; in section 4 it is shown how the Dupire procedure can be modified for a jump process; in section 5 some issues are discussed which may arise implementing the procedure discussed in section 4.

2 The Smile Effect

Black and Scholes formula is a typical example of option pricing: having specified a model and estimated the parameters, we are able to derive the option prices, which are unique provided that the market model is complete. In particular, the model assumes that the dynamics of the asset price S is given by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB \tag{1}$$

where dB is a standard Brownian motion and μ, σ are constant. Under the hypothesis that the market does not admit arbitrage, we can find the price of a contingent claim. In fact, no arbitrage implies the existence of a risk neutral probability measure, such that discounted prices are martingales. Then the price of the claim is determined as the discounted expectation of its future pay off with respect to the risk neutral measure. For instance, the price C of a European call option with strike price K and maturity T is given by

$$C(K,T) = \int_0^{+\infty} \max(S' - K, 0)\, \varphi_T(S')\, dS' \tag{2}$$

where $\varphi_T(S')$ is the risk neutral probability density at the maturity T; solving the integral we get the well known Black and Scholes formula

$$C(K,T) = SN(d_1) - Ke^{-r\tau} N(d_2) \tag{3}$$

(Merton, [15]), or stochastic volatility models (Hull and White, [11]). In general, this kind of model introduces non traded sources of risk, losing the completeness of the model.

It is possible, anyway, to find a process for the stock prices such that the model is compatible with the implied volatility given by the market and the model is complete. Given the price of an European call option $C(K,T)$ as a function of the strike price and the maturity, Dupire [8] suggests a way to find a process for the stock process in the form of diffusion:

$$dS_t = r(t)S_t\,dt + \sigma(S_t,t)S_t\,dB \tag{5}$$

We shall assume – to ensure the existence and uniqueness of the solution of the above stochastic differential equation – that the coefficients satisfy the slow growth condition.

There is an ambiguity in this approach. In general, the diffusion contains more information than the conditional law it generates at a fixed time, i.e. different diffusions can give the same conditional law. The knowledge of the *Smile surface* $C(K,T)$ corresponds only to the knowledge of the conditional law, and hence the process is not uniquely determined. This ambiguity can be removed if we restrict ourselves to observe the process under the risk neutral probability measure. In fact, no arbitrage condition in a complete market is equivalent to existence of a unique such measure.

Differentiating relation (2) twice with respect to K we get a relation between the risk neutral probability density and the Smile surface:

$$\varphi_T(K) = \frac{\partial^2 C(K,T)}{\partial K^2} \tag{6}$$

Hence, given the conditional probability density φ_t, we search for a diffusion of the type (5) – or, better, a risk neutral such a process – that generates it.

The inverse problem – i.e., given a process to deduce the evolution of the conditional probability density – is known to have solution. Since the backward and forward Kolmogorov equations characterizes, respectively, the for a certain real positive number γ – the left hand side in equation (8) has zero as a lower limit as S goes to infinity[1]. Hence, since the left hand side of the equation (8) is zero, and we assume that (9) holds, then the two integration constants α and β must be zero as well.

We finally get from (8):

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = \frac{\partial C}{\partial t}$$

which can be solved for σ

$$\sigma = \frac{1}{S}\sqrt{2\frac{\partial C}{\partial t} \Big/ \frac{\partial^2 C}{\partial S^2}} \tag{11}$$

Since σ must satisfies the slow growth condition (10), C must satisfies the following:

$$\frac{\partial C}{\partial t} \leq \frac{\gamma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} \tag{12}$$

where γ is as in (10). This latter condition simply means that it is not possible to recover a diffusion process from every Smile surface; C must fulfill some conditions. For instance, if a sudden, sharp change in time of the option price, e.g. a jump, happens – such that the latter condition is no more satisfied – then it is impossible to reproduce the Smile surface C by a diffusion.

Therefore, it is possible to recover a pure diffusion process whose volatility is given from the conditional distribution; hence, it is possible to infer a unique diffusion process from the option prices observed in the market. Those are essentially the results shown by Dupire [8]. We shall see in the following

[1] In fact, otherwise there would be a positive real number a such that:

$$0 < a < \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = \sigma^2 S^2 \varphi_t \leq \gamma^2 S^2 \varphi_t$$

where γ is as in (10). It would imply:

$$S\varphi_t \geq \frac{a}{\gamma^2 S}$$

in contradiction with the fact that φ_t has finite expectation.

4. the number of jumps in different (disjoint) time intervals are independent;

5. the distribution of the jumps in a given time interval depends only on the size of the interval;

where $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$ and λ (the mean number of jumps per unit time) is a real non negative constant. When a jump occurs, then its size is determined by drawing from a distribution Y. Hence, neglecting the continuous part, if only one jump occurs in a time interval Δt, then for the stock price S we have:

$$S_{t+\Delta t} = Y S_t$$

We assume, moreover, that $Y \geq 0$ and that the distributions for successive drawing are independent and identically distributed.

This process can be well described by the following stochastic differential equation:

$$dS_t = (\mu - \lambda k) S_t \, dt + \sigma S_t \, dB + S_t \, (Y - 1) dq \tag{13}$$

where dq is a Poisson process with λ events per unit time (on average), $k = \mathsf{E}_Y \, [Y - 1]$, μ and σ are non negative real constants. The process can be written in an equivalent way:

$$dS_t = \begin{cases} (\mu - \lambda k) S_t \, dt + \sigma S_t \, dB & \text{if the jump does not occur} \\ (\mu - \lambda k) S_t \, dt + \sigma S_t \, dB + (Y - 1) S_t & \text{if the jump occurs} \end{cases}$$

in fact, with probability one at most one jump occur in a time interval dt.

If we want to price a call option on an asset whose price follows the process (13) applying a Black and Scholes procedure, and we try to find a riskless portfolio of zero value, the first difficulty that we meet is that Ito's lemma for a discontinuous process has an additional jump term:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t \frac{\partial f(X_{s^-})}{\partial x} \, dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f(X_{s^-})}{\partial x^2} \, d[X_s^{cm}, X_s^{cm}] \\ &\quad + \sum_{0<s\leq t} \left(\Delta f(X_s) - \frac{\partial f(X_{s^-})}{\partial x} \Delta X_s \right) \end{aligned} \tag{14}$$

Hence, applying Ito's lemma, we get for the price $C(S,t)$ of the option:

$$
\begin{aligned}
dC &= \frac{\partial C}{\partial t}\,dt + \frac{\partial C}{\partial S}\,dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}S_t^2\sigma^2\,dt \\
&\quad +\Delta C - \frac{\partial C}{\partial S}\Delta S_t \\
&= \frac{\partial C}{\partial t}\,dt + \frac{\partial C}{\partial S}\left(\mu - \lambda k\right)S_t\,dt + \frac{\partial C}{\partial S}\sigma S_t dB + \frac{\partial C}{\partial S}S_t(Y-1)dq \\
&\quad +\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t\,dt + \Delta C - \frac{\partial C}{\partial S}\Delta S_t \\
&= \left[\frac{\partial C(S_t)}{\partial t} + \frac{\partial C(S_t)}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 C(S_t)}{\partial S^2}S_t^2\sigma^2\right. \\
&\quad \left. +\lambda\left(\mathsf{E}_Y\left[C(YS_t)\right] - C(S_t) - \frac{\partial C(S_t)}{\partial S}kS_t\right)\right] dt \\
&\quad +\frac{\partial C(S_t)}{\partial S}\sigma S_t dB \\
&\quad +\frac{\partial C(S_t)}{\partial S}S_t\left((Y-1)dq - \lambda k\,dt\right) \\
&\quad +\Delta C(S_t) - \frac{\partial C(S_t)}{\partial S}\Delta S_t - \lambda\left(\mathsf{E}_Y\left[C(YS_t)\right] - C(S_t) - \frac{\partial C(S_t)}{\partial S}kS_t\right) dt
\end{aligned}
\tag{16}
$$

Notice that the last three lines in equation (16) represent differential martingales.

The parameters of the process driving the option price can be derived in terms of the parameters of the stock price. In particular, for the option volatility we have:

$$\sigma_C = \frac{\frac{\partial C}{\partial S}\sigma S}{C}$$

The jump distribution for the option can be worked out: if the stock jumps from S to YS, then we have

$$C(YS,t) = Y_C C(S,t)$$

$$= \left(\mathsf{E}_Y\left[f(YS_t)\right] - f(S_t) - \frac{\partial f(S_t)}{\partial S}kS_t\right)\lambda\,dt$$

where $\mathsf{E}_Y\left[\cdot\right]$ is the expection taken with respect to the random variable Y.

We define $B(S, \tau, K, r, \sigma)$ as the solution of the usual Black and Scholes problem:

$$B(S, \tau, K, r, \sigma) = S\,N(d_1) - Ke^{-r\tau}\,N(d_2)$$

where d_1 and d_2 are as in (4) and $\tau = T - t$. Then, equation (17) has an exact solution, which is given by:

$$C(S,t) = \sum_{n=0}^{+\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} \mathsf{E}_{Y_n}\left[B(SY_n e^{-\lambda k\tau}, \tau, K, r, \sigma)\right]$$

where Y_n has the same distribution as the product of n independent and identically distributed random variable, each of them independent and identically distributed to Y, and $Y_0 = 1$.

In the special case Y has a lognormal distribution, i.e. $\log Y \equiv u \sim \mathcal{N}(\log(k+1) - \frac{s^2}{2}, s^2)$, then the solution has the form:

$$C(S,t) = \sum_{n=0}^{+\infty} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} B(S, \tau, K, r_n, \sigma_n) \tag{18}$$

where:

$$\begin{aligned}
\lambda' &= \lambda(1+k) \\
r_n &= r - \lambda k + \frac{n\log(1+k)}{\tau} \\
\sigma_n &= \sigma^2 + \frac{ns^2}{\tau}
\end{aligned}$$

Ceteribus paribus, an option on a stock whose price is driven by a process with jumps is more valuable than a stock whose price is driven simply by a geometric Brownian motion. Moreover, the formula (18), which assumes that the jumps have a lognormal distribution, explains (at least qualitatively) many discrepancies between the Black and Scholes formula and the option prices observed in the market, such as the fact that either deep in the money and deep out of the money options are underestimated by the Black and Scholes model.

4.1 Step 1: the Kolmogorov Backward Equation

The Kolmogorov backward equation characterizes the backward evolution of the conditional expectation $\nu(x,t) = \mathsf{E}\left[f(X_T)|X_t = x\right]$, given a process X_t. As we have seen in chapter 2, it is derived applying Ito's lemma to $\nu(X_t, t)$, and recognizing that it must be a martingale – since the process X_t satisfies the Markov property. The first difficulty we have to face is that for discontinuous process we have to apply Ito's lemma in its general form, which is given by equation (14). Problems can arise, of course, from the additional jump term (15); for this term we have already pointed out some useful facts in section 3.

Keeping in mind these statements , we can derive the Kolmogorov backward equation for process (19). First, we apply Ito's lemma to ν and we put in evidence the martingale part:

$$
\begin{aligned}
d\nu(S_t,t) = {} & \frac{\partial\nu(S_t,t)}{\partial t}\,dt + \frac{\partial\nu(S_t,t)}{\partial S}\,dS_t + \frac{1}{2}\frac{\partial^2\nu(S_t,t)}{\partial S^2}S_t^2\sigma^2(S_t,t)\,dt \\
& +\Delta\nu(S_t,t) - \frac{\partial\nu(S_{t^-},t)}{\partial S}\Delta S_t \\
= {} & \frac{\partial\nu(S_t,t)}{\partial t}\,dt + \frac{\partial\nu(S_t,t)}{\partial S}\left(\mu(S_t,t) - \lambda k\right)S_t\,dt \\
& +\frac{\partial\nu(S_t,t)}{\partial S}\sigma(S_t,t)S_t dB + \frac{\partial\nu(S_t,t)}{\partial S}S_t(Y-1)\,dq \\
& +\frac{1}{2}\frac{\partial^2\nu(S_t,t)}{\partial S^2}\sigma^2(S_t,t)S_t^2\,dt + \Delta\nu(S_t,t) - \frac{\partial\nu(S_{t^-},t)}{\partial S}\Delta S_t \\
= {} & \left[\frac{\partial\nu(S_t,t)}{\partial t} + \frac{\partial\nu(S_t,t)}{\partial S}\mu(S_t,t)S_t + \frac{1}{2}\frac{\partial^2\nu(S_t,t)}{\partial S^2}\sigma^2(S_t,t)S_t^2\right. \\
& \left.+\lambda\left(\mathsf{E}_Y\left[\nu(YS_t,t)\right] - \nu(S_t,t) - \frac{\partial\nu(S_t,t)}{\partial S}kS_t\right)\right]dt \\
& +\frac{\partial\nu(S_t,t)}{\partial S}\sigma(S_t,t)S_t dB \\
& +\frac{\partial\nu(S_t,t)}{\partial S}S_t\left((Y-1)\,dq - \lambda k\,dt\right) \\
& +\Delta\nu(S_t,t) - \frac{\partial\nu(S_t^-,t)}{\partial S}\Delta S_t
\end{aligned}
$$

where $(\cdot|\cdot)$ is the usual inner product in $\mathbf{L}^2(0,+\infty)$ and $f,g \in \mathbf{L}^2(0,+\infty)$. For the last term of equation (21) we have:

$$(\mathsf{E}_Y\left[f(YS)\right]|g) = \int_0^{+\infty}\int_Y f(yS)g(S)\,m(dy)\,dS$$

$$= \int_Y\int_0^{+\infty} f(yS)g(S)\,dS\,m(dy) = \int_Y\int_0^{+\infty} f(S)g\left(\frac{S}{y}\right)\frac{dS}{y}\,m(dy)$$

$$= \int_0^{+\infty} f(S)\int_Y g\left(\frac{S}{y}\right)\frac{1}{y}\,dS\,m(dy) = \int_0^{+\infty} f(S)\mathsf{E}_Y\left[g\left(\frac{S}{Y}\right)\Big/Y\right]\,dS$$

$$= \left(f\,\middle|\,\mathsf{E}_Y\left[g\left(\frac{S}{Y}\right)\Big/Y\right]\right)$$

where $m(dy)$ is the probability measure on Y. Moreover, we must impose $Y > 0$ (in order to divide by Y): from an economic point of view, we rule out the possibility that the firm whose shares are worth S_t can suddenly jump into the zero state, i.e. into bankruptcy.

It easy to see, then, that:

$$\begin{aligned}\mathbf{A}^{\dagger}f(S) &= -\frac{\partial}{\partial S}\left[(\mu(S,t)-\lambda k)Sf(S)\right] \\ &+\frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S)\right] + \lambda\mathsf{E}_Y\left[\frac{f(S/Y)}{Y}\right] - \lambda f(S)\end{aligned}$$

and finally we get the Kolmogorov forward equation:

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\partial}{\partial S}\left[(\mu(S,t)-\lambda k)Sf(S)\right] \\ &+\frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2 f(S)\right] + \lambda\int_Y \frac{f(S/y)}{y}m(dy) - \lambda f(S)\end{aligned} \qquad (22)$$

4.3 Step 3: the Equation for the Smile Surface

In order to get an equation for the Smile surface, the same procedure that Dupire in [8] adopted for diffusion process can be followed: the equation for the volatility σ and for the jumps can be derived, using relation (6) that links the Smile surface $C(S,t)$ to the risk neutral probability density $\varphi_t(S)$

$$= \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial z^2} + \left(\lambda k - \frac{\sigma^2}{2}\right)\frac{\partial C}{\partial z} - \lambda(k+1)C + \lambda\int_Y C(z - \log y, t)\, y\, m(dy) \tag{24}$$

Equations (24) and (23) are therefore the equations for the Smile surface C. Before going further, we can resume all the assumptions have been made in deriving them. They can be summarized, step by step, as follows:

1. in process (19) we assume:

   (a) $\sigma(S,t)$ is a sufficiently smooth function satisfying the slow growth assumption;

   (b) $Y \geq 0$ is the distribution, with finite mean and variance, from which is made a draw when a jump occur; it determines the jump size and successive drawings are independent;

   (c) λ is a real non negative constant;

2. deriving the Kolmogorov backward equation, no particular further assumption is made on σ or Y ;

3. deriving the Kolmogorov forward equation, we have assumed $Y > 0$, ruling out the chance to jump into bankruptcy;

$$\begin{aligned}
\frac{\partial^2 C}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right) = \frac{\partial}{\partial z}\left(S\frac{\partial C}{\partial S}\right) \\
&= \frac{\partial S}{\partial z}\frac{\partial C}{\partial S} + S\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial S}\right) \\
&= S\frac{\partial C}{\partial S} + S\frac{\partial^2 C}{\partial S^2}\frac{\partial S}{\partial z} \\
S^2\frac{\partial^2 C}{\partial S^2} &= \frac{\partial^2 C}{\partial z^2} - \frac{\partial C}{\partial z} \\
C\left(\frac{S}{y}, t\right) &= C(z - \log y, t)
\end{aligned}$$

If the argument of C is not specified, it is assumed $C(z,t)$; it should be clear from the contest whether $C(z,t)$ or $C(S,t)$ is meant.

where either σ or Y might be time dependent. Having an estimate for λ and k, given the jump distribution Y, we can solve explicitly for σ

$$\sigma = \sqrt{\frac{\lambda\Gamma \int_Y e^{-2\pi i x \log y}\, y\, m(dy) + 2\pi i \lambda k x \Gamma - \frac{\partial \Gamma}{\partial t} - \lambda(k+1)\Gamma}{(2\pi^2 x^2 \Gamma + \pi i x \Gamma)}}$$

4.4.2 Lognormal Jump Distribution: $\log y \sim N(-\frac{\alpha^2\sigma^2(t)}{2}, \alpha^2\sigma^2(t))$

It is quite natural to consider lognormal distribution for Y; we can assume, for instance, that the size of the jump has roughly the same size as the average change due to the continuous part, and that jump up are equally like jump down. This can be formalized assuming that the size of the jumps Y has a lognormal distribution, and $\log y \equiv u \sim N(-\frac{\alpha^2\sigma^2(t)}{2}, \alpha^2\sigma^2(t))$, where $\sigma(t)$ is the instantaneous variance of the continuous part of the process (19) and α is a real constant. Under this assumption we can solve explicitly (25)[7]:

$$\log \frac{\Gamma(x,t)}{\Gamma(x,0)} = -\int_0^t \left[\sigma^2 2\pi^2 x^2 + \sigma^2 \pi i x + \lambda - \lambda e^{-\frac{\alpha^2\sigma^2}{2}(4\pi^2x^2+2\pi i x)}\right] du$$

or, equivalently:

$$-\frac{1}{\Gamma}\frac{\partial \Gamma}{\partial t} = \sigma^2 2\pi^2 x^2 + \sigma^2 \pi i x + \lambda - \lambda e^{\frac{\alpha^2\sigma^2}{2}-(4\pi^2x^2+2\pi i x)} \tag{26}$$

we can then use this relation to estimate σ,α and λ from the observed Smile surface C. This approach does not explain anyway the dependance of implied volatility on the strike price of the option, since σ is allowed to be only time dependent.

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[7] In fact, the following relations hold true:

$$\begin{aligned}
k &= \mathrm{E}_Y[Y-1] = 0 \\
\int_Y e^{-2\pi i x \log y}\, y\, m(dy) &= \frac{1}{\sqrt{2\pi}\alpha\sigma}\int_{-\infty}^{+\infty} e^{-2\pi i u x}\, e^{u}\, e^{-\frac{(u+\alpha^2\sigma^2/2)^2}{2\alpha\sigma}}\, du \\
&= e^{-(1-2\pi i x)\frac{\alpha^2\sigma^2}{2}+\frac{1}{2}(1-2\pi i x)^2\alpha^2\sigma^2} \\
&= e^{-\frac{\alpha^2\sigma^2}{2}(4\pi^2x^2+2\pi i x)}
\end{aligned}$$

Since $k = \sum_{k=1}^{n}(y_k p_k) - 1$, we get :

$$\frac{\partial C}{\partial t} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + \lambda\left[\sum_{k=1}^{n}(y_k p_k) - 1\right] S\frac{\partial C}{\partial S} - \lambda\sum_{k=1}^{n}(y_k p_k) C + \lambda\sum_{j=1}^{n} C\left(\frac{S}{y_j}\right) y_j\, p_j \tag{28}$$

Again, if we have a good estimate for λ and probability $p_j\,(j = 1, \dots, n)$ – having fixed, for instance, the value for the jumps y_j – we are able to explicitate σ:

$$\begin{aligned}\sigma &= \frac{1}{S}\left\{\left[2\frac{\partial C}{\partial t} - \lambda\left(\sum_{k=1}^{n}(y_k p_k) - 1\right) S\frac{\partial C}{\partial S}\right.\right. \\ &\qquad \left.\left. + \lambda\sum_{k=1}^{n}(y_k p_k) C - \lambda\sum_{j=1}^{n} C\left(\frac{S}{y_j}\right) y_j\, p_j\right] \Big/ \frac{\partial^2 C}{\partial S^2}\right\}^{1/2}\end{aligned}$$

This approach has the advantage that avoids the main difficulty in solving equation (23) – i.e., the integral in the right hand side. Since this integral becomes a finite sum, we do not need all the manipulation we have gone through assuming Y continuously distributed, such as change of variable, Fourier transform, etc. In particular, we can allow σ to be dependent on S , but since we use equation (23) the jump size (i.e., neither y_j or p_j) cannot. On the other hand, there is a price we have to pay: the estimation of the n probability p_j associated to the jump y_j, which may turn out to be problematic.

5 Implementation of the Model

The main problem which arises implementing a model which includes jumps for the underlying process is that it is superabundant. If we assume that the underlying process is a diffusion, then the knowledge of the Smile surface C is sufficient to recover the volatility and hence the diffusion process – provided that C is sufficiently nice, i.e. that it satisfies the slow growth condition (10). The situation is more subtle for jump process: even if we define a priori the jump distribution Y we are not able to recover the process: the solution irregularly positioned data.

Therefore, there are mainly two kind of difficulties in solving this estimation problem. First, we have to estimate the jump parameters. To solve this

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one variable. Given a set data ($y_j = y(x_j)$) for $j = 1, \ldots, n$, the usual linear spline interpolation $\tilde{s}$ is given by

$$\tilde{s}(x) = \frac{(x_{j+1} - x)y_j + (x - x_j)y_{j+1}}{x_{j+1} - x_j}$$

for $x_j \leq x \leq x_{j+1}$, the function $\tilde{s}$ can be written in a more convenient way which is also easier to generalize:

$$\tilde{s}(x) = \sum_{j=1}^{n} \gamma_j |x - x_j|$$

where the coefficients γ_j are defined by the relation:

$$\tilde{s}(x_j) = y_j$$

The obvious generalitation in the multidimensional case is

$$\tilde{s}(\mathbf{x}) = \sum_{j=1}^{n} \gamma_j ||\mathbf{x} - \mathbf{x}_j||$$

where $\mathbf{x}_j \in \mathsf{R}^d$, $||\cdot||$ is the Euclidian norm and the coefficient γ_j are defined by an analogue relation. We can generalize this approximation further, requiring that $\tilde{s}$ is a linear combination of translates of function spherically symmetric about the origin. Hence, the general form for a radial basis function is

$$\tilde{s}(\mathbf{x}) = \sum_{j=1}^{n} \gamma_j \varphi\left(||\mathbf{x} - \mathbf{x}_j||\right)$$

where $\varphi : \mathsf{R}^+ \rightarrow \mathsf{R}$ is fixed. We have different kind of radial basis interpolation according to which function φ we choose:

1. linear radial basis, $\varphi(r) = r$;
2. cubic radial basis, $\varphi(r) = r^3$;
3. thin plate spline radial basis, $\varphi(r) = r \log r$;
4. Gaussian radial basis, $\varphi(r) = e^{-r^2}$;
5. multiquadric radial basis, $\varphi(r) = \sqrt{r^2 + c^2}$ with $c \in \mathsf{R}$;
6. inverse multiquadric radial basis, $\varphi(r) = 1 / \sqrt{r^2 + c^2}$ with $c \in \mathsf{R}$.

The advantage of this kind of approximation is that there is no constraint on the position of the data points and it is straithforward to compute the value of $\tilde{s}$ and of its derivatives for any $\mathbf{x} \in \mathsf{R}^d$.

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