Calibration of Kennedy and Multi-Factor Gaussian HJM to Caps and Swaptions Prices

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Abstract

We calibrate a Gaussian Random Field Term Structure model of Kennedy (1994) to market caps and swaptions prices using an exact cap pricing formula and an approximate swaption pricing formula. The calibrated Kennedy model is used as an indirect way to calibrate multi-factor Gaussian Heath, Jarrow and Morton interest rate models. We compare the resulting HJM model with previous calibrations. The indirect calibration of Gaussian HJM via the Kennedy model is preferred to a direct calibration.

In this paper we examine the issue of calibrating Gaussian versions of Heath, Jarrow and Morton (1992), HJM, and Kennedy (1994) to market prices of liquid instruments. We calibrate to caps and swaptions prices. Ideally, the calibrated model would produce either a stationary volatility term structure where zero-rate volatilities depend on their maturities only, or an evolving volatility term structure that maintains similar shapes across yield maturities. This is supported by historical data and any interest rate term structure models that do not replicate the empirical volatility structures would be susceptible to errors when pricing volatility dependent derivatives.

There are a number of reasons why the calibration is difficult. Some models, as Rebonato and Cooper (1996) illustrate, cannot match simultaneously market caps and swaptions prices because of their failure to reproduce empirical and implied correlations between yield changes. Others such as the Markovian short rate models of Black, Derman and Toy (1990), Black and Karasinski (1991) and Hull and White (1993 and 1994) imply non-stationary volatility term structures that can evolve uncontrollably through time to produce volatility term structures that are very different from the initial structures. HJM models do not suffer from these problems but their calibration is also difficult. Rebonato and Cooper (1996) illustrate that implementations based on a Principal Components Analysis (PCA) of the covariance matrix of historical interest rate changes would require many factors to reproduce the correlations accurately. This implies typical PCA implementations of HJM would not be able to price correlation sensitive instruments accurately since, in practice, we can only accept the first two factors as the higher factors are difficult to estimate accurately. In any case, even if we were able to obtain all the factors accurately, there is no guarantee that the resulting model would produce prices that would be consistent with the market. This is because there will be some amount of mis-specification error in the model. Furthermore, measurements based on historical data cannot react to structural shifts without a delay. We have to find implied volatility functions that will allow the model to be consistent with market prices in the same way that the famous Black and Scholes formula needs an implied volatility to price equity options. However implementing standard approaches may require specifications for the functional forms and for the number of factors and proves to be unsatisfactory because we do not know how appropriate our choices would be.

To overcome the problems highlighted, in this paper we adopt a non-parametric approach that does not require us to specify the number of factors. Our aim is to calibrate HJM but we prefer to do it by first calibrating the Gaussian Random Field Term Structure model of Kennedy (1994) to caps and swaptions and then finding an HJM approximation to the calibrated Kennedy model. Kennedy (1994) requires as input an initial interest rate term structure and a covariance function that gives the covariance between any two instantaneous forward rates. Our estimation is non-parametric in the sense that we do not assume a functional form for the covariance function. Kennedy (1994) is a very flexible model and we would therefore expect that flexibility would enable it to fit market caps and swaptions prices simultaneously. The Gaussian assumption allows for many closed form solutions to option pricing problems. This is an essential property when we calibrate a model. Kennedy (1994) provides a cap pricing formula and we derive an approximate European Swaption pricing formula that can also be used for any of the existing Gaussian interest rate term structure models in the literature. The formulae allow us to calibrate Kennedy (1994) to a wide range of quoted caps and swaptions prices rapidly.

Kennedy (1994), however, does not allow for easy pricing of more sophisticated derivatives such as the path dependent variety. We can tackle such pricing and hedging problems more easily in the HJM framework and it is for this reason that we use Kennedy as a device for calibrating HJM. With this in mind we focus on a particular specification of Kennedy (1994) that can be approximated easily by a multi-factor Gaussian HJM model with stationary volatility structures. This, we believe, provides for better calibration of HJM models than by direct calibration. The resulting HJM models have stationary volatility structures that are also stable between re-calibration across calendar time. We examine the fitted HJM model and find that quoted caps and swaptions prices produce two or possibly three significant factors.

We begin by giving a brief review of Kennedy (1994) in Section 1 and examining the Kennedy covariance function we will fit in this paper in Section 2. Section 3 provides the key motivation factor for our choice of Kennedy covariance function; it shows how our chosen Kennedy model can be approximated by a stationary HJM model. Section 4 shows how contingent claims are priced in the Kennedy model and produces formulae that are used to calibrate the Kennedy model to caps and swaptions. We review the data we have employed in Section 5 and discuss the results and implications of the fitted model in Section 6. Section 7 summarises and makes a few concluding remarks.

1. REVIEW OF THE KENNEDY GAUSSIAN RANDOM FIELD TERM STRUCTURE MODEL

Unlike the majority of other interest rate modelling and derivative pricing models where interest rate processes are specified as stochastic differential equations for which one would have to solve to find the relevant risk-adjusted distributions, Kennedy (1994) instead specifies the risk-adjusted distributions such that no arbitrage opportunities exist. Kennedy (1994) requires as inputs the covariance of all future instantaneous forward rates and the initial instantaneous forward rate curve. The joint distribution of all future forward rates in the risk-adjusted measure is then given by standard no-arbitrage arguments. Derivative prices are given simply by finding the appropriate expectations of the discounted payoffs under the risk-neutral measure. Kennedy's approach is complementary to the more traditional approach of specifying an interest rate process. Indeed Kennedy's model encompasses all other diffusion models in which interest rates are normally distributed. Thus, for example, it is possible to find specifications for the Kennedy (1994) that corresponds to the models of Vasicek (1977), the Extended Vasicek model of Hull and White (1990), the multi-factor model of Langetieg (1980) and any Gaussian HJM. Not only does the Kennedy model encompass all Gaussian interest rate models in the current literature, but it also allows for infinite factor models.

We adopt Kennedy's notation and use Kennedy to refer to Kennedy (1994) models. Let $F_{s,t}$, or $F(s,t)$ denote the instantaneous forward interest rate an investor can contract to borrow or lend at time s for time t. The model assumes

$$F_{s,t} = \mu_{s,t} + X_{s,t}, \tag{1.1}$$

$0 \leq s \leq t$, where $X_{s,t}$ is a mean-zero continuous Gaussian random field. Kennedy (1995) shows the Gaussian and no arbitrage assumptions imply that $Cov(F_{s_1,t_1}, F_{s_2,t_2})$ must take the form $c(s_1 \wedge s_2, t_1, t_2)$, where $s_1 \wedge s_2$, denotes $min(s_1,s_2)$, and that the random field has independent increments in the s-direction. That is for any $0 \leq s \leq s' \leq t$, the random variable $X_{s',t} - X_{s,t}$ is independent of the σ-field $\Im_s = \sigma\{X_{u,v}: u \leq s, u \leq v\}$. Furthermore, the absence of arbitrage implies that under the risk-neutral measure, the means of the future instantaneous forward rates must be related by

$$\mu_{s,t} = \mu_{0,t} + \int_0^t c(s \wedge v, v, t) dv \text{ for all } 0 \leq s \leq t. \tag{1.2}$$

Readers are referred to Kennedy (1994, 1995) for more details. Note that since zero-rates are averages of instantaneous forward rates, we can readily reformulate Kennedy (1994, 1995) in terms of zero rates. Indeed, all the formulae and our calibration procedure in this paper can be adapted for the case when zero-rates are modelled as a Gaussian Random Field. To illustrate Kennedy (1994) consider the following two examples.

1.1 Hull-White Model

The risk-neutral process for the short rate is assumed to be

$$dr = \left[\phi(t) - a(t)r\right]dt + \sigma(t)dz$$

where $\phi(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of time usually chosen such that the model is made consistent with an observed yield curve, a yield volatility term structure and perhaps prior believes on short rate volatilities. Hull and White (1990) show that under the risk-neutral process, the time t price of a pure discount bond with maturity T is given by

$$P(t,T) = A(t,T)\exp[-B(t,T)r(t)]$$

where

$$B(t,T) = \frac{B(0,T) - B(0,t)}{\partial B(0,t)/\partial t}$$

$$B(0,t) = \frac{R(0,t)\sigma_R(0,t)t}{\sigma(0)}$$

$\sigma_R(0,t)$ is the proportional volatility of a t-maturity zero rate at time 0, $R(0,t)$ is the t-maturity zero rate at time 0, $\sigma(0)$ is the short rate proportional volatility at time 0 and $A(t,T)$ is a deterministic function of the initial term structure given by

$$\ln A(t,T) = \ln\frac{A(0,T)}{A(0,t)} - B(t,T)\frac{\partial}{\partial t}\ln A(0,t) - \frac{1}{2}\left[B(t,T)\frac{\partial B(0,t)}{\partial t}\right]^2 \int_0^t \left[\frac{\sigma(\tau)}{\partial B(0,\tau)/\partial \tau}\right]^2 d\tau$$

It is shown in Appendix 8.1 that the covariance between the two instantaneous forward rates F_{s_1,t_1} and F_{s_2,t_2} is given by

$$Cov[F(s_1,t_1),F(s_2,t_2)] = \frac{\partial B(0,t_1)}{\partial t_1}\frac{\partial B(0,t_2)}{\partial t_2}\int_0^{s_1 \wedge s_2}\left[\frac{\sigma(u)}{\partial B(0,u)/\partial u}\right]^2 du\,.$$

Thus the Hull and White (1990) model can be put into the Kennedy framework by specifying that $c(s_1 \wedge s_2, t_1, t_2) = f(s_1 \wedge s_2) g(t_1, t_2)$ with the functions $f(s)$ and $g(t_1, t_2)$ defined by equations 1.3 and 1.4 respectively.

$$f(s) = \int_0^s \left[\frac{\sigma(u)}{\partial B(0,u)/\partial u} \right]^2 du \tag{1.3}$$

$$g(t_1, t_2) = \frac{\partial B(0,t_1)}{\partial t_1} \frac{\partial B(0,t_2)}{\partial t_2}. \tag{1.4}$$

1.2 Gaussian Heath, Jarrow and Morton

The risk-neutral instantaneous forward rate process is given by

$$dF_{s,u} = \alpha(s,u)dt + \underline{\sigma}(s,u) \cdot d\underline{z}$$

where $s \leq u$, $\underline{\sigma}(s,u)$ is a n element column vector of deterministic volatilities, $d\underline{z}$ is a n element column vector of independent Brownian increments and the $\cdot$ denotes the inner product. HJM (1992) shows that $\alpha(s,u)$ is constrained by no-arbitrage arguments to be given by

$$\alpha(s,u) = \underline{\sigma}(s,u) \cdot \int_s^u \underline{\sigma}(s,v)dv.$$

Assuming that the volatilities satisfy the appropriate integrability conditions of HJM (1992), then the instantaneous forward rate $F_{s,t}$ is given by

$$F_{s,t} = F_{0,t} + \int_o^s \alpha(u,t)du + \int_0^s \underline{\sigma}(u,t) \cdot d\underline{z}(u).$$

It follows that the equivalent specification for Kennedy follows immediately since

$$Cov[F_{s_1,t_1}, F_{s_2,t_2}] = c(s_1 \wedge s_2, t_1, t_2) = \int_0^{s_1 \wedge s_2} \underline{\sigma}(u,t_1) \cdot \underline{\sigma}(u,t_2)du. \tag{1.5}$$

We have seen how all Gaussian HJM models can be put into the Kennedy framework. Equation 1.5 also suggests we can approximate Kennedy with a finite factor Gaussian HJM. We show how this can be achieved easily for a particular class of Kennedy covariance functions in Section 3. Before that we specify our non-parametric covariance function and provide sufficient conditions for the continuity and smoothness of the term structure of forward rates.

2. COVARIANCE FUNCTION, CONTINUITY AND SMOOTHNESS

There are various ways in which one can utilise market data to calibrate interest rate derivative pricing models. In the context of Kennedy, we need to ascertain what the covariance structure of future instantaneous rates are. Given the current interest rate term structure, the means are then determined by standard no-arbitrage arguments. We need to estimate the covariance function $c(s,u,v)$ and ensure that the fitted model will be well-behaved. This section examines the conditions that will ensure that the term structure of the forward rates will be continuous and smooth.

As we explained in the introduction, we want to find a Kennedy model that can be readily approximated by a finite factor Gaussian HJM model. We are interested in stationary volatility structures, so it follows from

$$Cov[dF_{s,t_1}, dF_{s,t_2} | \Im_s] = \lim_{t \to s^+} \frac{Cov\left[F_{t,t_1} - F_{s,t_1}, F_{t,t_2} - F_{s,t_2} \middle| \Im_s\right]}{t-s} ds$$

$$= \lim_{t \to s^+} \frac{Cov\left[F_{t,t_1}, F_{t,t_2} \middle| \Im_s\right]}{t-s} ds$$

$$= \lim_{t \to s^+} \frac{c(t,t_1,t_2) - c(s,t_1,t_2)}{t-s} ds$$

$$= \frac{\partial}{\partial s^+} c(s,t_1,t_2) ds$$

that

$$\frac{\partial}{\partial s^+} c(s,t_1,t_2) = g(t_1 - s, t_2 - s)$$

for some non-negative definite function $g(u,v)$. Integrating gives

$$c(s,t_1,t_2) = \int_0^s g(t_1 - u, t_2 - u)du + h(t_1,t_2).$$

We therefore assume that the Kennedy covariance function of this paper satisfies:

Assumption 1:

$c(s,t_1,t_2) = \int_0^s g(t_1 - u, t_2 - u)du$ for all t_2, $t_1 \geq s \geq 0$ with $g(u,v)$ satisfying the conditions of Proposition 1.

We assume $h(t_1, t_2) = 0$ for all t_1 and t_2 so that $c(0, t_1, t_2) = 0$ for all t_1 and t_2. This corresponds to assuming that $\Im_0$ is the trivial σ-field. Kennedy (1995) examines interesting cases where $\Im_0$ is non-trivial. We call the covariance function of Assumption 1 the stationary Kennedy covariance function. Proposition 1 provides a sufficient condition for the continuity of the forward rate term structure.

Proposition 1: *For $Cov(F_{s_1,t_1}, F_{s_2,t_2}) = c(s_1 \wedge s_2, t_1, t_2) = \int_0^{s_1 \wedge s_2} g(t_1 - u, t_2 - u)du$, $0 \leq s_1 \leq t_1 < \infty$, $0 \leq s_2 \leq t_2 < \infty$, the Gaussian Random Field has continuous sample functions, with probability one, when $g(u,v)$ is continuous and bounded.*

Proposition 1 is proved in Appendix 8.2. The forward rate surface will be continuous when $g(u,v)$ of the stationary Kennedy covariance function is continuous and bounded. The conditions of Proposition 1 are very natural so they impose no significant constraints on the type of covariance functions we may want to use. Proposition 2 provides a sufficient condition for the smoothness of the forward rate term structure.

Proposition 2: *For Cov(* F_{s_1,t_1}, F_{s_2,t_2} *) =c(s₁∧s₂,t₁,t₂) =* $\int_0^{s_1 \wedge s_2} g(t_1 - u, t_2 - u)du$*, for finite* s_1*,* s_2*,* t_1 *and* t_2*, the term structure of instantaneous forward rates is smooth, with probability one, when* $\frac{\partial^2 g(\tau_1,\tau_2)}{\partial\tau_1\partial\tau_2}$ *is continuous and bounded.*

Proof: Consider

$$Cov\left[\frac{dF_{s_1,t_1}}{dt_1}, \frac{dF_{s_2,t_2}}{dt_2}\right]$$

$$= \underset{\Delta t_1 \to 0}{Lim}\, \underset{\Delta t_2 \to 0}{Lim}\, Cov\left[\frac{F_{s_1,t_1+\Delta t_1} - F_{s_1,t_1}}{\Delta t_1}, \frac{F_{s_2,t_2+\Delta t_2} - F_{s_2,t_2}}{\Delta t_2}\right]$$

$$= \underset{\Delta t_1 \to 0}{Lim}\frac{1}{\Delta t_1}\underset{\Delta t_2 \to 0}{Lim}\left[\frac{c(s_1 \wedge s_2, t_1+\Delta t_1, t_2+\Delta t_2) - c(s_1 \wedge s_2, t_1+\Delta t_1, t_2)}{\Delta t_2} - \frac{c(s_1 \wedge s_2, t_1, t_2+\Delta t_2) - c(s_1 \wedge s_2, t_1, t_2)}{\Delta t_2}\right]$$

$$= \underset{\Delta t_1 \to 0}{Lim}\frac{1}{\Delta t_1}\left[\frac{\partial c(s_1 \wedge s_2, t_1+\Delta t_1, t_2)}{\partial t_2} - \frac{\partial c(s_1 \wedge s_2, t_1, t_2)}{\partial t_2}\right]$$

$$= \frac{\partial^2 c(s_1 \wedge s_2, t_1, t_2)}{\partial t_1 \partial t_2} = \int_0^{s_1 \wedge s_2} \frac{\partial^2 g(t_1 - u, t_2 - u)}{\partial t_1 \partial t_2} du .$$

The covariance function has a similar functional form to that of Proposition 1. Therefore the proof of Proposition 1 can be used here also to provide the result that $\frac{dF_{s,t_1}}{dt_1}$ is continuous when $\frac{\partial^3 c(s,t_1,t_2)}{\partial s \partial t_1 \partial t_2} = \frac{\partial^2 g(t_1 - s, t_2 - s)}{\partial t_1 \partial t_2}$ is continuous and bounded. **QED.**

Propositions 1 and 2 provide constraints on the function $g(u,v)$ that we are going to fit to caps and swaptions prices. There is a compromise between the ease with which $g(u,v)$ can be fitted and an ideal. With the aim of simplifying the estimation of the covariance function we make the following assumption:

Assumption 2: *$g(u,v)$ is symmetric and piecewise-triangular: For node times t_i, $i = 1..n$, $t_i < t_j$ for $i < j$, the four corners (t_i,t_j), (t_{i+1},t_j), (t_i,t_{j+1}) and (t_{i+1},t_{j+1}) of the surface $g(u,v)$ define two piecewise triangular sections of the surface with the triangles being joined along the line running from (t_i,t_j) to (t_{i+1},t_{j+1}).*

Assumption 2 allows us to calibrate the stationary Kennedy model much more quickly than otherwise. It is non-parametric and sufficiently flexible to approximate any continuous surface well. However, it does not meet the criteria of Proposition 2 for a smooth term structure of forward rates. However, this is more of a theoretical rather than a practical problem for we can assume that the edges where the triangular planes of $g(u,v)$ meet are rounded off to provide continuous and bounded second derivatives. We will see in Section 4 that the rounding off of the edges have little effect on caps and swaptions prices since those prices depend on the volume beneath $g(u,v)$. The HJM approximation prescribed in Section 3 will give the rounded edges. When we fit the function in Section 6 we impose the condition that the matrix $G = [g(t_i,t_j)]_{i,j=1..n}$ be positive definite[1]. We calibrate the Kennedy covariance function of Assumption 1 to market caps and swaptions prices in Section 6.

3. IMPLIED COVARIANCE OF ZERO-RATE CHANGES AND HJM CALIBRATION

It will be interesting to see what the implied covariances of zero-rate changes would be when we have fitted Kennedy to market caps and swaptions prices. This section also explains how we can extract the volatility structures for the HJM approximation to the fitted Kennedy model.

The t-s maturity zero-rate at time s is given by

$$Y_{s,t} = \frac{1}{t-s}\int_s^t F_{s,u}du.$$

For any $t_1, t_2 \geq s$ we have

$$Cov\left[dY_{s,t_1}, dY_{s,t_2}\middle|\Im_s\right] = \frac{1}{(t_1-s)(t_2-s)} Cov\left[\int_s^{t_1} dF_{s,u}du, \int_s^{t_2} dF_{s,v}dv \middle| \Im_s\right]$$

$$= \frac{1}{(t_1-s)(t_2-s)} \int_s^{t_1}\int_s^{t_2} Cov[dF_{s,u}, dF_{s,v}|\Im_s]dvdu$$

$$= \frac{1}{(t_1-s)(t_2-s)} \int_s^{t_1}\int_s^{t_2} \frac{\partial}{\partial s^+}c(s,u,v)dvdu$$

$$= \frac{1}{(t_1-s)(t_2-s)} \int_s^{t_1}\int_s^{t_2} g(u-s, v-s)dvdu$$

$$= \frac{1}{\tau_1\tau_2}\int_0^{\tau_1}\int_0^{\tau_2} g(u,v)dvdu \qquad (3.1)$$

where $\tau_1 = t_1 - s$ and $\tau_2 = t_2 - s$ are the maturities of the two yields considered. The volatility term structure is stationary. Our choice of the covariance function is motivated by this result. It also allows for a very simple HJM approximation to the fitted Kennedy. Suppose we want to construct the HJM volatilities at n distinct maturities, τ_i, $i = 1..n$, $\tau_i > \tau_j$ if $i > j$. Then the HJM approximation is characterised by

$$\sigma_i(\tau_j) = \sqrt{\Lambda_{ii}} M_{ji}$$

where $\sigma_i(\tau_j)$ is the ith stationary volatility factor for a τ_j maturity zero-rate; Λ and M are respectively the eigenvalues and eigenvectors of the covariance matrix C where

$$C_{ij} = Cov\left[dY_{0,\tau_i}, dY_{0,\tau_j} \middle| \Im_0\right] = \frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} g(u,v) dv du. \tag{3.2}$$

The volatility factors can be completed by interpolating, for example using cubic-splines, between the chosen maturities and the accuracy of the approximation improved by increasing n. Our approach is related to the use of PCA in some popular implementations of Gaussian HJM models. Our approach conducts a PCA on the implied covariance matrix of zero-rate changes whereas in more common approaches, a PCA is conducted on a covariance matrix of historical zero-rate changes. The latter implementations have not been successful at matching market cap and swaption prices as discussed earlier. Our calibration procedure provides volatility structures that are implied from market caps and swaptions prices. Typically, we can choose a large number for n to ensure the HJM approximation is good. No numerical tractability is lost since the HJM approximation will typically have at most three significant factors. We believe our procedure is preferable to working with the HJM model directly. We do not have to pre-specify the functional form for the volatilities and do not have to pre-specify the number of volatility factors required.

It is possible to extend the flexibility of the Kennedy covariance function further. If the covariance function is assumed to be

$$c(s,t_1,t_2) = \int_0^s f(u) g(t_1 - u, t_2 - u) du$$

where $f(s)$ is a non-negative function, then it follows that equation 3.1 becomes

$$Cov\left[dY_{s,t_1}, dY_{s,t_2} \middle| \Im_s\right] = \frac{f(s)}{\tau_1 \tau_2} \int_0^{\tau_1} \int_0^{\tau_2} g(u,v) dv du$$

where τ_1 and τ_2 are defined as before. The zero-rate volatility factors maintain their shapes through time but the volatility levels are permitted to shift up and down to reflect changing volatility levels. The volatility term structures are still given by the eigenvectors of the matrix C defined by equation 3.2 and the approximating HJM model is characterised by

$$\sigma_i(s, \tau_j) = \sqrt{f(s)\Lambda_{ii}} M_{ji}$$

where $\sigma_i(s,\tau_j)$ is the ith volatility factor for a τ_j zero-rate at time s.

4. THE PRICING OF CONTINGENT CLAIMS

As shown by Geman, El Karoui and Rochet (1995) and Jamshidian (1989), it is now well known that for the purpose of deriving derivative pricing formulae, a change of numeraire or probability measure can simplify the process greatly. For the pricing of caps and swaptions, it is convenient to use a suitably chosen pure discount bond, (PDB), as the numeraire. The standard results of Harrison and Kreps (1979) and Harrison and Pliska (1981) then allow us to use the equivalent martingale measure (EMM) that renders prices measured in our chosen numeraire a martingale to price contingent claims consistently.

We use $P_{s,t}$ or $P(s,t)$ to denote the time s price of a PDB which matures at time t. When the chosen numeraire is a t maturity PDB, we call the EMM the t-measure and use $E^t[]$ to indicate expectations taken with respect to the t-measure. Note that changing the numeraire does not change the forward rate covariances. Therefore the Kennedy covariance function is invariant in the following propositions.

Proposition 3: *The following two statements are equivalent:*

a) *For the chosen numeraire,* P_{s,t_1} *, the process* $\{ P_{s,t_2} / P_{s,t_1}, \Im_s, 0 \leq s \leq t_1 \leq t_2\}$ *is a martingale;*

b) $\mu_{s_1,t_2} = \mu_{s,t_2} + \int_{t_1}^{t_2} c(s_1,t_2,v) - c(s,t_2,v)dv$ *for* $0 \leq s \leq s_1 \leq t_1 \leq t_2$.

Proof: We have from the definition of forward rates that

$$\frac{P_{s_1,t_2}}{P_{s_1,t_1}} = \frac{P_{s,t_2}}{P_{s,t_1}} \left\{ \exp - \int_{t_1}^{t_2} F_{s_1,u} - F_{s,u} du \right\} \text{ where } s \leq s_1$$

so that

$$E^{t_1}\left[\frac{P_{s_1,t_2}}{P_{s_1,t_1}} \middle| \Im_s \right] = \frac{P_{s,t_2}}{P_{s,t_1}} E^{t_1}\left[\exp - \int_{t_1}^{t_2} F_{s_1,u} - F_{s,u} du \middle| \Im_s \right] = \frac{P_{s,t_2}}{P_{s,t_1}} E^{t_1}\left[\exp - \int_{t_1}^{t_2} F_{s_1,u} - F_{s,u} du \right]$$

where the second equality follows since the random variable $\{ F_{s_1,u} - F_{s,u}, s \leq s_1 \leq u\}$ is independent of $\Im_s$. Now since

$$E^{t_1}\left[-\int_{t_1}^{t_2} F_{s,u} - F_{s_1,u} du \right] = \int_{t_1}^{t_2} \mu_{s_1,u} - \mu_{s,u} du$$

and

$$Var\left[\int_{t_1}^{t_2} F_{s_1,u} - F_{s,u} du \right] = \int_{t_1}^{t_2}\int_{t_1}^{t_2} c(s_1,u,v) - c(s,u,v) dv du$$

it follows that the process $\{ P_{s,t_2} / P_{s,t_1}, \Im_s, 0 \leq s \leq t_1 \leq t_2\}$ is a martingale when

$$\int_{t_1}^{t_2} \mu_{s_1,u} - \mu_{s,u} du = \int_{t_1}^{t_2}\int_{t_1}^{u} c(s_1,u,v) - c(s,u,v) dv du$$

since $c(.,u,v)$ is symmetric in u and v. Differentiating with respect to t_2 gives (b).

Now consider $\frac{P_{s_1,t_2}}{P_{s_1,t_1}} = \exp - \int_{t_1}^{t_2} F_{s_1,u} du$. Using (b) we have that

$$E^{t_1}\left[\int_{t_1}^{t_2} F_{s_1,u} du \middle| \Im_s \right] = \int_{t_1}^{t_2} F_{s,u} du + \int_{t_1}^{t_2}\int_{t_1}^{u} c(s_1,u,v) - c(s,u,v) dv du$$

and

$$Var\left[\int_{t_1}^{t_2} F_{s_1,u} du \middle| \Im_s \right] = \int_{t_1}^{t_2}\int_{t_1}^{t_2} c(s_1,u,v) - c(s,u,v) dv du$$

so that

$$E^{t_1}\left[\frac{P_{s_1,t_2}}{P_{s_1,t_1}} \middle| \Im_s \right] = \exp\left\{-\int_{t_1}^{t_2} F_{s,u} du\right\} = \frac{P_{s,t_2}}{P_{s,t_1}}. \text{ QED.}$$

Proposition 4: *The following two statements are equivalent:*

a) *For the chosen numeraire, P_{s,t_2}, the process $\{ P_{s,t_1} / P_{s,t_2}, \Im_s, 0 \le s \le t_1 \le t_2\}$ is a martingale;*

b) $\mu_{s_1,t_1} = \mu_{s,t_1} - \int_{t_1}^{t_2} c(s_1,u,t_1) - c(s,u,t_1) du$ *for $0 \le s \le s_1 \le t_1 \le t_2$.*

Proof: The proof follows from the same type of calculations as that of Proposition 3. **QED.**

4.1 Pricing of Caps

Kennedy (1994) derives the time $s \le t$ price of a caplet that has payoff at time $t + \delta$,

$$[\exp(\delta Y_{t,t+\delta}) - \exp(\delta k)]^+,$$

where

$$Y_{t,t+\delta} = \frac{1}{\delta}\int_{t}^{t+\delta} F_{t,u}du$$

is the δ maturity zero-rate at time t and k is the cap rate by taking expectations with respect to the risk-neutral measure. We derive the same formula to illustrate that a well chosen numeraire can simplify the calculations greatly. We chose as numeraire the PDB with maturity $t + \delta$. Thus the value of the caplet at time $s \leq t$ is given by the expectation

$$cpt(s) = P(s,t+\delta)E^{t+\delta}\left[\left(\exp\int_{t}^{t+\delta} F_{t,v}dv - \exp(k\delta)\right)^{+}\middle|\Im_s\right]$$

Proposition 4 gives

$$\mu_{t,u} = \mu_{s,u} - \int_{u}^{t+\delta} c(t,v,u) - c(s,v,u)dv$$

so that

$$\begin{aligned} E^{t+\delta}\left[\int_{t}^{t+\delta} F_{t,u}du \middle| \Im_s\right] &= \int_{t}^{t+\delta} F_{s,u}du - \int_{t}^{t+\delta}\int_{u}^{t+\delta} c(t,v,u) - c(s,v,u)dvdu \\ &= \int_{t}^{t+\delta} F_{s,u}du - \int_{t}^{t+\delta}\int_{t}^{u} c(t,v,u) - c(s,v,u)dvdu \end{aligned}$$

where the second equality follows since $c(\cdot,v,u)$ is symmetric in u and v and

$$Var\left[\int_{t}^{t+\delta} F_{t,u}du \middle| \Im_s\right] = \int_{t}^{t+\delta}\int_{t}^{t+\delta} c(t,v,u) - c(s,v,u)dvdu = \sigma^2 .$$

Taking expectations now gives

$$cpt(s) = P(s,t)N(d_1) - P(s,t+\delta)e^{\delta k}N(d_2)$$

where

$$d_1 = \frac{\ln\left[\frac{P(s,t)}{P(s,t+\delta)}\right] - k\delta}{\sigma} + \frac{\sigma}{2}.$$

$$d_2 = d_1 - \sigma$$

Note that the market convention in London is to quote rates on a quarterly basis with payoff at time $t + ¼$ of $¼\left[f_t^{t,t+¼} - k\right]^+$ where $f_t^{t,t+¼}$ is the floating rate for borrowing or lending over the period $[t, t + ¼]$ and k the cap rate both expressed quarterly. However since

$$f_t^{t,t+¼} = 4\left[\exp\left(\frac{F_t^{t,t+¼}}{4}\right) - 1\right]$$

then denoting k' for the cap rate in continuous terms we have that the

$$\text{payoff} = \left[\exp\left(\frac{F_t^{t,t+¼}}{4}\right) - \exp\left(\frac{k'}{4}\right)\right]^+$$

with $k' = 4\ln(1 + k/4)$. Thus we can use the caplet formula provided we convert the cap rate to continuous compounding.

4.2 Pricing of European Swaptions

In the London market, the settlement value for an European Payer Swaption, with maturity s, on an n year USD swap is given by

$$[w(s) - k]^+ \sum_{i=1}^{n} \frac{1}{[1 + w(s)]^i} \tag{4.1}$$

where $w(s)$ is the swap rate at time s for an n year USD swap and k is the strike rate. We do not have an exact closed form formula for the price of the swaption above but the next proposition allows us to estimate the price efficiently by numerically evaluating the expectation using Quasi Monte Carlo Methods as in Joy, Boyle and Tan (1995). Proposition 5 allows us to test the accuracy of the approximate European swaption pricing formula of Proposition 6. Let the time s_i denote $s + i$.

Proposition 5: *The time u price of an European payer swaption with maturity $s \geq u$ and strike k on an year swap with the payoff*

$$\left[w(s)-k\right]^{+}\sum_{i=1}^{n}\frac{1}{\left[1+w(s)\right]^{i}}$$

is given by

$$Swpt(u) = P(u,s)E^{s}\left[\left[w(s)-k\right]^{+}\sum_{i=1}^{n}\frac{1}{\left(1+w(s)\right)^{i}}\middle|\Im_{u}\right] \tag{4.2a}$$

where

$$w(s) = \frac{1-P(s,s_n)}{\sum_{j=1}^{n}P(s,s_j)}; \tag{4.2b}$$

Defining X_i such that

$$P(s,s_i) = exp(X_i), \tag{4.2c}$$

then under the s-measure, the random variable $\{X_i \mid \Im_u\}$ is normally distributed with mean μ_i and variance σ_i^2 given by

$$\mu_i = \ln\left[\frac{P(u,s_i)}{P(u,s)}\right] - \frac{1}{2}\sigma_i^2 \tag{4.2d}$$

and

$$\sigma_i^2 = \int_s^{s_i}\int_s^{s_i} c(s,a,b) - c(u,a,b)dbda. \tag{4.2e}$$

Moreover

$$Corr(X_i, X_j \mid \Im_u) = \frac{\int_s^{s_i}\int_s^{s_j} c(s,a,b) - c(u,a,b)dbda}{\sqrt{\int_s^{s_i}\int_s^{s_i} c(s,a,b) - c(u,a,b)dbda \int_s^{s_j}\int_s^{s_j} c(s,a,b) - c(u,a,b)dvdu}}. \tag{4.2f}$$

Proof: Brace and Musiela (1994) show that the swap rate on an n year USD swap at time s would be given by

$$w(s) = \frac{1 - P(s, s_n)}{\sum_{j=1}^{n} P(s, s_j)}.$$

It follows that, under the *s*-measure induced by taking the time *s* maturity PDB as the numeraire, the value of the payer swaption at any time $u \le s$ is given by the expectation of equation 4.2a. The distribution of $\{P(s,s_i) \mid \Im_u\}$ and hence $\{X_i \mid \Im_u\}$ follows from using Proposition 3 and simplifying to give equations 4.2d and 4.2e. Equation 4.2f follows from using

$$Cov[X_i, X_j | \Im_u] = \int_s^{s_i} \int_s^{s_j} c(s,a,b) - c(u,a,b) dbda \qquad \textbf{QED.}$$

The proposition allows us to value European Payer Swaptions using

$$Swpt(0) = \sum_{j=1}^{N} Swpt_j(0)$$

where the *jth* sample value is given by

$$Swpt_j(0) = P(0,s)\left[w(s) - k\right]^+ \sum_{i=1}^{n} \frac{1}{\left[1 + w(s)\right]^i}$$

with $w(s)$ of equation 4.2b calculated using

$$\left\{P(s,s_i)\middle|\Im_0\right\} = \exp\left[\mu_i + \sigma_i \varpi_i(j)\right]$$

where μ_i and σ_i are given by equations 4.2d and 4.2e respectively, $\varpi_i(j)$ is the *i*th element of $\underline{\varpi}(j)$ which is the *j*th realisation of a random vector of standard normal random variates with $Corr(\varpi_i, \varpi_k) = Corr(X_i, X_k \mid \Im_0)$ as given by equation 4.2f, and N is the total number of sample values. We generate the correlated normal variates, $\underline{\varpi}(j)$, using the transformation $\underline{\varpi}(j) = M\Lambda^{1/2}\underline{\varepsilon}(j)$ where $\underline{\varepsilon}(j)$ is a sample random vector of uncorrelated standard normal variates, and M and Λ are respectively matrices with the eigenvectors and eigenvalues of the required correlation matrix. $\Lambda^{1/2}$ is a diagonal matrix with the elements set equal to the square root of the respective elements in Λ. The standard uncorrelated normal variates are generated from an n-dimensional Faure sequence. Joy, Boyle and Tan (1995) shows that the estimate has an error of $O(1/N)$.

4.3 An Approximate European Swaption Pricing Formula

To derive our approximate formula we make two assumptions. The first assumes, at maturity s, the payer swaption has payoff given by

$$[w(s)-k]^{+}\sum_{i=1}^{n}P(s,s_i) \tag{4.3}$$

instead of the USD market convention of equation 4.1 where $w(s)$ is the swap rate for an n year USD swap and $s_i = s + i$. Equation 4.1 discounts future cashflows with the time s par yield of an n year coupon bond whereas equation 4.3 discounts the cashflows with the time s zero-rates. With this simplifying assumption, a result of Brace and Musiela (1994) allows us to write the payoff as

$$\left[1-\sum_{i=1}^{n}k_i P(s,s_i)\right]^{+}$$

where $k_i = k$ for $i = 1..n\text{-}1$ and $k_n = 1{+}k$. Thus the payoff to the payer swaption is the same as that of a put option on a coupon bond with strike one. We next assume that the underlying coupon bond at the maturity of the swaption is lognormally distributed. The two assumptions allow Proposition 6:

Proposition 6: *An approximate formula for the time u price of an European Payer Swaption with maturity $s \geq u$ and strike rate k on an n year swap with the maturity payoff*

$$[w(s)-k]^+\sum_{i=1}^{n}\frac{1}{[1+w(s)]^i}$$

is given by

$$P(u,s)\left\{N\left[-\frac{a}{b}\right]-e^{a+\frac{b^2}{2}}N\left[-\frac{a+b^2}{b}\right]\right\} \tag{4.4a}$$

where

$$a=\ln\left[\frac{m}{\sqrt{1+v/m^2}}\right], \tag{4.4b}$$

$$b^2=\ln\left[1+v/m^2\right], \tag{4.4c}$$

$$m=\sum_{i=1}^{n}\frac{k_iP(u,s_i)}{P(u,s)}, \tag{4.4d}$$

$$v=\underline{K}^TC\underline{K}, \tag{4.4e}$$

$$\underline{K}^T=[k\ \ k\ \ \cdots\ \ k\ \ 1+k] \tag{4.4f}$$

k_i is the ith element of the vector $\underline{K}$ and C is an n x n matrix with the element (i,j) given by

$$C_{ij}=\frac{P(u,s_i)P(u,s_j)}{P(u,s)^2}\left\{\exp\left(\int_s^{s_i}\int_s^{s_j}c(s,x,y)-c(u,x,y)dydx\right)-1\right\}. \tag{4.4g}$$

Proof: Under the s-measure and with our simplifying assumptions, the time u price of the swaption is given by

$$payer(u)=P(u,s)E^s\left[\left(1-\sum_{i=1}^{n}k_iP(s,s_i)\right)^+\middle|\Im_u\right]$$

Under the s-measure, we have

$$E^s[P(s,s_i)|\Im_u]=\frac{P(u,s_i)}{P(u,s)}$$

$$Cov\left[P(s,s_i),P(s,s_j)\middle|\Im_u\right]=\frac{P(u,s_i)P(u,s_j)}{P(u,s)^2}\left\{\exp\left(\int_s^{s_i}\int_s^{s_j}c(s,a,b)-c(u,a,b)dbda\right)-1\right\}$$

so that the mean and variance of the coupon bond at time s is given by m and v, defined by equations 4.4d and 4.4e respectively. As the coupon bond price is assumed to be lognormal, we can write

$$\left\{\sum_{i=1}^{n}P(s,s_i)\middle|\Im_u\right\}=e^Z,Z\sim N(a,b^2).$$

Matching the means and variance of the distribution gives a and b defined by equations 4.4b and 4.4c respectively. Finally evaluating $P(u,s)E^s\left[(1-e^Z)^+|\Im_u\right]$ gives the approximate formula, equation 4.4a. **QED.**

Corollary: *An approximate formula for the time u price of an European Receiver Swaption with maturity $s\geq u$ and strike k on an n year swap with the maturity payoff*

$$\left[k-w(s)\right]^+\sum_{i=1}^{n}\frac{1}{\left[1+w(s)\right]^i}$$

is given by

$$P(u,s)\left\{e^{a+\frac{b^2}{2}}N\left[\frac{a+b^2}{b}\right]-N\left[\frac{a}{b}\right]\right\}$$

where a and b are as defined in Proposition 6.

Proof: Follows from recognising that the receiver swaption is given by

$$P(u,s)E^s\left[(e^Z-1)^+|\Im_u\right]$$

with Z as defined in Proposition 6, and that the expectation can be obtained by making the same changes one makes to an European Put formula to obtain an European Call on a lognormal underlying. Alternatively, evaluate the expectations directly. **QED**.

The approximate swaption pricing formula of Proposition 6 allows us to calibrate Kennedy far more quickly than would have been possible using a numerical integration. The approximation performs very well for close to the money swaptions. For typical covariance structures, the differences between the two prices are very small. To illustrate this, we use the implied covariance function of 31st May 1996 in Section 6, shown in Table 5. Table 1 compares the European Payer Swaption prices produced by a numerical integration of Proposition 5 with those by the approximation formula of Proposition 6. The numerical integration uses one million samples so the errors are of $O(10^{-6})$. The approximation formula of Proposition 6 performs well for near the money swaptions. This result is not unexpected as the distribution of the coupon bond is the convolution of a number of lognormal distributions. For small coupons, the present value of the principal dominates and so the distribution of the coupon bond would be approximately lognormal. We can conclude that since quoted swaptions are at-the-money, the approximation formula would be appropriate for the calibration in Section 6. Note that the approximation would perform even better for swaptions with maturity payoffs defined by equation 4.3. Table 1 took about two hours to produce on a UNIX terminal, so clearly calibrating with the numerical integration to market prices would be unfeasible.

We are now ready to proceed with the calibration. Before we present the results of the calibration we review our data.

5. DATA

Our data set was kindly provided by Martin Cooper of Tokai Bank Europe, London. The data consists of contemporaneous money market rates and 2, 3, 5, 7 and 10 year swap rates, for a variety of currencies, together with quoted Black (1976) volatilities for at-the-money caps and swaptions for much of the period 21st June 1995 - 31st May 1996. In more detail, the data set provide Black (1976) volatilities for 1, 2, 3, 4, 5, 7 and 10 years caps, and Black (1976) volatilities for swaptions with maturities 3 months, 6 months, 1, 2, 3, 4, and 5 years on 1, 2, 3, 4, 5, 7 and 10 years swaps. We only use the USD data.

6. CALIBRATION TO CAPS AND SWAPTIONS PRICES

We fit to caps and swaptions prices by optimising $g(u,v)$ at the points $(u,v) = \{0,2,4,6,8,10\}^2$ using standard optimisation software to minimise the residual sum of squared differences between model and market prices. The cap and approximation swaption pricing formulae of Section 4 depend on triple integrals of the form

$$\int_s^{s_i}\int_s^{s_j} c(s,x,y)dydx = \int_0^s \int_{s-u}^{s_i-u}\int_{s-u}^{s_j-u} g(x,y)dydxdu\,. \tag{6.1}$$

We can obtain an approximation to the integral rapidly using the following procedure. Our choice of caps and instruments imply that we only need to evaluate the integral for s, s_i and s_j that are multiples of a quarter. We interpolate $g(u,v)$ on $\{0, 2, 4, 6, 8, 10\}^2$ to provide values to $\{0, ¼, ½, ¾, 1, 1¼, \ldots, 9¾, 10\}^2$. This allows us to evaluate

$$I_s^{s_i,s_j}(u) = \int_{s-u}^{s_i-u}\int_{s-u}^{s_j-u} g(x,y)dydx$$

analytically when s, s_i, s_j and u are multiples of a quarter. The triple integral of equation 6.1 is approximated using the trapezium rule by evaluating $I_s^{s_i,s_j}(u)$ at $u = 0, ¼, ½, ¾, \ldots, s - ¼$, *s*. With this simplification for the triple integral, we find that the optimisation can be completed in about fifteen minutes on a 75Mhz Pentium PC.

We fit the covariance function simultaneously to market caps and swaptions prices for all dates in May 1996 for which we have data for: 1, 9, 10, 13, 14, 15, 16, 17 20, 21, 22, 23, 28, 29, 30 and 31. There were two national holidays in the UK in May 1996 and the missing dates are clustered around them. We fit to 2, 3, 4, 5, 7 and 10 year caps and 2x1, 2x2, 2x3, 2x4, 2x5, 2x7, 3x1, 3x2, 3x3, 3x4, 3x5, 3x7, 4x1, 4x2, 4x3, 4x4, 4x5, 5x1, 5x2, 5x3, 5x4 and 5x5 swaptions.

The fitted cap and swaptions prices for 31st May 1996 are shown in Tables 2 and 3. The price differences are small. Table 4 shows the Black volatilities for the Kennedy swaptions prices. Except for swaptions on one-year swaps, the fitted Kennedy prices give Black volatilities that are within bid-ask spreads of ± ¼ %. We minimised an equally weighted residual sum of squared pricing differences and because the swaptions on one-year swaps are worth little, their percentage errors are larger and so some Black volatilities are outside the bid-ask range. This can be easily remedied by adding extra weights to low valued swaptions. We can conclude the fit on 31st May is good. The fit for the other days of May 1996 are similarly good. Figure 1 shows how the residual sum of squares varied.

The fitted function $g(u,v)$ for 31st May 1996 is plotted in Figure 2 and tabulated in Table 5. We find that the fitted functions $g(u,v)$ maintain similar shapes throughout May 1996. Since $g(u,v)$ is the instantaneous covariance of changes to instantaneous forward rates of maturities u and v, some fluctuation through time would be expected. Figure 3 plots the square root of $g(u,u)$, $u = 0..10$ throughout May 1996. The figure shows the volatility term structures of instantaneous forward rates across May 1996. The forward rate volatilities term structure maintain a similar shape through time and often exhibit a humped structure often observed in the market. Rather than plotting $g(u,v)$ throughout May 1996, which would be difficult to compare, we examine the implied zero-rate covariance structure instead. We construct the matrix C defined by equation 3.2

$$C_{ij} = Cov\left[dY_{0,\tau_i}, dY_{0,\tau_j} \middle| \Im_0\right] = \frac{1}{\tau_i \tau_j} \int_0^{\tau_i} \int_0^{\tau_j} g(u,v) dv du \tag{3.2}$$

for τ_i, τ_j = ¼, ½, ¾, ... , 9¾, 10 and extract the eigenvalues and eigenvectors for each day in May 1996. The first three eigenvectors are plotted in Figure 4 and the fourth to the sixth are plotted in Figure 5. All six eigenvectors maintain similar shapes throughout May 1996. This is in marked contrast to a corresponding PCA on a historical covariance matrix that produces very unstable higher order eigenvectors. Table 6 shows that for 31st May 1996, the 1st factor account for 95.8% of the variation, the 2nd factor account for 3.2% and the 3rd account for only 0.8%. The first three factors affect zero-rates in the precisely the same way that the factors produced by a more traditional PCA decomposition of a historically estimated covariance matrix affect zero-rates. The first factor corresponds to a change in level; the second factor corresponds to a change of slope and the third corresponds to a change of curvature. Figure 6 shows how the first three eigenvalues varied in May 1996 and so the proportions of the zero-rate covariance structure explained by each of the first three factors are also stable. An HJM approximation taking just the first three factors would be sufficiently well calibrated to the cap and swaptions prices. This is illustrated in Tables 7 and 8. Table 7 shows the swaptions prices produced by a three-factor HJM approximation to the fitted Kennedy model. The three factor HJM prices are very close to both the market quotes and Kennedy prices. Thus we have a well calibrated HJM model. Table 8 shows the swaptions prices in terms of Black volatilities. All Black volatilities, except for the swaptions on one year swaps are within bid-ask spreads of ± ¼ %. The swaptions on one year swaps have a larger error for the same reason as explained earlier. Tables 7 and 8 also show prices and Black volatilities produced by an one-factor approximation to the fitted Kennedy. The one-factor HJM approximation does not price the swaptions badly. Black volatility errors are greatest for low valued swaptions for the same reason as explained above. Note that it would be possible to make an one factor Gaussian HJM model fit the prices better than the one-factor HJM approximation here. The optimisation routine optimises the fit for Kennedy and not an one factor HJM model. Indeed, Brace, Marek and Musiela (1995) show that one-factor Gaussian HJM models can be fitted to cap and swaption prices simultaneously. In the context of our approach, a one-factor fit would require us to constrain the Kennedy covariance function to be consistent with an one-factor model. This is in contrast to our aim of letting the covariance matrix be unconstrained. Table nine shows the implied correlation of instantaneous zero-rate changes.

7. SUMMARY

We have how Kennedy can be calibrated rapidly and accurately to quoted caps and swaptions prices using an approximate European Swaption pricing formula that we have shown to be accurate for near-the-money swaptions. Our approximate European Swaption pricing formula applies to other diffusion Gaussian interest rate models and so offer them quicker calibration than otherwise. We have shown how our calibrated Kennedy model can be approximated easily and accurately by a Gaussian multi-factor HJM. The 3-factor HJM approximation prices caps and swaptions consistently with the market. It also possesses attractive attributes; the volatility factors are stationary and stable between re-calibration across trading days in May 1996. Our indirect calibration of HJM using Kennedy as an intermediate step is superior to conventional methods.

8. APPENDIX

8.1 Hull White (1990) Covariance Function

Hull and White (1990) assume that $r(t)$ follows the risk-neutral process

$$dr(t) = [\phi(t) - a(t)r(t)]dt + \sigma(t)dz(t).$$

Using a deterministic time change for the Brownian motion $Z(t)$, it is possible to show that $r(t)$ has distribution

$$r(t) \sim c(t) + b(t)\tilde{Z}_{f(t)} \tag{A1}$$

where $\tilde{Z}_{f(t)}$ is another Brownian motion and

$$b(t) = e^{-\int_0^t a(u)du} \tag{A2}$$

$$c(t) = e^{-\int_0^t a(u)du} \int_0^t \phi(u) e^{-\int_0^u a(s)ds} du + r(0)e^{-\int_0^t a(u)du} \tag{A3}$$

$$f(t) = \int_0^t \left[\frac{\sigma(u)}{b(u)}\right]^2 du \tag{A4}$$

Hull and White show that pure discount bond prices are given by

$$P(t,T) = A(t,T)\exp(-B(t,T)r(t)) \tag{A5}$$

where

$$B(t,T) = \frac{B(0,T) - B(0,t)}{\partial B(0,t)/\partial t} \tag{A6}$$

$$B(0,t) = \frac{R(0,t)\sigma_R(0,t)t}{\sigma(0)} \tag{A7}$$

$$\ln A(t,T) = \ln\frac{A(0,T)}{A(0,t)} - B(t,T)\frac{\partial}{\partial t}\ln A(0,t) - \frac{1}{2}\left[B(t,T)\frac{\partial B(0,t)}{\partial t}\right]^2 \int_0^t \left[\frac{\sigma(\tau)}{\partial B(0,\tau)/\partial \tau}\right]^2 d\tau \tag{A8}$$

Equations (A5) and (A6) imply

$$Cov[F(s_1,t_1),F(s_2,t_2)] = \frac{\partial B(0,t_1)/\partial t_1}{\partial B(0,s_1)/\partial s_1}\frac{\partial B(0,t_2)/\partial t_2}{\partial B(0,s_2)/\partial s_2}Cov[r(s_1),r(s_2)]. \tag{A9}$$

Hull and White (1990) show

$$a(t) = -\frac{\partial^2 B(0,t)/\partial t^2}{\partial B(0,t)/\partial t} \tag{A10}$$

which gives by substituting equation (A10) into (A2)

$$b(t) = \exp\int_0^t \frac{\partial^2 B(0,u)/\partial u^2}{\partial B(0,u)/\partial u}du = \frac{\partial B(0,t)}{\partial t}. \tag{A11}$$

Equation (A1) gives

$$Cov[r(s_1),r(s_2)] = b(s_1)b(s_2)Cov[\tilde{Z}_{f(s_1)},\tilde{Z}_{f(s_2)}] = b(s_1)b(s_2)f(s_1 \wedge s_2) \tag{A12}$$

and substituting equations (A11), (A12) into equation (A9) finally gives

$$Cov[f(s_1,t_1),f(s_2,t_2)] = \frac{\partial B(0,t_1)}{\partial t_1}\frac{\partial B(0,t_2)}{\partial t_2}\int_0^{s_1 \wedge s_2}\left[\frac{\sigma(u)}{\partial B(0,u)/\partial u}\right]^2 du.$$

8.2 Continuity of the Gaussian Random Field

We have from Adler (1981) Theorem 3.4.1 that our real, zero-mean Gaussian Random Field with a continuous covariance function has, with probability one, continuous sample functions over I_0, if there exists some $0 < C < \infty$ and some $\varepsilon > 0$ such that

$$E\left|X(s_1,t_1) - X(s_2,t_2)\right|^2 \le \frac{C}{\left|\log\sqrt{(s_2-s_1)^2+(t_2-t_1)^2}\right|^{1+\varepsilon}}$$

where

$$I_0 = \{s_1, s_2, t_1, t_2 : (s_2 - s_1)^2 + (t_2 - t_1)^2 < 1, 0 \le s_1 \le t_1, 0 \le s_2 \le t_2\}.$$

We first proof that when $Cov(F_{s_1,t_1}, F_{s_2,t_2}) = c(s_1 \wedge s_2, t_1, t_2) = \int_0^{s_1 \wedge s_2} g(t_1 - u, t_2 - u)du$, $0 \le s_1 \le t_1 < 1$, $0 \le s_2 \le t_2 < 1$, of Assumption 1 the Gaussian Random Field has continuous sample functions, with probability one, when $g(t_1-u,t_2-u)$ is continuous and bounded.

We have by assumption $c(s,t_1,t_2)=\int_0^s g(t_1-u,t_2-u)du$.

Since $Var[dF_{s,t}|\Im_s]=Var[dF_{s,s+\tau}|\Im_s]=g(\tau,\tau)$, this implies that $g(\tau,\tau)\geq 0$.

Suppose $t_2 \geq t_1$ and $s_2 \geq s_1$. We have that $Var[F_{s_1,t_1}-F_{s_2,t_2}]$

$$=c(s_1,t_1,t_1)-2c(s_1,t_1,t_2)+c(s_2,t_2,t_2)$$

$$=\int_0^{s_1} g(t_1-u,t_1-u)du-2\int_0^{s_1} g(t_1-u,t_2-u)du+\int_0^{s_2} g(t_2-u,t_2-u)du$$

$$=\int_0^{s_1} g(t_1-u,t_1-u)du-2\int_0^{s_1} g(t_1-u,t_2-u)du+(1+k(s_1,s_2,t_2)(s_2-s_1))\int_0^{s_1} g(t_2-u,t_2-u)du$$

for some $\infty > k(s_1,s_2,t_2)\geq 0$ since $g(t_2-u,t_2-u)\geq 0$ and bounded by assumption

$$=k(s_1,s_2,t_2)(s_2-s_1)\int_0^{s_1} g(t_2-u,t_2-u)du+$$
$$\int_0^{s_1} g(t_1-u,t_1-u)+g(t_2-u,t_2-u)-2g(t_1-u,t_2-u)du$$

$$\leq k(s_1,s_2,t_2)(s_2-s_1)\int_0^{s_1} g(t_2-u,t_2-u)du.$$
$$+2\int_0^{s_1}\left[g(t_1-u,t_1-u)\vee g(t_2-u,t_2-u)\right]-g(t_1-u,t_2-u)du.$$

Now note firstly that the non-negative semi-definiteness of $g(u,v)$, implies

$$g(t_1-u,t_1-u)g(t_2-u,t_2-u)\geq g(t_1-u,t_2-u)^2$$

$$g(t_1-u,t_1-u)\vee g(t_2-u,t_2-u)\geq g(t_1-u,t_2-u)$$

which implies

$$g(t_1-u,t_1-u)\vee g(t_2-u,t_2-u)-g(t_1-u,t_2-u)$$

$$=g(t_1-u,t_1-u)\vee g(t_2-u,t_2-u)\frac{M(u,t_1,t_2)}{2}(t_2-t_1)$$

for some $\infty > M(u,t_1,t_2)\geq 0$ since $g(u,v)$ is continuous by assumption.

Thus

$$Var[F_{s_1,t_1} - F_{s_2,t_2}]$$

$$\leq k(s_1,s_2,t_2)(s_2 - s_1)\int_0^{s_1} g(t_2 - u, t_2 - u)du$$

$$+\int_0^{s_1} g(t_1 - u, t_1 - u) \vee g(t_2 - u, t_2 - u) M(u,t_1,t_2)(t_2 - t_1)du$$

$$\leq k(s_1,s_2,t_2)(s_2 - s_1)\int_0^{s_1} g(t_2 - u, t_2 - u)du$$

$$+ N(s_1,t_1,t_2)(t_2 - t_1)\int_0^{s_1} g(t_1 - u, t_1 - u) \vee g(t_2 - u, t_2 - u)du$$

where $N(s_1,t_1,t_2) = \underset{u \leq s_1}{Max}\, M(u,t_1,t_2)$

$$\leq \{k(s_1,s_2,t_2)(s_2 - s_1) + N(s_1,t_1,t_2)(t_2 - t_1)\}\int_0^{s_1} g(t_1 - u, t_1 - u) \vee g(t_2 - u, t_2 - u)du$$

$\leq P(s_1,s_2,t_1,t_2)[(s_2 - s_1) + (t_2 - t_1)]$ where $0 \leq P(s_1, s_2, t_1, t_2) < \infty$ and

$$P(s_1,s_2,t_1,t_2) = \{k(s_1,s_2,t_2) \vee N(s_1,t_1,t_2)\}\int_0^{s_1} g(t_1 - u, t_1 - u) \vee g(t_2 - u, t_2 - u)du$$

$$\leq P(s_1,s_2,t_1,t_2)(rCos\theta + rSin\theta)$$

$\leq Qr$ where $0 \leq Q < \infty$ and $Q = \underset{s_1,s_2,t_1,t_2}{Max}\, P(s_1,s_2,t_1,t_2)$.

Note that $r = \sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} < 1$, so we only need to show further that

$$Qr \leq \frac{C}{|\log r|^{1+\varepsilon}} \qquad \text{(CC)}$$

for some $0 < C < \infty$ and some $\varepsilon > 0$ for all $0 \leq r < 1$. Now

$$\underset{0 \leq r < 1}{\text{Min}}\left[\frac{d}{dr}\frac{C}{|\log r|^{1+\varepsilon}}\right] = \frac{C(1+\varepsilon)^{2+\varepsilon}}{(2+\varepsilon)^{2+\varepsilon}} > 0$$

so there exists some $0 < C < \infty$ such that

$$\frac{C(1+\varepsilon)^{2+\varepsilon}}{(2+\varepsilon)^{2+\varepsilon}} > Q$$

and so the proposition follows from observing that (CC) holds with equality at $r = 0$ and with strict inequality for $0 < r < 1$. Exchanging t_2 for t_1 and or s_2 for s_1 and adjusting the definition of r and or θ appropriately leads to the same conclusion.

To extend the range over which the Proposition applies, we only need to find some T, $\infty > T > max(t_1,t_2)$. Then we can scale time by $1/T$ and the proposition applies. **QED**.

9. REFERENCES

1. Adler R.J., 1981, *The Geometry of Random Fields*, (New York: Wiley).
2. Black F., March 1976, "The Pricing of Commodity Contracts", *Journal of Financial Economics* 3.
3. Black F., Derman E. and Toy W., 1990, "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options", *Financial Analysts Journal* January-February 1990, 33-39.
4. Black F. and Karasinski P., 1991, "Bond and Option Pricing when Short Rates are Lognormal", *Financial Analysts Journal* July-August 1991, 52-59.
5. Brace A. and Musiela M., 1994, "Swap Derivatives in a Gaussian HJM Framework", *Working Paper, School of Mathematics, UNSW.*
6. Brace A., Gatarek D. and Musiela M., 1995, "The Market Model of Interest Rate Dynamics", *Working Paper, USNW.*
7. Carverhill A. and Pang K., September 1995, "Efficient and Flexible Bond Option Valuation in the Heath, Jarrow and Morton Framework", *Journal of Fixed Income* Vol 5 No 2.

8. Geman H., El Karoui N., Rochet J-C., 1995, "Changes of Numeraire, Changes of Probability Measure and Option Pricing", *J. Appl. Prob.* 32, 443-458.
9. Harrison J.M. and Kreps D.M., 1979, "Martingales and Arbitrage in Multiperiod Securities Markets", *J. Econ. Theory* 20, 381-408.
10. Harrison J.M. and Pliska S.R., 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stoch. Proc. Appl.* 11, 215-260.
11. Heath D.C., Jarrow R.A. and Morton A., 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", *Econometrica* 60, 77-105.
12. Hull J. and White A., 1990, "Pricing Interest-Rate Derivative Securities", *Review of Financial Studies* 3, 573-592.
13. Hull J. and White A., 1993, "One-Factor Interest Rate Models and the Valuation of Interest-Rate Derivative Securities", *Journal of Financial and Quantitative Analysis* 28, 235-254.
14. Hull J. and White A., 1994, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models", *The Journal of Derivatives* Winter 1994, 37-48.
15. Jamshidian F., March 1989, "An Exact Bond Option Formula", *Journal of Finance* Vol XLIV No 1.
16. Joy C., Boyle P.P. and Tan K.S., February 1995, "Quasi Monte Carlo Methods in Numerical Finance", *Working Paper, University of Waterloo.*
17. Kennedy D.P., 1995, "Characterizing and Filtering Gaussian Models of the Term Structure of Interest Rates", *Working Paper, University of Cambridge.*
18. Kennedy D.P., July 1994, "The Term Structure of Interest Rates as a Gaussian Random Field", *Mathematical Finance* Vol 4 No 3, 247-258.

19. Langetieg T.C., 1980, "A Multivariate Model of the Term Structure", *Journal of Finance* Vol XXXV No 1, 71-97.

20. Press,W.H., Teukolsky S.A., Vetterling W.T. and Flannery B.P., 1992, *Numerical Recipes in C*, (Cambridge University Press).

21. Rebonato R. and Cooper I., 1996, "The Limitations of Simple Two-Factor Interest Rate Models", *Journal of Financial Engineering* Vol 5 No 1.

22. Vasicek O., 1977, "An Equilibrium Characterization of the Term Structure", *J. Financial Econ.* 5, 177-188.

10. FOOTNOTES

1 We do not know whether the interpolation rule of Assumption 2 and the positive definiteness of G are sufficient conditions for the positive definiteness of the fitted function $g(u,v)$. The function $g(u,v)$ generated by the approximating HJM model of Section 3 will however be positive definite.

Figure 1: Residual Sum of Squares

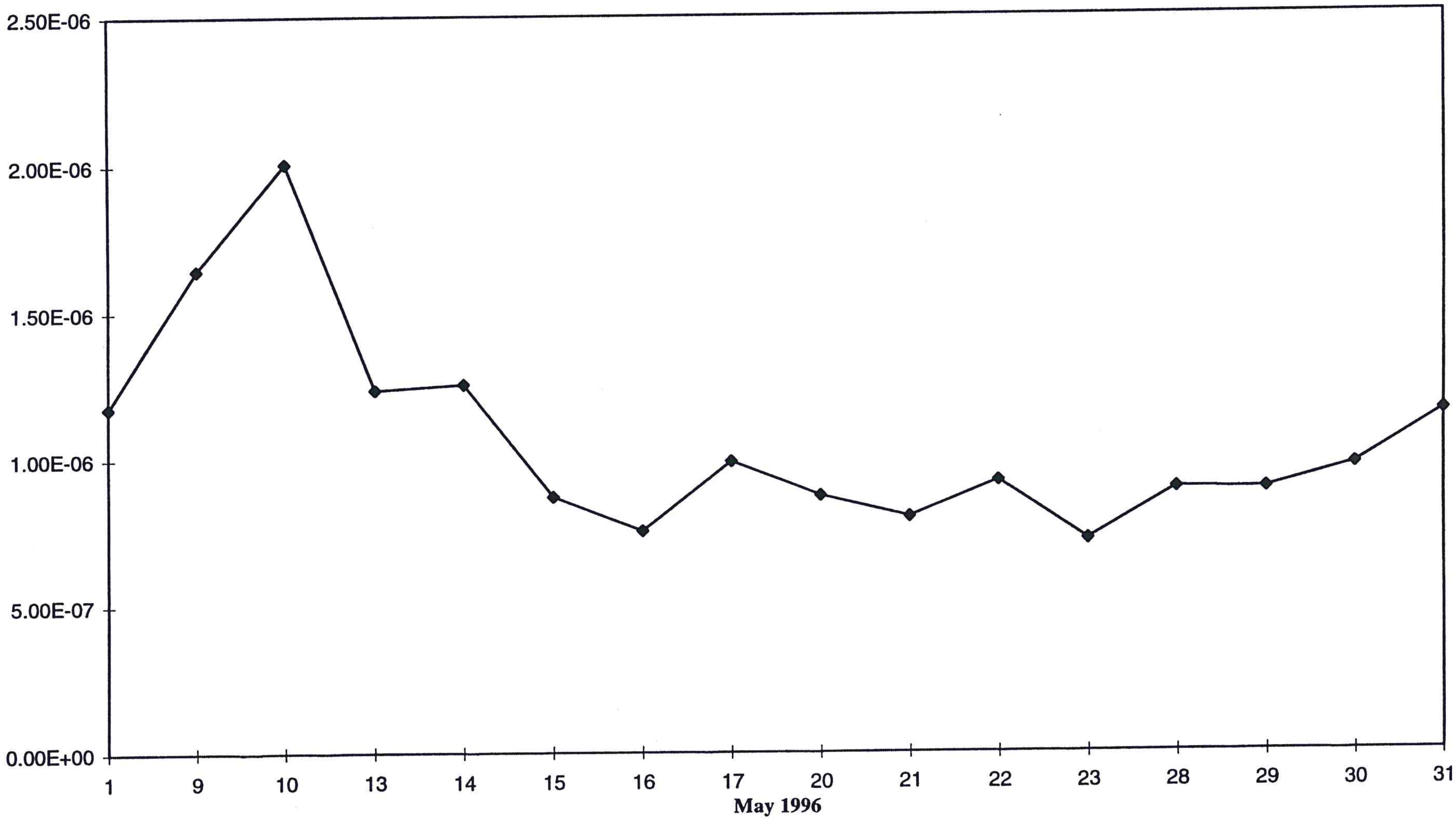


Figure 2: g(u,v), 31st May 1996

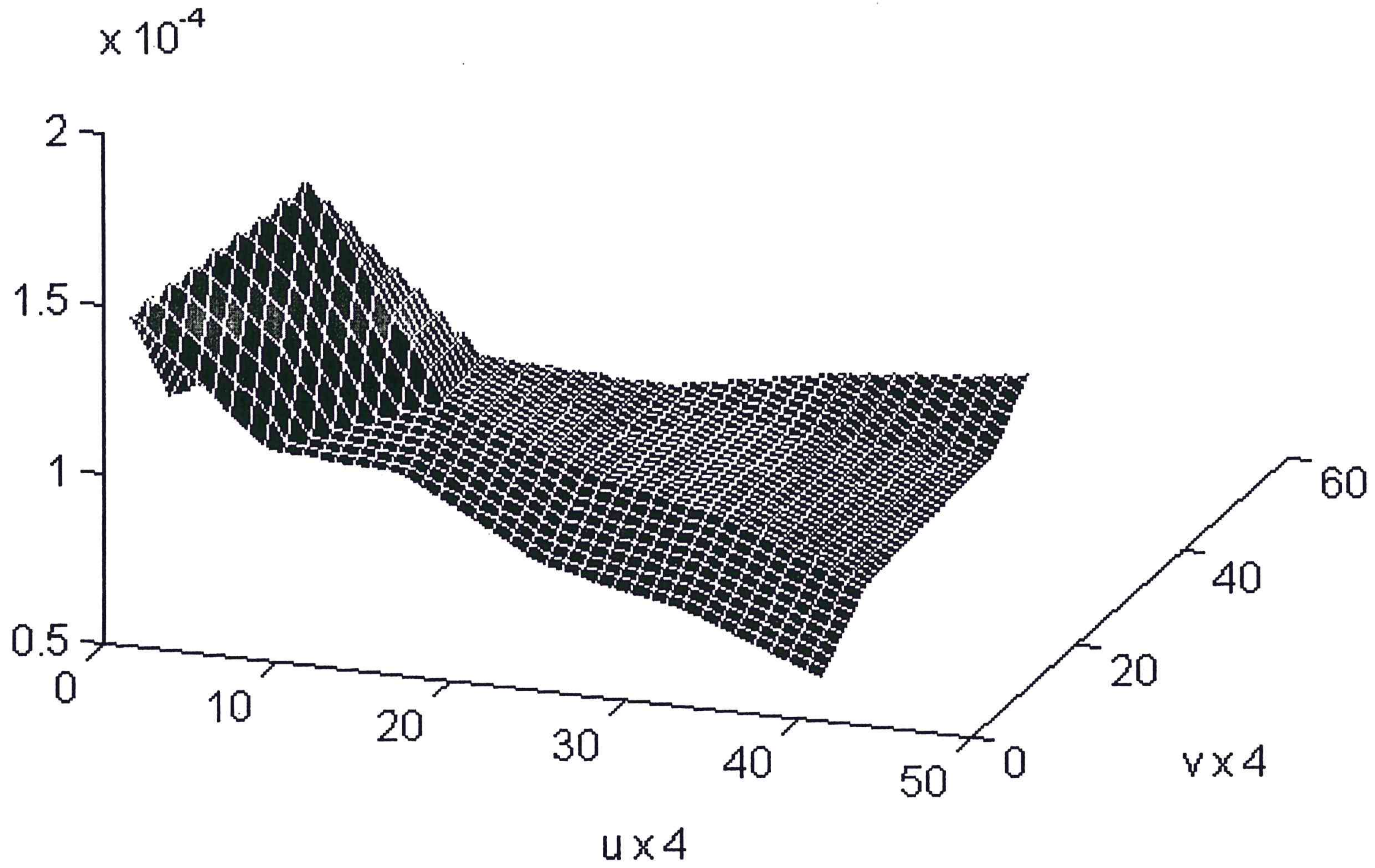


Figure 3: Implied Instantaneous Forward Rate Volatilities
1, 9, 10, 13, 14, 15, 16, 17, 20, 21, 22, 23, 28, 29, 30, 31 May 1996.

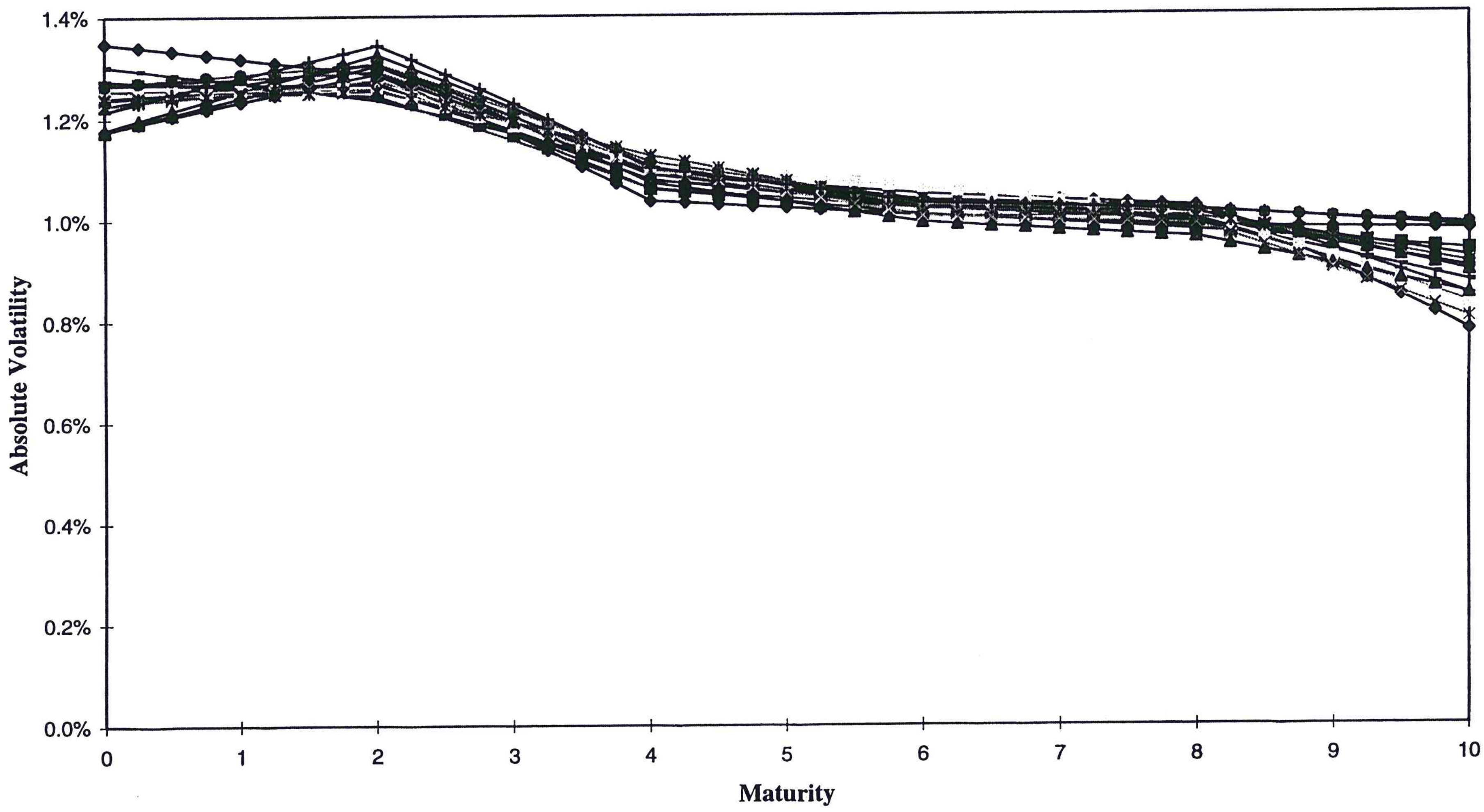


Figure 4: Implied Zero-Rate Factors, 1st - 3rd
1, 9, 10, 13, 14, 15, 16, 17, 20, 21, 22, 23, 28, 29, 30, 31 May 1996.

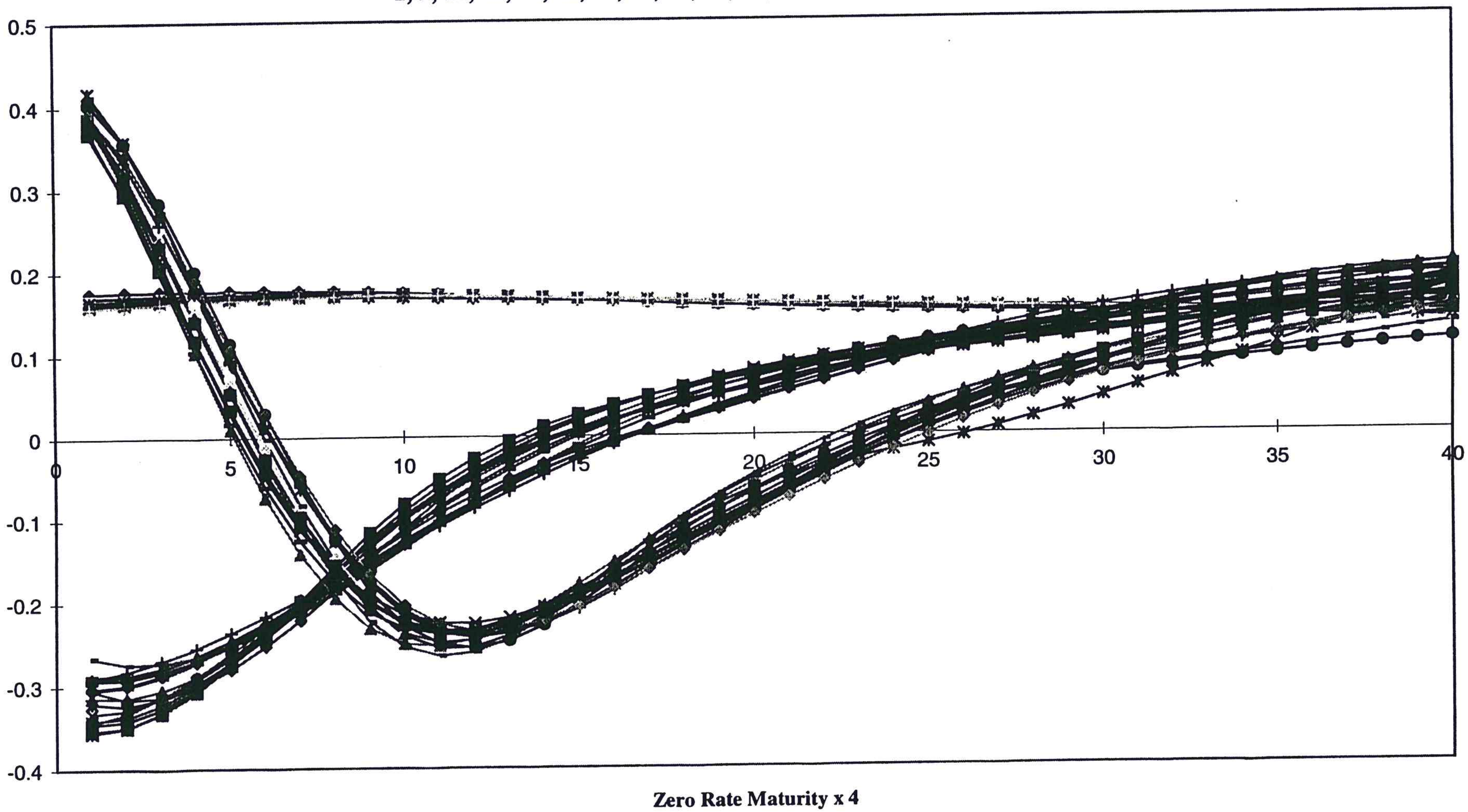
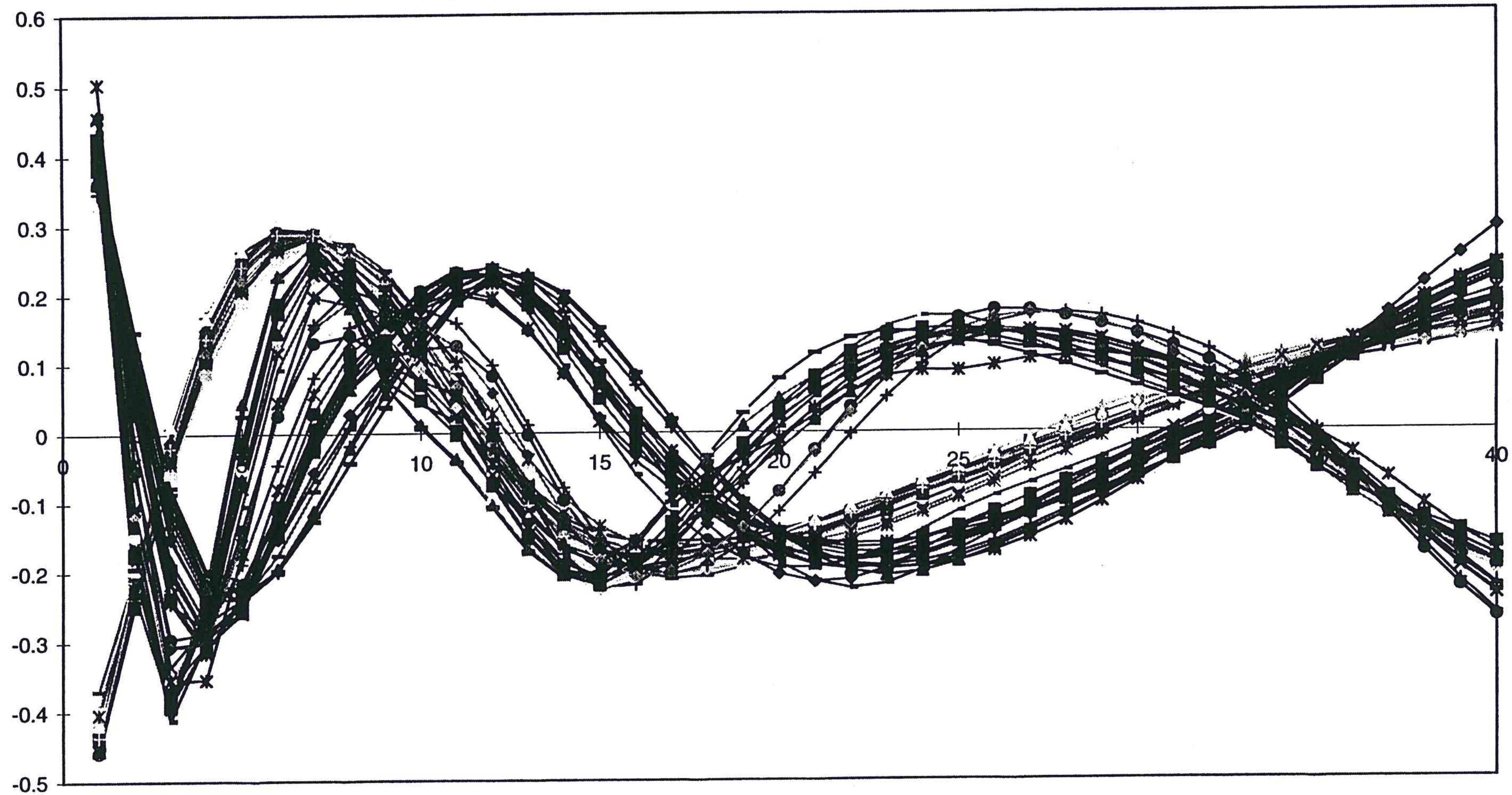


Figure 5: Implied Zero-Rate Factors, 4th - 6th
1, 9, 10, 13, 14, 15, 16, 17, 20, 21, 22, 23, 28, 29, 30, 31 May 1996.



Zero Rate Maturity x 4

Figure 6: Principal Eigenvalues

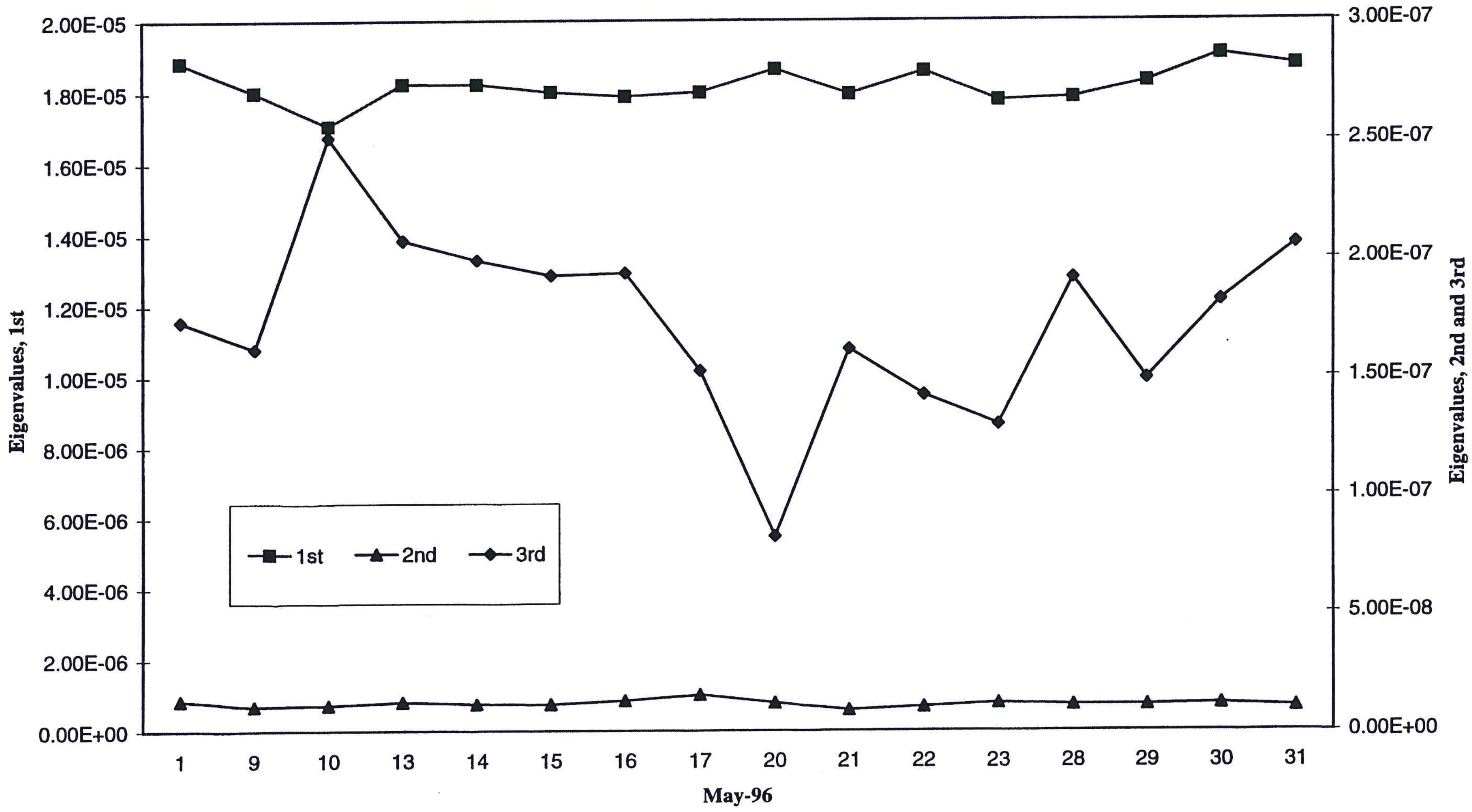


Table 1: Comparison of Numerical Integration andApproximation Formula for European Payer Swaption Values, 31st May 1996.

		Swap Tenor					
		1	2	3	4	5	7
Moneyness[1] 0.9							
2 Year Swaptions	Num. Integration[2]	0.009644	0.018358	0.026098	0.033044	0.039402	0.050541
	Approximation	0.009643	0.01836	0.026112	0.033059	0.039432	0.050592
3 Year Swaptions	Num. Integration	0.010197	0.019305	0.027318	0.034554	0.041128	0.052601
	Approximation	0.010196	0.019313	0.027322	0.034564	0.041134	0.052635
4 Year Swaptions	Num. Integration	0.010400	0.019576	0.027717	0.035031	0.041689	
	Approximation	0.010399	0.019571	0.027714	0.035018	0.041678	
5 Year Swaptions	Num. Integration	0.010277	0.019414	0.027492	0.03478	0.041377	
	Approximation	0.010276	0.019417	0.027489	0.034777	0.041384	
Moneyness 1.0							
2 Year Swaptions	Num. Integration	0.006305	0.011822	0.016423	0.020421	0.023955	0.029973
	Approximation	0.006304	0.011824	0.016433	0.020434	0.023981	0.030031
3 Year Swaptions	Num. Integration	0.007098	0.013176	0.01834	0.022843	0.026875	0.033664
	Approximation	0.007097	0.013181	0.018344	0.022855	0.026892	0.033727
4 Year Swaptions	Num. Integration	0.007433	0.013828	0.019296	0.024124	0.028425	
	Approximation	0.007433	0.013825	0.019297	0.024122	0.028435	
5 Year Swaptions	Num. Integration	0.007544	0.014057	0.019691	0.024659	0.029064	
	Approximation	0.007544	0.014060	0.019692	0.024667	0.02909	
Moneyness 1.1							
2 Year Swaptions	Num. Integration	0.003823	0.007019	0.009436	0.011432	0.013093	0.015791
	Approximation	0.003822	0.007021	0.009446	0.01145	0.013131	0.015885
3 Year Swaptions	Num. Integration	0.004686	0.008473	0.011535	0.014071	0.016293	0.019826
	Approximation	0.004686	0.008477	0.011543	0.014092	0.016332	0.019942
4 Year Swaptions	Num. Integration	0.00508	0.009307	0.012744	0.015704	0.018263	
	Approximation	0.005079	0.009306	0.01275	0.015719	0.018303	
5 Year Swaptions	Num. Integration	0.005335	0.009769	0.013494	0.01668	0.019423	
	Approximation	0.005334	0.009772	0.013501	0.016703	0.019477	

1 Moneyness is defined as strike / forward swap rate for the underlying swap tenor.
2 Based on 1 million samples. Error has $O(10^{-6})$.

Table 2: Fitted Caps Prices, 31st May 1996		
Cap Maturity	**Market Cap Price**	**Model Cap Price**
2	0.010331	0.010606
3	0.017984	0.017971
4	0.025852	0.025804
5	0.034419	0.033964
7	0.050143	0.050034
10	0.072310	0.072806

Table 3: Fitted Swaptions Prices, 31st May 1996						
2 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	0.006329	0.011687	0.016512	0.020442	0.024142	0.030126
Kennedy	0.006304	0.011824	0.016433	0.020434	0.023981	0.030031
3 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	0.006958	0.013153	0.018291	0.023031	0.026725	0.034146
Kennedy	0.007097	0.013181	0.018344	0.022855	0.026892	0.033727
4 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	0.007459	0.013744	0.019442	0.024021	0.028558	-
Kennedy	0.007433	0.013825	0.019297	0.024122	0.028435	-
5 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	0.007396	0.013831	0.019451	0.024347	0.029145	-
Kennedy	0.007544	0.014060	0.019692	0.024667	0.029090	-

Table 4: Fitted Swaptions, Black Volatilities, 31st May 1996						
2 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	19.2	18.2	17.5	16.7	16.2	15.3
Kennedy	19.12	18.41	17.42	16.69	16.09	15.25
3 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	18.2	17.5	16.7	16.2	15.5	15
Kennedy	18.57	17.54	16.75	16.08	15.60	14.81
4 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	17.5	16.7	16.2	15.5	15.2	-
Kennedy	17.44	16.80	16.08	15.57	15.13	-
5 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	16.7	16	15.5	15	14.8	-
Kennedy	17.04	16.27	15.69	15.12	14.77	-

	Table 5: Fitted $g(u,v)x10^{-5}$, 31st May 1996					
	0	**2**	**4**	**6**	**8**	**10**
0	14.78					
2	11.05	18.04				
4	10.74	11.93	12.14			
6	8.50	10.24	10.36	10.63		
8	7.66	9.31	9.44	9.56	10.32	
10	6.07	7.58	7.83	7.95	8.07	9.60

	Table 6: Zero Rate Factors, 31st May 1996									
	1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**9th**	**10th**
Eigenvalue	1.19e-3	3.95e-5	1.03e-5	2.02e-6	5.54e-7	1.70e-7	6.36e-8	3.11e-8	1.39e-8	5.00e-9
%	95.77%	3.17%	0.83%	0.16%	0.04%	0.01%	0.01%	0.00%	0.00%	0.00%
Cum. %	95.8%	98.9%	99.8%	99.9%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
1	0.327	-0.624	0.602	0.306	-0.176	-0.097	-0.082	-0.035	-0.007	0.000
2	0.340	-0.432	-0.122	-0.604	0.406	0.261	0.257	0.135	0.034	-0.002
3	0.340	-0.178	-0.488	-0.215	-0.273	-0.380	-0.494	-0.310	-0.093	0.012
4	0.333	-0.025	-0.392	0.286	-0.408	-0.045	0.387	0.518	0.251	-0.062
5	0.324	0.070	-0.207	0.372	0.042	0.426	0.255	-0.426	-0.490	0.199
6	0.315	0.156	-0.058	0.293	0.360	0.282	-0.352	-0.129	0.515	-0.415
7	0.306	0.227	0.065	0.139	0.399	-0.199	-0.301	0.439	-0.129	0.576
8	0.298	0.282	0.166	-0.039	0.216	-0.493	0.264	0.013	-0.370	-0.553
9	0.290	0.325	0.246	-0.213	-0.124	-0.194	0.333	-0.421	0.481	0.363
10	0.282	0.355	0.304	-0.352	-0.460	0.441	-0.271	0.219	-0.193	-0.119

Table 7: Fitted Swaptions Prices, 31st May 1996						
2 Year Swaptions Prices						
Tenure	1	2	3	4	5	7
Market	0.006329	0.011687	0.016512	0.020442	0.024142	0.030126
Three-Factor HJM approx.	0.00626	0.01182	0.016425	0.020435	0.02398	0.030034
One-Factor HJM approx.	0.005797	0.011348	0.016175	0.020361	0.024024	0.030076
3 Year Swaptions Prices						
Tenure	1	2	3	4	5	7
Market	0.006958	0.013153	0.018291	0.023031	0.026725	0.034146
Three-Factor HJM approx.	0.007048	0.013177	0.018336	0.022856	0.026892	0.03373
One-Factor HJM approx.	0.006549	0.012686	0.018071	0.022775	0.026935	0.033775
4 Year Swaptions Prices						
Tenure	1	2	3	4	5	-
Market	0.007459	0.013744	0.019442	0.024021	0.028558	-
Three-Factor HJM approx.	0.007423	0.01382	0.019297	0.024121	0.028437	
One-Factor HJM approx.	0.007062	0.01359	0.019254	0.024173	0.028465	
5 Year Swaptions Prices						
Tenure	1	2	3	4	5	-
Market	0.007396	0.013831	0.019451	0.024347	0.029145	-
Three-Factor HJM approx.	0.007532	0.014055	0.019692	0.024665	0.029091	
One-Factor HJM approx.	0.007196	0.013835	0.019647	0.024715	0.029118	

Table 8: Fitted Swaptions, Black Volatilities, 31st May 1996						
2 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	19.2	18.2	17.5	16.7	16.2	15.3
Three-Factor HJM approx.	18.99	18.41	17.41	16.69	16.09	15.25
One-Factor HJM approx.	17.58	17.67	17.14	16.63	16.12	15.27
3 Year Swaptions						
Tenure	1	2	3	4	5	7
Market	18.2	17.5	16.7	16.2	15.5	15
Three-Factor HJM approx.	18.44	17.53	16.74	16.08	15.60	14.82
One-Factor HJM approx.	17.12	16.87	16.50	16.02	15.62	14.84
4 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	17.5	16.7	16.2	15.5	15.2	-
Three-Factor HJM approx.	17.41	16.79	16.08	15.57	15.13	
One-Factor HJM approx.	16.56	16.51	16.04	15.60	15.15	
5 Year Swaptions						
Tenure	1	2	3	4	5	-
Market	16.7	16	15.5	15	14.8	-
Three-Factor HJM approx.	17.01	16.26	15.69	15.12	14.77	
One-Factor HJM approx.	16.24	16.00	15.66	15.15	14.79	

Table 9: Zero-Rate Instantaneous Change Correlations, 31st May 1996.

	1 yr.	2yr.	3yr.	4yr.	5yr.	6yr.	7yr.	8yr.	9yr.	10yr.
1 yr.	1									
2 yr.	0.970	1								
3 yr.	0.928	0.984	1							
4 yr.	0.914	0.968	0.994	1						
5 yr.	0.908	0.957	0.985	0.997	1					
6 yr.	0.897	0.944	0.973	0.989	0.997	1				
7 yr.	0.883	0.930	0.961	0.979	0.991	0.998	1			
8 yr.	0.870	0.916	0.948	0.969	0.983	0.993	0.998	1		
9 yr.	0.857	0.903	0.935	0.958	0.975	0.986	0.994	0.998	1	
10 yr.	0.844	0.890	0.923	0.947	0.965	0.979	0.988	0.994	0.998	1