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Abstract

Under perfect market assumptions dynamic rebalancing is costless and the underlying asset trades continuously. In reality, asset prices have jumps, volatility changes randomly, there are transactions cost and trading can only take place at discrete points in time. For standard options optimal and sub-optimal delta hedging strategies have been suggested in the literature however for exotics options all these strategies could be very expensive to implement. These path dependent options are characterised by high and rapidly changing gamma and this implies a high frequency and therefore costly rebalancing that makes it unattractive to hedge dynamically even when optimal strategies are used. To overcome this problem we extend Clewlow and Hodges (1994) simulation approach to search for optimal methods of hedging based on heuristics which are consistent with myopic strategies. We use Monte Carlo simulations to study the efficiency of this model, static replication portfolios and other delta hedging strategies under transaction costs and jumps in the underlying process. We also show that mathematical optimisation is a useful way of constructing replicating portfolios. It allows replication under general asset pricing models involving jumps and stochastic volatility, trading restrictions can be considered and the replicating portfolio can be constructed in terms of the strikes and maturities available in the market.

A COMPARISON OF ALTERNATIVE METHODS FOR HEDGING EXOTICS OPTIONS

1. INTRODUCTION

In the Black-Scholes (1973) theory, option valuation is based on the argument that a given derivative asset payoff can be replicated by an appropriate continuous trading strategy. A call option can be hedged perfectly by shorting a dynamic replicating portfolio consisting of a long position in the risky asset and a short position in bonds, which is equal in value to the price of the option. As time passes, the weights of this portfolio are continuously rebalanced so that it replicates the payoff of the option contract at maturity.

Under perfect market assumptions the underlying asset trades continuously and continuous dynamic rebalancing is costless. In reality, asset prices have jumps, volatility changes randomly, there are transactions cost and trading can only take place at discrete points in time. For standard options this leads to Black-Scholes delta hedging and higher order hedges such a delta-gamma hedging being sub-optimal and dominated by optimal strategies which take account of the market imperfections (see for example Hodges and Neuberger (1989) and Edirisinghe et al (1993) among others). For exotic options reasonable risk reduction implies such frequent rebalancing of the hedge that transaction costs make the Black-Scholes hedging approach impractical. These options are characterized by high and rapidly changing gamma and this implies a high frequency and hence costly rebalancing that it makes unattractive to hedge dynamically even when optimal strategies are used. To overcome this problem Bowie and Carr (1994) and Derman *et al.* (1995) propose static hedges based on portfolios of standard options that replicate the payoff of a target option for a chosen range of futures times and market levels.

This paper analyses the performance of a variety of alternative hedging strategy for exotic options including static and dynamic. We propose a methodology based on mathematical optimisation techniques to construct static replicating portfolios that are optimal respect to some predetermined criteria and under realistic constrains. We also describe an extension of Clewlow and Hodges (1994) myopically optimal gamma hedging strategy and study its efficiency for the hedging of barrier options.

There are very few papers that we are aware of which study the problem of hedging exotic options under realistic market conditions. Dolbear (1992) used a simulation approach to compare the performance of delta hedging and hedges based on straddles for exchange rate lookback options under stochastic volatility and interest rates. Both methods used simple filters to control the level of transaction costs. The straddle based method was found to perform significantly better than the delta hedging for realistic levels of transaction costs in terms of cost/risk trade-off. Kat (1996) uses a sophisticated stochastic simulation model to study the performance of delta hedging of lookback and Asian options on S&P 500 index. A set of heuristic filter based rules are used to control the level of transaction costs; rebalancing after a certain number of days, rebalancing after the index has moved by a certain percentage and rebalancing when delta is outside an error interval. The main conclusions were that the gamma characteristics of the option position to be hedged, volatility misestimation and computing hedge ratios based on Black-Scholes assumptions was an important source of hedging error standard deviation.

This paper is organized as follows; in Section 2 we review some literature on dynamic hedging strategies under transaction costs and in Section 3 we describe our methodology for implementing a heuristic delta-gamma hedging strategy. Section 4 presents a brief review of the static hedging and Section 5 describes how mathematical programming techniques can be used to calculated the static replicating portfolio under more realistic conditions. Section 6 describes our simulation analysis of alternative hedging approaches under a variety of assumptions and the results of the analysis by simulation are presented in Section 7. The final section summarises our work and sketches some ideas for further research in this area.

2. DYNAMIC STRATEGIES

The most common strategy for immunizing a portfolio to small changes in the underling asset is to approximately apply the Black-Scholes delta hedging strategy. In a Black-Scholes type economy where trading is discrete, the delta hedge is rebalanced at discrete points in time t_j and the risky asset S follows a Brownian motion diffusion process given by,

$$S_{t_j} = S_{t_0} \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)\left(t_j - t_0\right) + \sigma z\left(t_j - t_0\right)\right) \tag{1}$$

where z is a standard Brownian motion, r is the risk-free interest rate, and σ is the risky asset volatility. At time t_0, the calendar time when the hedge is initiated, an option hedger writes an European option V with maturity at time t_n, and buys the replicating portfolio in order to follow a delta hedging strategy. We have that the hedged portfolio is one for which at each rebalancing period the quantity held in the risky asset is given by the option delta

$\Delta_{t_j} = \left.\frac{\partial V}{\partial S}\right|_{S=S_{t_j}}$ while the value invested in the risk-free security is given by $B_{t_j} = V - V_S S$.

Thus, the realised hedging error at time t_{j+1} is defined as the difference between the value of the hedge portfolio and the value of the option:

$$H_{t_{j+1}} = e^{r(t_{j+1}-t_j)} B_{t_j} + \Delta_{t_j} S_{t_{j+1}} - V_{t_{j+1}} \tag{2}$$

H is a random variable with mean zero and variance of order Δt^2 whose distribution has been analysed by Boyle and Emanuel (1980) and subsequent papers by Leland (1985), Toft (1996), Boyle and Vorst (1992), etc. When transactions costs exists, the delta hedging error grows with the frequency of the hedging rebalance overwhelming the profit margin of the option and traders have to compromise between cost and accuracy of the replication . Leland (1985) is the first to consider option replication under transaction. He shows in a continuous framework that perfect replication can be obtained in the limit as the time between rebalancing points goes to zero if a transactions costs adjusted hedging volatility is used to construct the hedging portfolio. Basically, Leland (1985) proposes a delta hedging strategy under proportional transactions costs using a Black-Scholes option price based on a modified variance. His results are important in the sense that allows to construct a replicating portfolio inclusive of transaction costs, where the error is uncorrelated with the market and with a variance that goes to zero with the length of the rebalancing interval. Toft (1996) provides exact closed form expressions for the expectation and variance of the hedging error of Leland's variance adjusted delta hedging strategy. Merton (1990) sets up the problem in a two period time economy where he replicates the option value under proportional transactions costs. Later, Boyle and Vorst (1992) generalise his model to a multiperiod economy. They employ no-arbitrage arguments in discrete time framework to construct a portfolio of the risky asset and riskless bonds that replicates a long and short European call. They also derived an analytical approximation for the long call price in the presence of proportional transactions costs when there is a very large number of revision times and low costs rate. Like in Leland (1985), this approximation results in a Black-Scholes option pricing formula with an adjusted variance.

So far we have described hedging strategies under transactions costs where the hedger has to compromise between accuracy of the replication and costs of the hedging without any criteria of optimality. Optimality can be defined in different but related ways. For example, Hodges and Neuberger (1989), Davis et al. (1993) and Hodges and Clewlow (1996) maximise the expected utility of the terminal cash flow net of transactions costs generated at the end of the hedging period. They characterise the intertemporal dynamic programming problem that can be constructed when the hedger has exponential preferences and continuous trading is possible. The nature of the optimal solutions to this problem is that with proportional transaction costs the optimal hedging strategy is to transact only as much as is necessary to maintain the asset holding within a region which can be computed. This region takes the form of a band around the preferred delta, which itself differs from the Black-Scholes delta[1] . It is optimal under this assumptions to rebalance the hedge portfolio only when the option delta moves outside a non-trading region. Whalley and Wilmott (1996) use asymptotic analysis of the Hodges and Neuberger (1989) model to derive approximations for the bounds of this non-trading region for small level of transaction cost. Bensaid et al (1992) and Edirisinghe et al (1993) utilise a different optimality criteria. In a discrete time framework they minimise the initial cost of obtaining a terminal payoff that is at least as large as that from the option being hedged. They show that the cost of the super-replicating portfolio may be smaller than the cost of the exact replication portfolio.

All these delta hedging strategies aim to control the exposure to the change in the underlying asset price. When hedging exotics option this may not be enough. Exotics options are characterised by large - in absolute value - and rapidly changing gamma so the delta neutral portfolios are likely to quickly become to exposed to the underlying asset, resulting in significant transaction costs in order to maintain the delta neutrality. Hodges et al. (1993) confirmed that gamma hedging provided a significant risk reduction compared to delta hedging, and did so even when rebalancing was implementing daily. The reason why the improvement is so marked appears to be because real asset processes involve jumps and stochastic volatility. However, the study also showed that this kind of standard gamma hedging involved high turnover of options positions - whose transaction costs would make the hedge prohibitively costly. Clewlow and Hodges (1994) proposed a convenient methodology for delta-gamma hedging with transaction costs under either jumps or stochastic volatility in the underlying asset process. In this paper we adopt and extend this approach to exotics options. Next section describes the Clewlow and Hodges (1994) methodology for implementing a quasi optimal gamma hedging strategy and our extension to consider optimal delta strategies.

3. A QUASI OPTIMAL DELTA-GAMMA HEDGING STRATEGY

The optimal gamma hedging problem could be formally stated in a dynamic programming framework but the dimensionality of the problem formulation would make it difficult to

[1] Clewlow and Hodges (1996) show that the Black-Scholes delta can even lie outside the optimal region.

implement[2]. Clewlow and Hodges (1994) propose a close to optimal strategy to manage gamma under transactions costs. They assume that the cost of been mishedged is a quadratic function of the gamma and the speed of the portfolio[3], i.e., at each rebalancing date, they solve the following problem:

$$\begin{aligned} &Min\ y'Cy + e|x| \\ &s.t. \quad y = y_0 + Gx \end{aligned} \tag{3}$$

where $C = \begin{bmatrix} c_g & 0 \\ 0 & c_s \end{bmatrix}$,

y_0 is the vector of the current gamma and speed of the portfolio, c_g and c_s are the weights of gamma and speed relative to the transaction costs, G is the matrix of gamma and speed of the available standard options[4], x is the vector of adjustments to make in our holdings and e is the vector of transaction costs. The parameters c_g and c_s are chosen by simulating the hedging strategy and searching for the values which maximise the expected utility of the terminal wealth w_T:

$$U(w_T) = -\exp(-\lambda w_T) \tag{4}$$

with λ a coefficient of risk aversion.

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[2] The gamma hedging problem is a complex problem in that there are many different candidate instruments that could be used to modify the gamma of a hedged portfolio.
[3] A lesser known greek called speed (see Garman (1992)) measures the change in gamma with respect to the underlying asset it is the third derivative of the option pricing formula respect to the underlying asset.
[4] The gamma of an option peaks at spot prices close to the present value of the strike price. It is therefore usually advisable to gamma hedge using an at the money option, for if we neutralise gamma by buying or selling options a long way from the money, our gamma hedge is likely to be very sensitive to changes in the underlying asset price (see Figure 1).

During the implementation of this myopically optimal gamma hedge we must simultaneously delta hedge the resulting portfolio. This can be done naively adjusting delta at each rebalancing point or implementing an optimal delta hedging strategy with our myopic optimization. For simplicity in the simulations we adopt the Whalley and Wilmott (1996) approximation for the Hodges and Neuberger (1989) utility maximization model. They derived a hedging bandwith around the Black-Scholes delta given by:

$$\Delta_t \pm \left[\frac{3\alpha S_t e^{-r(T-t)} \Gamma_t^{\,2}}{2\lambda}\right]^{1/3} \tag{5}$$

where Δ_t and Γ_t are the Black-Scholes delta and gamma of the hedged portfolio and α is the proportional transaction cost variable. We call the myopically optimal delta-gamma hedging strategy a quasi optimal delta-gamma hedging strategy when it is simultaneously implemented with the utility maximization strategy described above.

4. STATIC HEDGING OF EXOTIC OPTIONS

A static replication portfolio consists of a set of standard options, with different strikes and maturities, whose value matches the payoff of a given target option at maturity and along the boundaries. The usefulness of a replicating portfolio is twofold. First, the market value of the replicating portfolio is a good and realistic approximation of the value of the target option. This value is derived from the added value of actively traded standard options so it reflects transactions costs, volatility smile effects and other market conditions. Second, the replicating portfolio can be used to hedge the position in the target option. Since path-dependent options have large gammas, a reasonable risk reduction would imply a frequent rebalance of the hedge that could be very expensive in the presence of transactions costs. A static hedge for exotics options may be considerably easier and cheaper than a dynamic hedge.

In general, it would be necessary to hold options with an infinite number of strikes and maturities to obtain a perfect replicating portfolio. Let $V(S_t, T-t)$ denote the value at time t of a contingent claim on asset price S_t and maturity at time T. The value of the replication portfolio that matches the value of V in all possible states of the world during the life of the option is given by:

$$V(S_t, T-t) = \int_t^T \int_0^\infty x(K,\tau) C(K,\tau - t) d_K d_\tau \tag{6}$$

where $C(K,\tau)$ is the price of a standard option (call or put) with strike price K and time to maturity τ and $x(K,\tau)$ represents a density of holdings with respect to time τ. In order to solve for the holdings, we define a subset of the space (S_t, t) where the portfolio and the target are to be matched. There is not a unique replicating portfolio and there are many ways to construct it. Obviously, Equation 6 cannot be applied in practice where only finite number of strikes and maturities are available.

Carr et al. (1994, 1997) shows how to construct static replicating portfolios for a wide range of exotics options with a small number of standard options. Based on strong assumptions on the underlying process [5], they derive a relationship between calls and puts denominated "Put-

[5] The underlying process must be a diffusion with zero drift and a symmetry volatility structure, ruling out jumps and stochastic volatility in the price process.

Call Symmetry", an extension of the Put-Call parity to different strikes prices [6]. They use this relationship to set replication portfolios that provide an exact match of the target for all times and market levels. Although their results are an important contribution to the understanding of static hedging, their assumptions restrict their applicability to real markets.

Derman et al. (1995) proposes a different and more general method to calculate a replicating portfolio in a binomial tree framework. It is based on the argument that a sufficient condition to replicate an exotic option is to match its value along its natural boundary. As a result, the resulting replicating portfolio would have the same value that the target in all the interior points of the boundary. Derman et al. (1995) reformulates the problem with a discrete approximation of Equation 6:

$$\tilde{V}\left(b\left(t_j\right), t_N - t_j\right) = \sum_{i}^{N-1} x_i C_i\left(b_i, t_i - t_j\right) + x_N C_T\left(b_T, t_N - t_j\right) \tag{7}$$

where $t = t_0,\ldots t_N = T$ represents a set of discrete points in time, $b\left(t_j\right)$ is the point of the exotic boundary at the time t_j, $\underline{x} = x_1,\ldots x_N$ is the vector of weights of the options that match the target at the boundary and $C_i\left(b_i, t_i - t_j\right)$ is a standard option at time t_j, struck at the boundary at time t_i so the inclusion of this option in the replicating portfolio does not interfere with the cash flow inside the boundary. Equation 7 shows that the subset of the space $\left(S_t, t\right)$ and the set of strikes to be included in the portfolio are reduced to the points along the boundary of the exotic option under consideration. Derman et al. (1995) shows that

[6] See Carr et al. (1997) for a detailed exposition of the assumptions and the intuition of the Put-Call Symmetry relationship.

following this method it is possible to construct a portfolio with a small numbers of options that satisfactorily replicate a path-dependent option in a binomial world. In more realistic terms implied trees should be used to reflect the conditions prevailing in the market such as volatility smiles [7]. These trees are constructed to return the market prices of standard European options and they therefore embed the risk neutral distribution of the underlying asset for various horizons.

While in theory, the method proposed by Derman et al. (1995) may work very well, in practice, it could present some difficulties. There is only a small number of strikes and maturities available in real markets to construct the replicating portfolio. This gives us the problem of requiring options with maturities and strikes corresponding to every date and state in the tree to construct the hedge. Obviously, this problem could be overcome by interpolating the required values from the available prices, but this cannot be done without loss in the accuracy and effectiveness of the replication. On the other hand, given the presence of jumps and stochastic volatility in the market, it is not guaranteed that the sensitivities of the replicating portfolio are equal to the ones of the exotic option and it is not clear that boundary points are the only places where the match should be obtained. In the following section we propose optimization techniques as a more robust and efficient methodology to construct a static replicating portfolio.

5. MATHEMATICAL OPTIMIZATION AND STATIC HEDGES

Working backward in the tree to solve for the holdings in a static replicating portfolio is a procedure rather limited by the implied tree representation of the world. It is assumed that the

[7] See Derman (1994) (1996), Dupire (1993) and Rubinstein (1994) for a detailed exposition of Implied Trees.

hedger has a wide range of strikes and maturities to match at each node of the tree on the boundaries. In reality this is not the case and it is necessary to interpolate prices or volatility smiles to construct the portfolio. In this section we describe how the use of mathematical optimization can overcome this problem by working only with the existing strikes and maturities available in the market. We construct a portfolio that it is optimal respect to some predetermined criteria and under realistic constraints given by the market and hedger's environment. e.g., position limits.

As before, we want to closely replicate the characteristics of the exotic option in a subset of the space (S_t, t). Using a mathematical optimization formulation the problem can be established in the following way:

P_1:

Find the vector of holdings $\underline{x}$ that

Minimise the initial cost: $f = \sum_{i=1}^{N} x_i C(K_i, 0)$

subject to: (8)

$$V(S_t, T-t) = \sum_{i=1}^{N} x_i C(S_t, K_i, t_i - t)$$

for a predefined subset of the space (S_t, t).

Given that we can only consider a limited number of options, we reformulate the problem P1 to make it feasible and well-posed. We relax the restrictions and impose constraints that allow for an absolute hedging mismatch:

P_2:

Find the vector of holdings $\underline{x}$ that

$$\text{Minimise: } f = \sum_{i=1}^{N} x_i C(K_i, 0) + \sum_{j=1}^{M} |Z_j| \tag{9}$$

subject to

$$V(S_j, t_j) = \sum_{i=1}^{N} x_i C(S_j, K_i, t_i - t_j) + Z_j \quad \text{for } j = 1 \text{ .. M (scenarios)}$$

Here Z_j represents the mismatch of the static portfolio respect to the exotic option at the sample points (S_j, t_j) from all futures market levels during the life of the option. P_2 is a non-linear optimization problem so in order to maintain things as simple as possible we work with the following transformation:

$$Z_j = Z_j^+ - Z_j^- \tag{10}$$

and consequently

$$f = \sum_{i=1}^{N} x_i C(S, K_i, t_i - t) + \sum_{j=1}^{M} (Z_j^+ - Z_j^-) \tag{11}$$

where $Z_j^+ * Z_j^- = 0$ at the optimum and P_2 can be solved using linear optimization. We now discuss some of the advantages of using mathematical optimization to construct a static replicating portfolio:

1. **Scenario simulation.** There are several ways to generate the scenarios to be considered in the optimization problem. We could use Monte Carlo simulations to evolve the underlying asset and generate a sample of the asset price at different times during the life of the option or we could define a discrete economy representing by a standard or implied risk neutral tree. In any case it is possible to consider an underlying process that includes jumps and stochastic volatility [8].

2. **Trading Restrictions.** Trading is not continuous so the investor is usually subject to lot size constraints and position limits. This restrictions can be easily representing in the optimization model via the following constraints:

   a) $x_i = k\alpha \quad \forall i$ and some integer k would reflect that trading can only be done in multiples of lots of size α.

   b) $x_{\inf} \le x_i \le x_{\sup} \quad \forall i$ would state the lower and upper bounds for the trading positions.

   Other constraints can be defined to set restrictions respect to the sensitivities of the portfolio, the amount of borrowing, etc.

3. **Sensitivities.** If we work under the standard assumptions of the Black-Scholes theory the value of the sensitivities of the static replicating portfolio are equal to the theoretical value and sensitivities of the target option. Under most realistic assumptions, e.g., jumps and

[8] See Amin (1993) for a lattice representation of the Merton (1976) Jump Diffusion Model and Naik (1993) and Derman (1997) for discrete representations of stochastic volatility models.

stochastic volatility, the hedge provided by replicating the portfolio could be out of line for small changes in the volatility or jumps in the asset. This risk could be substantially reduced if the optimization criteria outlined by P_2 is extended to replicate not just the value of the target in the selected scenarios but also the target sensitivities.

We now present an example of the use of mathematical optimization to hedge an Up and Out (UAO) call option with fixed times for monitoring the barrier. By definition, a UAO call option[9] with strike K and barrier $H > K$, has the same payoff of a standard call option if the underlying asset does not reach the barrier during the life of the option or pays nothing otherwise [10]. Let us assume an initial asset price S_0 equal to 100, strike price K equal to 90, barrier H set at 110 and volatility for the asset process equal to 18%. The option maturity is 3 months and the barrier is monitored daily. Interest rate and dividends yields are set equal to zero. We use the Amin (1993) lattice tree approximation of a jump diffusion process to generate the scenarios for the optimization. This allow us to compare the hedging effectiveness under Black-Scholes (1973) and Merton (1976) worlds, generate an exhaustive representation of the states of the world and work out the probability of each path of the asset to optimise respect to the expected hedging error in the tree. Figures 2, 3 and 4 show the value, delta and gamma of the UAO barrier in a Black-Scholes world.

[Insert Figures 2, 3 and 4 here]

Using a set of 30 standard options: five maturities from 13 days to 65 days (assuming a year of 260 days) and six strikes per maturity from 90 to 115, we solve for the fixed holdings in a

[9] See Reiner and Rubinstein (1991) for a complete description and formulae of barrier options in the Black-Scholes model.

replicating portfolio. Figure 5 and 6 show the values of the replicating portfolio and the relative mismatch respect to the value of the barrier option. The mismatch is the difference between the replicating portfolio and the target option, divided by the initial value of the target.

[Insert Figures 5 and 6 here]

Table 1 and Table 2 show the hedging performance of six different replicating portfolios in terms of the hedging error expectation and standard deviation. We consider sets of 15, 48 and 72 standard call options to replicate the UAO call option in the Black-Scholes world and the Jump-Diffusion world. As we could expect, the static hedging performance improves as we increase the number of options in the replication portfolios. Generally speaking, even under the presence of jumps in the asset process the static hedges perform quite well. Tables 3, 4 and 5 show the holdings by strike and maturity for the portfolios under the Black-Scholes assumptions.

The optimization problem can be extended to match the sensitivities of the target. Figures 7 and 8 show the gamma of three replicating portfolios compared to the gamma of the target option . The first portfolio was calculated only optimising with respect to the value of the target, i.e., delta replication . A second portfolio was calculated optimising respect to the value and gamma of the target option and a third portfolio respect to the value, gamma and speed of the target option. It can be observed that this simple transformation of the problem produces a significant risk reduction in terms of delta-gamma performance.

[10] Knock-Out barrier options could pay a rebate if the asset hit the barrier and the option becomes worthless.

[Insert Figures 7 and 8 here]

We followed the above described methodology to construct the static hedge for our analysis by simulation. Next section describes the structure and presents the results of our simulations.

6. THE STRUCTURE OF THE SIMULATIONS

We compare the performance of five different hedging strategies under three different assumptions for the nature of the world and under a variety of methods for implementing the hedges. The strategies are: Black-Scholes delta hedge, optimal delta hedge, myopically optimal gamma hedge, quasi optimal delta-gamma hedge and static hedge. The three worlds are: a Black-Scholes world where the asset follows a geometric Brownian motion, a Jump-Diffusion world based on the Merton (1976) model, and a Stochastic Volatility world based on the Hull and White (1988) model. We have chosen two different examples to evaluate the performance of the proposed hedging strategies. Example 1 considers the hedge of a short one year UAO call at the money option, with initial price of 100 and a barrier at 120. We gamma hedge using standard options with an expiry of three months from our starting date and we look at the hedges performance one month before the maturity of the target option. This is to avoid the problems associated with the gamma and speed changing rapidly near to expiry, which we consider a separate problem. We also rollover the hedge options into the next maturity when they reach one months to expiry for similar reasons. For the myopically optimal gamma hedge we also obtained separate optimal gamma and speed weights for each sub-period between the rollover dates. This gives an idea of the time dependence of the weights. Example 2 considers the hedge of a short six months UAO call at the money option, with initial price of 100 and a barrier at 110. We gamma hedge with two months standard options which are rollover two weeks before maturity. The hedge performances is also evaluated two weeks before the maturity of the target option.

In each of the worlds we have low level of cost for transacting in the underlying at 0.5%. For buying or selling options to adjust the gamma exposure the transaction cost is a fixed spread chosen so that it is 0.2% of the premium on a three months at the money standard option for the Example 1 and 0.2% of the premium on a two months at the money standard option for the Example 2. Each of the worlds has a risk free interest rate of 5% and gives the underlying asset a volatility of either 20% or reverting to 20%. In the jump diffusion world the intensity parameter is 5, i.e., 5 jumps per year on average, and it jumps to a distribution with standard deviation of 0.05. The volatility of the diffusion is chosen so that the overall annualised standard deviation is 20%. In the stochastic volatility world (which takes the CIR square root form for the variance) we have a mean reversion parameter of 1, and a volatility parameter of 0.30 as the volatility of volatility. Table 6 summarises the properties of the asset return empirical distributions we obtain under these parameters values. Notice that the excess kurtosis under the jump-diffusion model falls off rapidly and has disappeared by the one month horizon. In contrast, for the stochastic volatility model the kurtosis builds up gradually as we look at longer horizons. If we combine the two models we obtain values typical for equity indices.

The simulations generated daily prices and each strategy was run 1000 times for each world. The naïve Black-Scholes delta and the optimal delta strategy were implemented with a range of rebalancing frequency from one day up to 10 days. The static replicating portfolios were calculated optimising with respect to the value and gamma of the target option. We consider sets of 15 (5 strikes and 3 maturities), 48 (8 strikes and 6 maturities) and 72 (8 strikes and 9 maturities) standard call options to replicate the UAO call option in the Black-Scholes world and the jump-diffusion world. Similarly, we optimised our myopic strategies for the Black-Scholes and jump-diffusion world only. If the volatility assumed for the optimizations changes then the hedge could become badly out of line. We then consider the stochastic volatility world as a test for the performance of the different hedging strategies when there is a misspecification with respect to the volatility. All the strategies use Black-Scholes models for computing prices and sensitivities.

7. RESULTS OF THE SIMULATIONS.

Figures 9 to 11 show the results in each of the three worlds for the long maturity barrier. In each case we have plotted the expected level of transactions cost against the standard deviation of the replication error. Our myopic optimization approach has been run on a daily rebalancing and we have then computed its subsequent performance on an out of sample basis.

[Insert Figures 9, 10 and 11 here]

Given our low level of transaction cost, the delta hedging approach produces less replication variance as we rebalance more frequently, and the expected cost of this only increases fairly slowly, up to the rebalancing every four days. At this point the increasing transaction costs variance begins to dominate the reduction in replication variance. As expected, it performs less well once the Black-Scholes assumptions are violated by jumps, or to a lesser extent by stochastic volatility.

Our myopically optimal gamma hedge produces a significant improvement respect to the naïve Black-Scholes delta hedge even though we are delta hedging our portfolio on a daily basis. This clearly shows the stabilising role of the heuristic gamma hedge. When we optimise delta as well as gamma, we get a important reduction in the expected costs. The sub-period optimized myopic gamma strategy (only calculated for the Black-Scholes world) has a much reduced standard deviation of hedging cost but higher expected cost reflecting the fact that it rebalances more often. The asymptotic approximation for the optima delta hedge strategy performs very well in each of the three worlds. Interestingly, in terms of performance, the asymptotic approximation seems to be a half way strategy between the myopically optimal gamma hedge and the quasi optimal delta-gamma hedge. Again it can be observed the effect of the gamma optimization with respect to the optimal delta hedging strategy, in particular for the jump diffusion and stochastic volatility worlds.

Our static replication portfolios significantly dominate the other strategies. As we increase the number of options in the portfolio (from 15 to 72) it produces less replication variance and more expected hedging cost. The figures show that increasing the number of options from 48 to 72 it does not produce a significant improvement in terms of risk reduction. Indeed, the 48 options replicating portfolio dominates the other two indicating the existence of an "optimal size" for the replicating portfolio.

The stochastic volatility world is a very important example of the robustness of the hedges. As we mentioned before, the parameters of our optimal strategies and the static portfolios were computed for the Black-Scholes and jump diffusion worlds. We then test the performance of these "optimal strategies" in the stochastic volatility world to see how a change in the volatility could affect our hedges. Figures 11 show that our simulations have enabled us to calibrate reasonable and robust parameter values for the myopic optimization policies and the static hedges. In particular the quasi optimal delta-gamma hedge and the static portfolios give a significant advantage over the other strategies, particularly in the robustness of the performance across the different worlds. It is important to notice that given the relationship between the gamma and the vega of an option, and because we gamma hedge with portfolios of options with the same maturity, our myopically optimal delta-gamma hedge is equivalent to a myopically optimal delta-vega hedge.

[Insert Figures 9, 10 and 11 here]

Figures 12, 13 and 14 show the results of the simulations for the hedge of the 6 months maturity barrier. In general the results are similar to the one year example but with a more clear advantages of the static portfolios over the dynamic strategies. In particular the heuristic gamma hedging strategies do not perform very well. Given that our static portfolios have been calculated matching the value and the gamma of the target option, it seems to indicate that a dominance of the static replication over the dynamic hedging when we are delta-gamma hedging short maturities exotics.

[Insert Figures 15 and 16 here]

Finally, Figures 15 and 16 illustrates the nature of the myopic strategies we obtain. Figure 15 illustrate how revisions are made to the portfolio in gamma-speed space. It will be noted that they correspond to a kind of control region strategy, but where the exact location of the optimal region varies through time depending on the opportunities available in the option market. Figure 16 shows the actual adjustments the strategy makes to holdings in the standard options through time for some typical simulation runs. Note how the strategy, particularly in the early period, makes quite infrequent discrete adjustments to the holdings when good opportunities present themselves.

8. CONCLUSIONS

We have described a simulation methodology which allows the construction of almost optimal dynamic delta-gamma hedges for barrier options under realistic market conditions such as jumps, stochastic volatility and transaction costs. Our simulations have confirmed that gamma hedging is particularly useful in realistic situations but can be dangerous if applied in a naïve way. We also show that simultaneously optimising delta and gamma provides a great improvement respect to the optimal delta and optimal gamma strategies. Hodges et al. (1993) and Hodges and Clewlow (1994) found that the choice of volatility input and the model were not important when delta-gamma hedging long dated standard options. But for exotics, particularly barrier options, the choice of the model and/or the estimation/forecast of volatility does appear to be important. (Kat (1996) also found this to be the case for lookback and Asian options). This is probably due to the increased gamma sensitivity of exotics options.

Static hedging for exotics options has several advantages when compare to dynamic hedging. It is easier to construct and manage and it avoids the periodical rebalancing of the hedge that could be prohibitively expensive for the high gamma path dependent options.

Mathematical optimisation is a useful way of constructing replicating portfolios. It allows us to work with general asset pricing models involving jumps and stochastic volatility, trading restrictions can be considered and the replicating portfolio can be constructed in terms of the strikes and maturities available in the market. It also enables us to work with very complicated types of exotics options, or portfolios of options. We simply require that the payoff value along every simulated path can be calculated. In principle this extends to situations with multiple states variables. However this can lead to some practical difficulties of sufficiently sampling the state variable space.

Further work remains to be done. Investigation of the benefit of more accurate option pricing models and volatility forecasts. More realistic costs structures so that the myopic strategies choose from the complete set of available options of all maturities as well as strikes. This should provide a better prescription for rolling over the hedge options. The optimal size for a given static replication portfolio should be also explored. Finally, more refined prescriptions for conserving transaction cost, and also to extend the analysis to a world with both jumps in the underlying asset process and stochastic jump-diffusion volatility would be interesting.

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Figure 1: Gamma Hedging an Up and out call option with a standard call option
Parameters: So=100, Strike=100, Barrier=120.
Standard option strike=70.

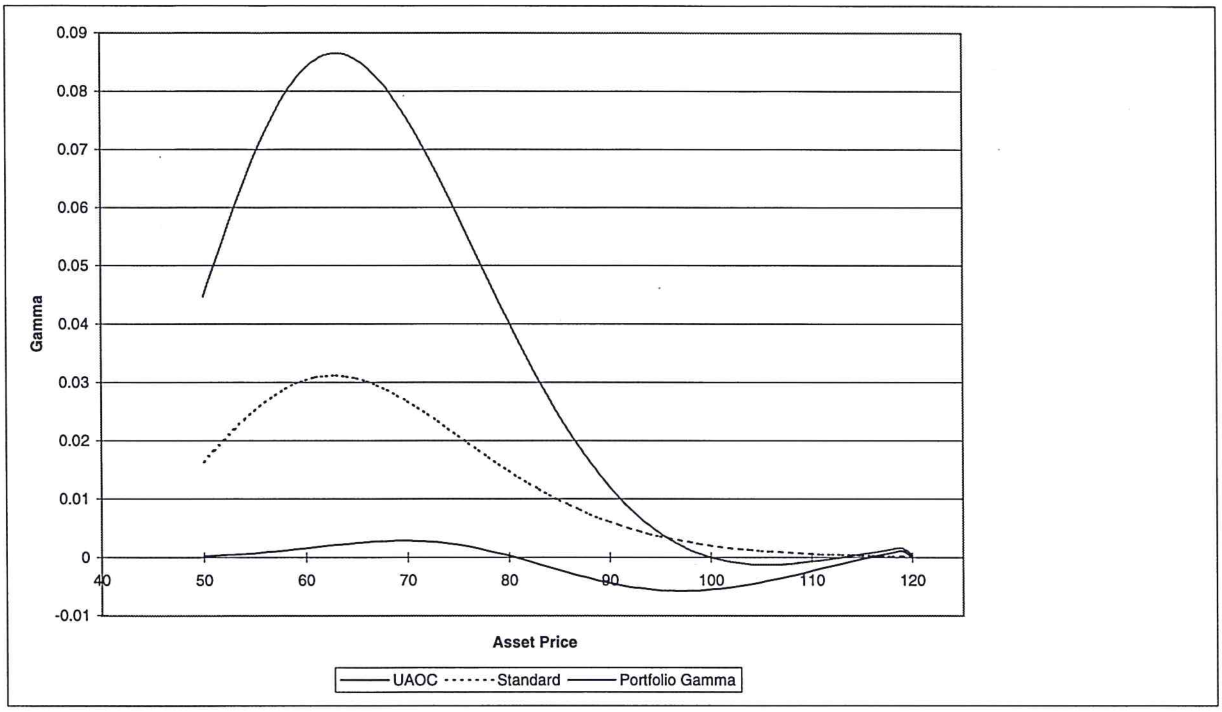

Figure 2: Up and Out Call Option Value
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

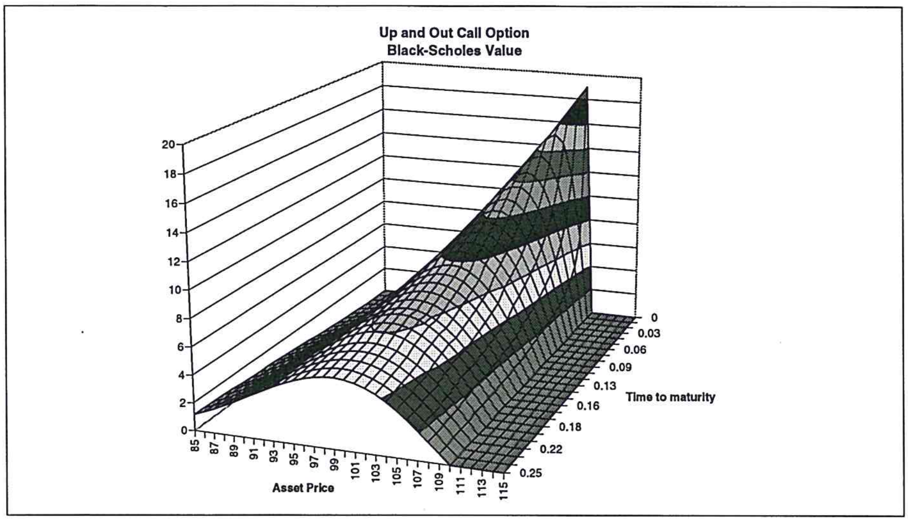

Figure 3: Up and Out Call Option Delta
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

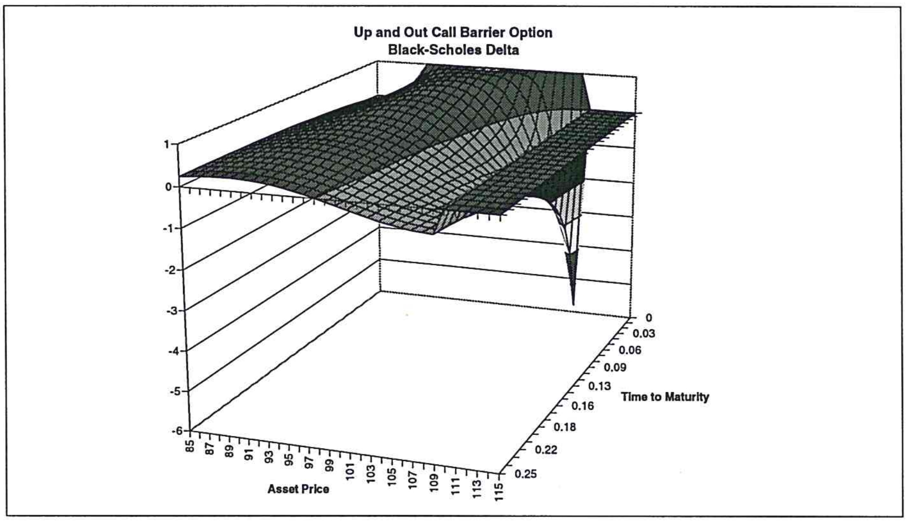

Figure 4: Up and Out Call Option Gamma
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

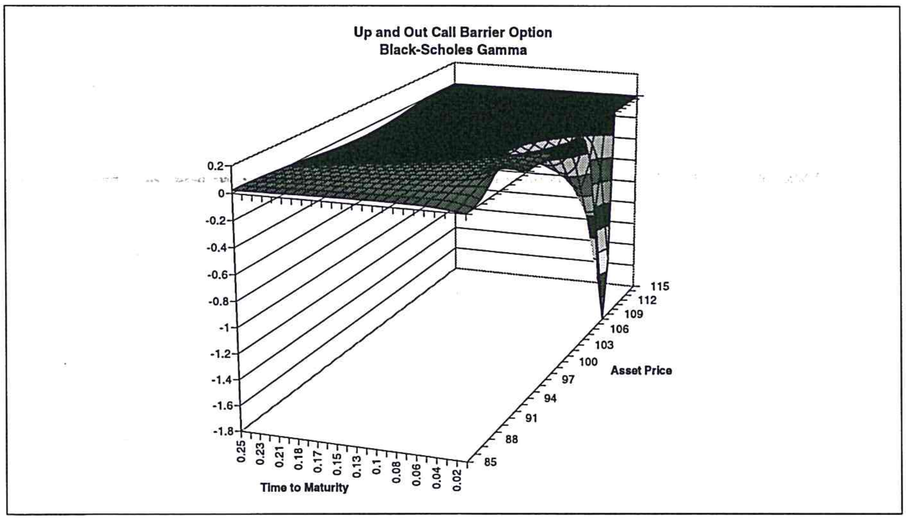

Figure 5: Up and Out Call Option.
Static Portfolio Replication Value
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

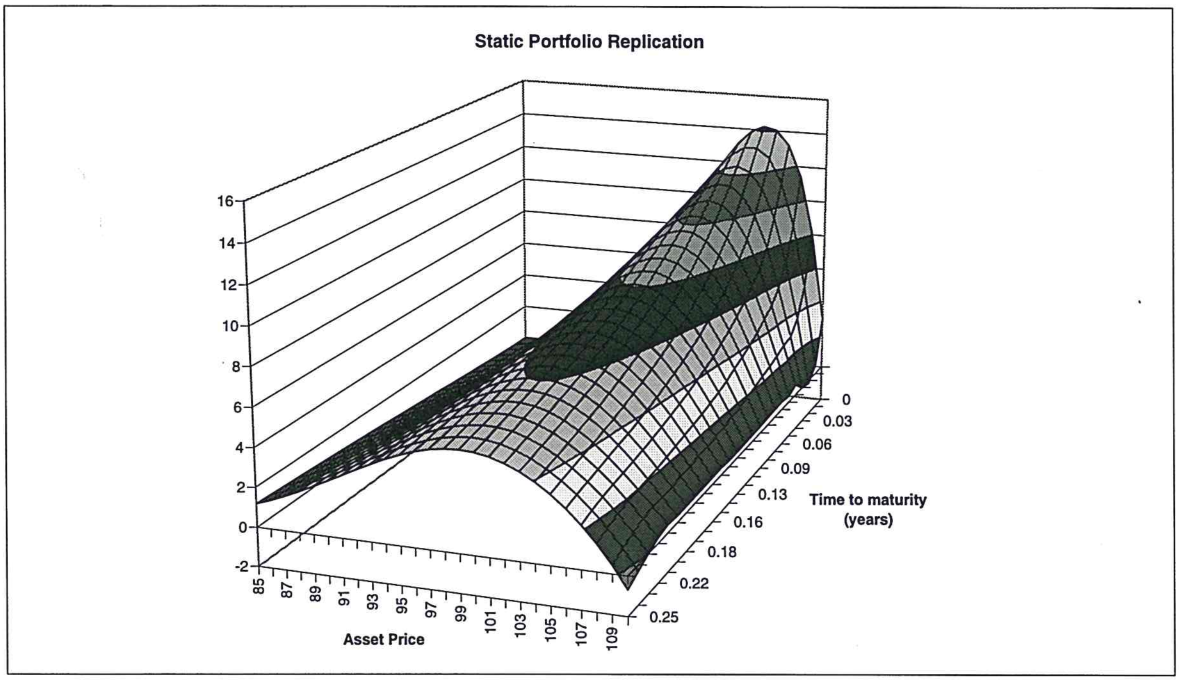

Figure 6: Relative Mismatch Between the Value of an Up and Out Call Option and Its Static Portfolio Replication.
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

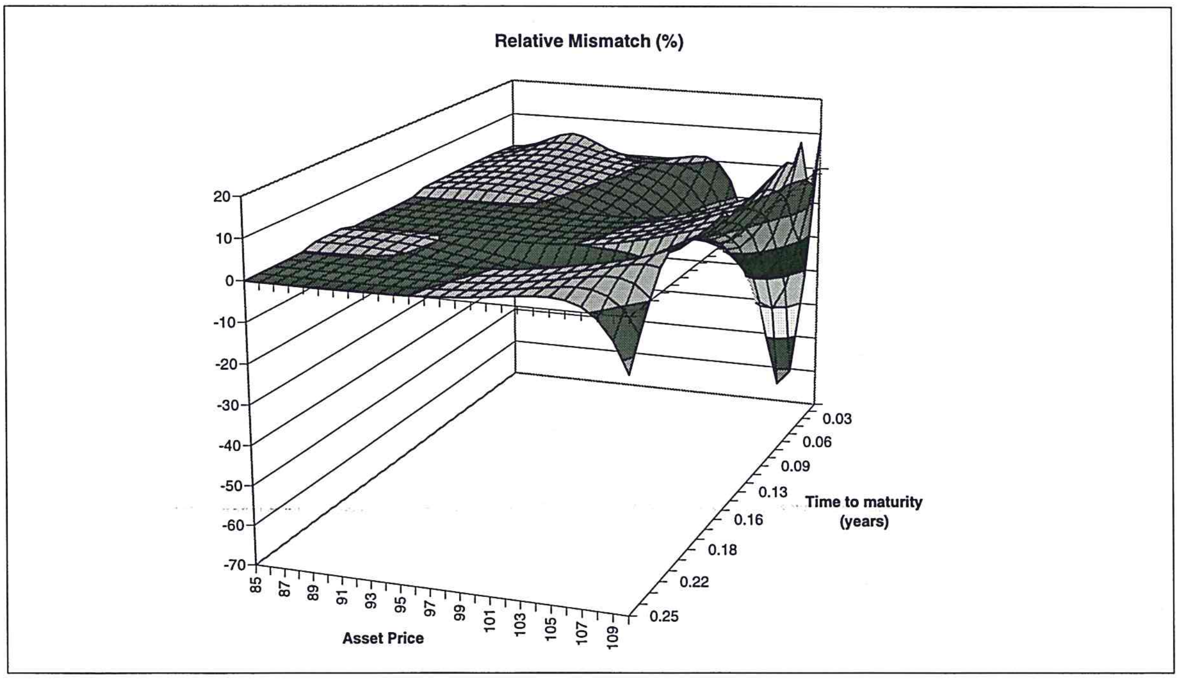

Figure 7: Up and Out Call Option.
Static Portfolio Replication Gamma Profile
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

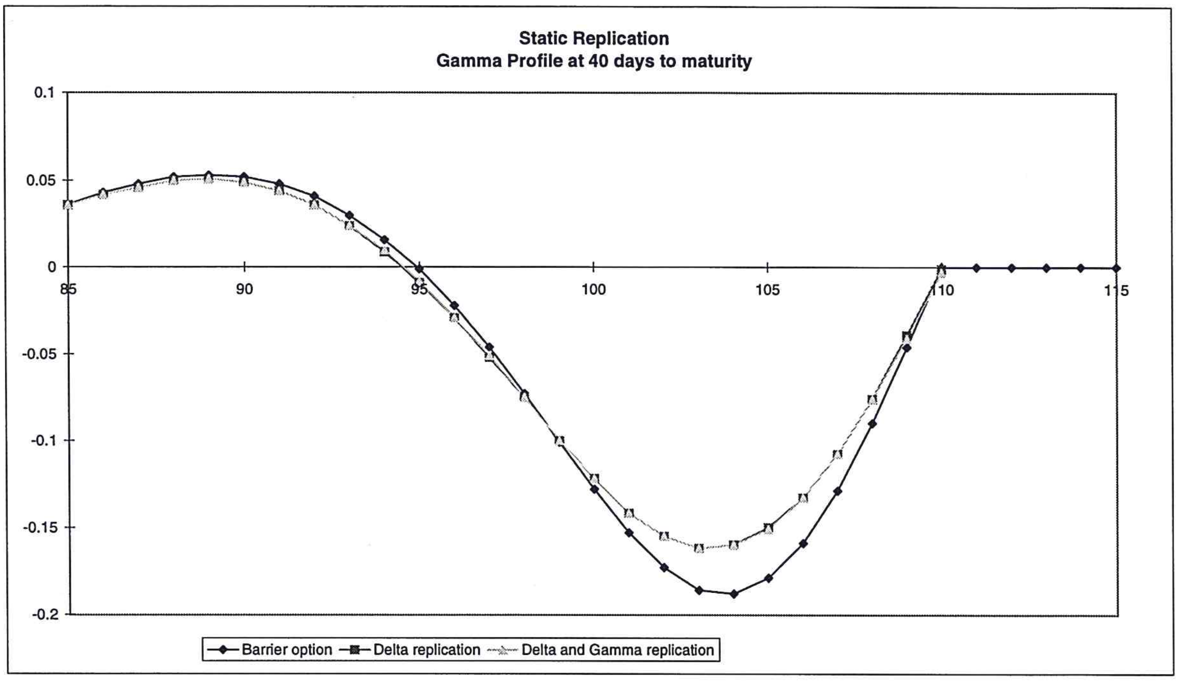

Figure 8: Up and Out Call Option.
Static Portfolio Replication Gamma Profile
Parameters: Strike=90, Barrier=110, interest rate and dividend yield equal to zero.

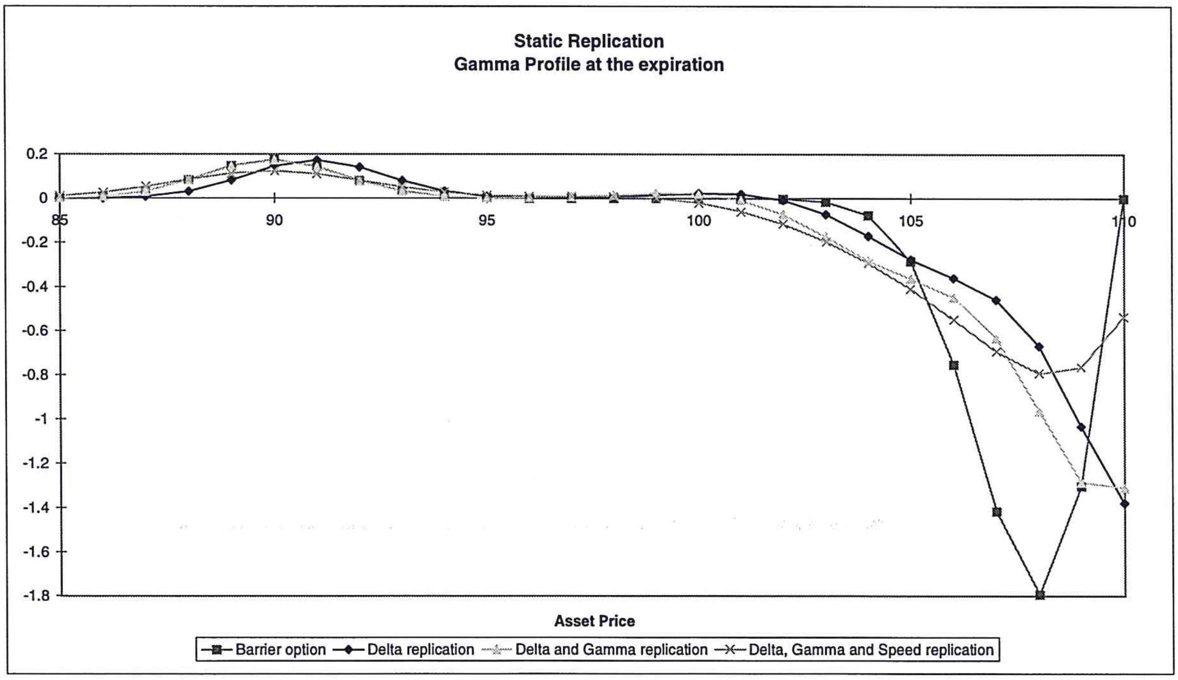

Figure 9: Hedging in a Black-Scholes World
Up and out Call Option
Parameters: Strike=100, Barrier=120, Maturity=1 year

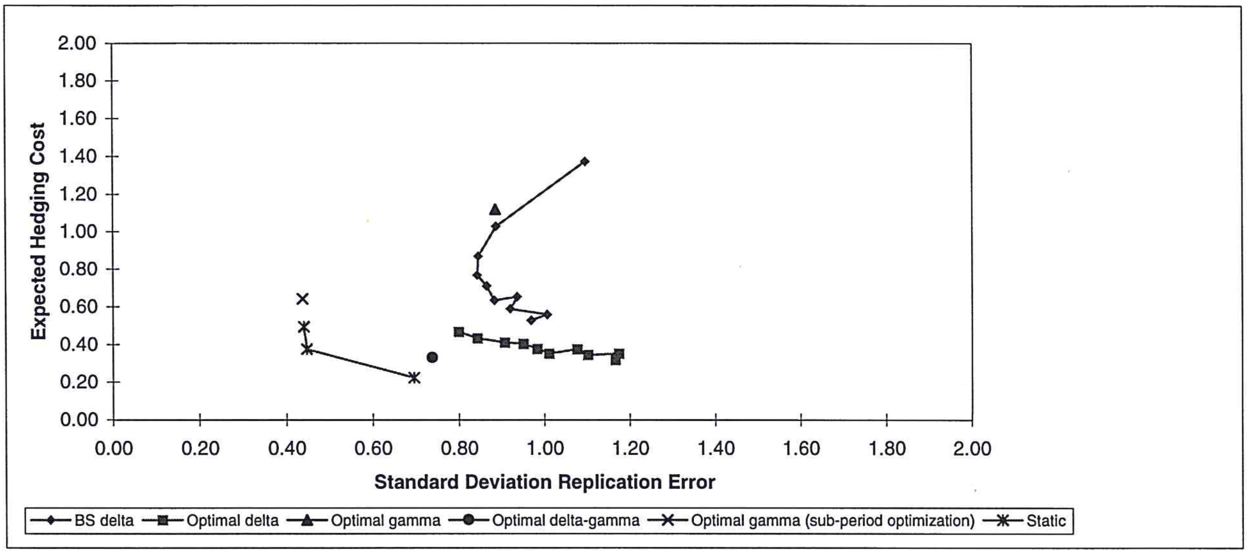

Figure 10: Hedging in a Jump Diffusion World
Up and out Call Option
Parameters: Strike=100, Barrier=120, Maturity=1 year

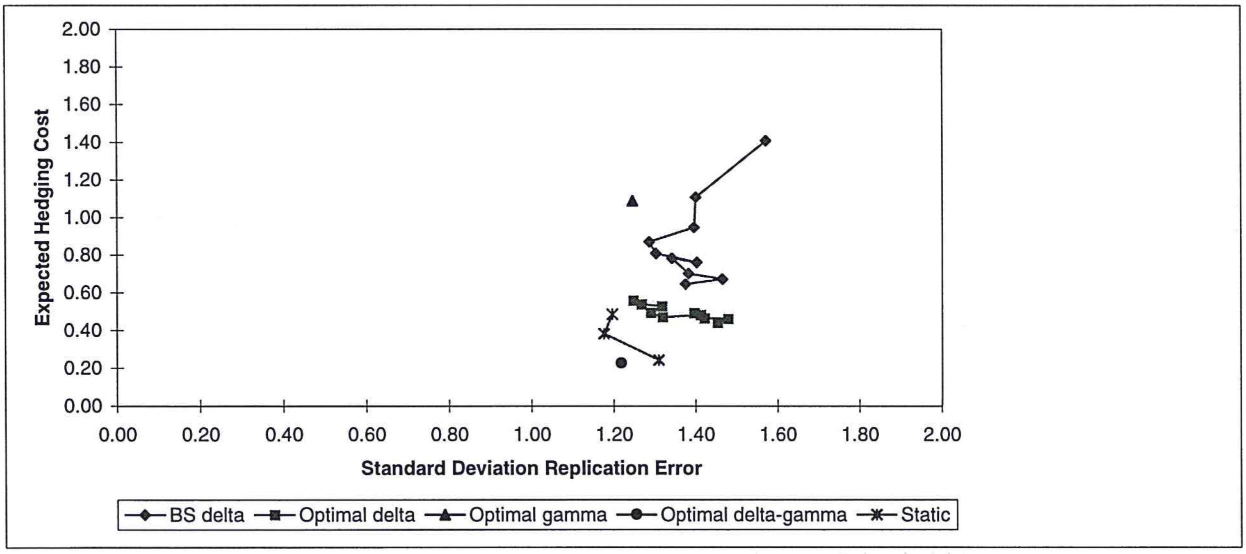

Figure 11: Hedging in a Stochastic Volatility World
Up and out Call Option
Parameters: Strike=100, Barrier=120, Maturity=1 year

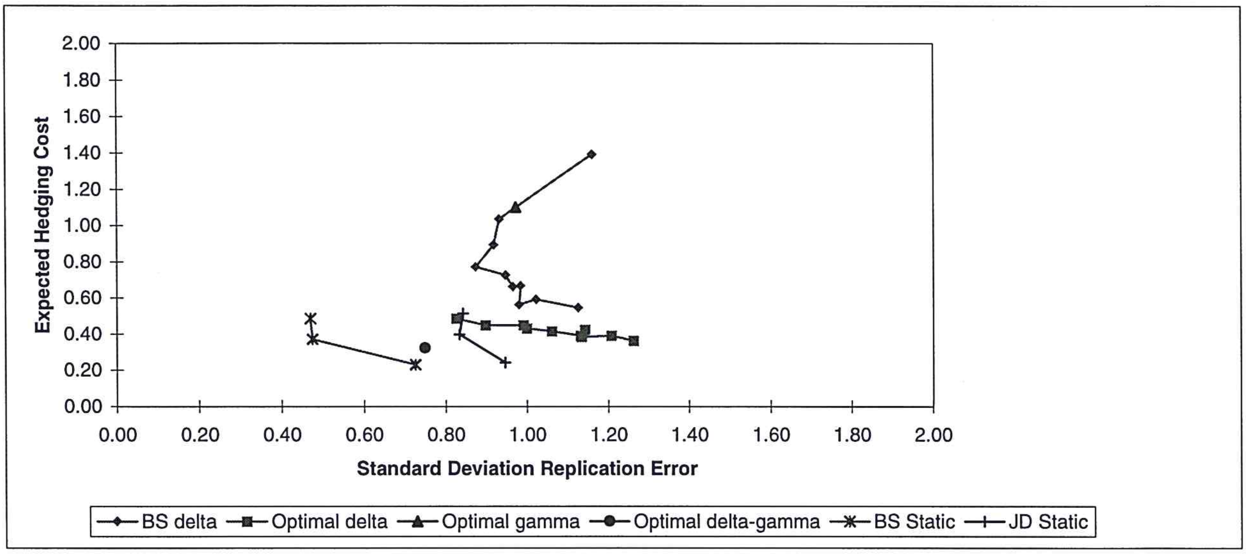

Figure 12: Hedging in a Black-Scholes World
Up and out Call Option
Parameters: Strike=100, Barrier=110, Maturity=6 months

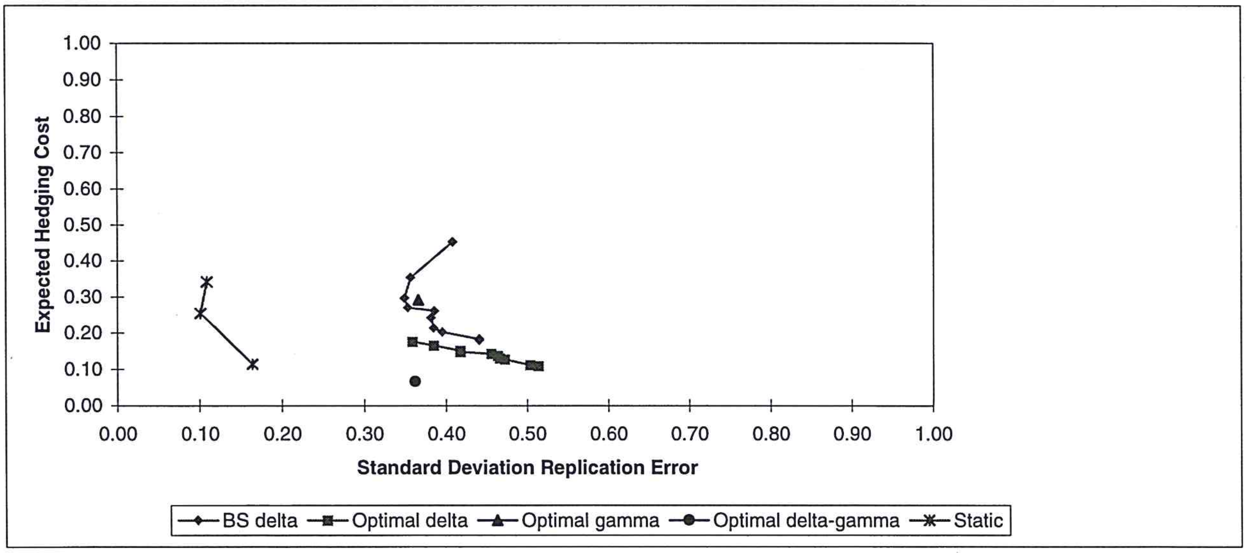

Figure 13: Hedging in a Jump Diffusion World
Up and out Call Option
Parameters: Strike=100, Barrier=110, Maturity=6 months

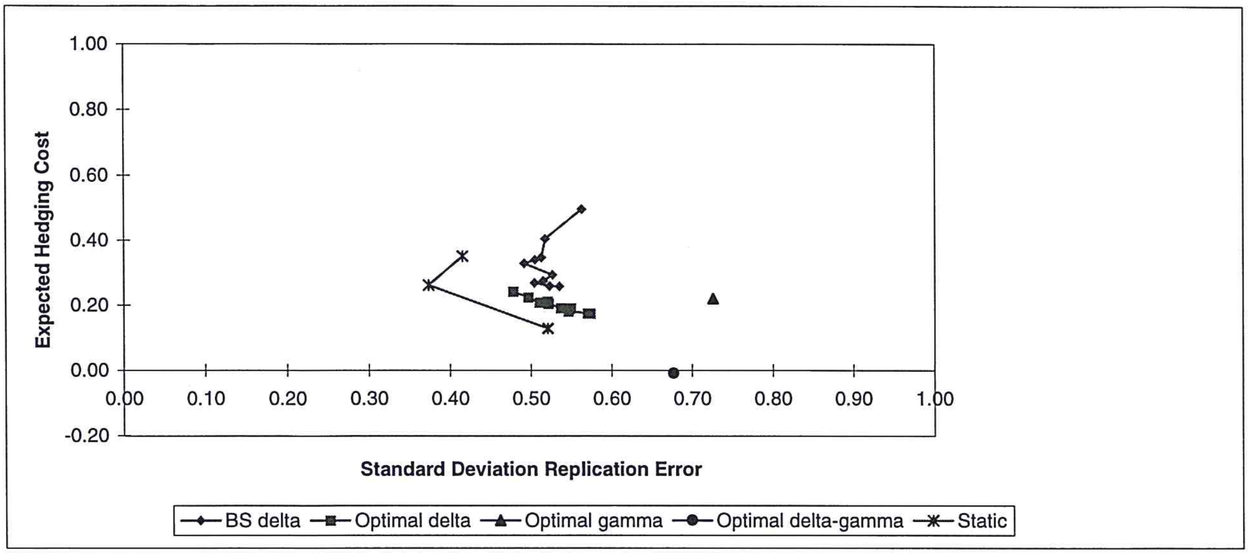

Figure 14: Hedging in a Stochastic Volatility World
Up and out Call Option
Parameters: Strike=100, Barrier=110, Maturity=6 months

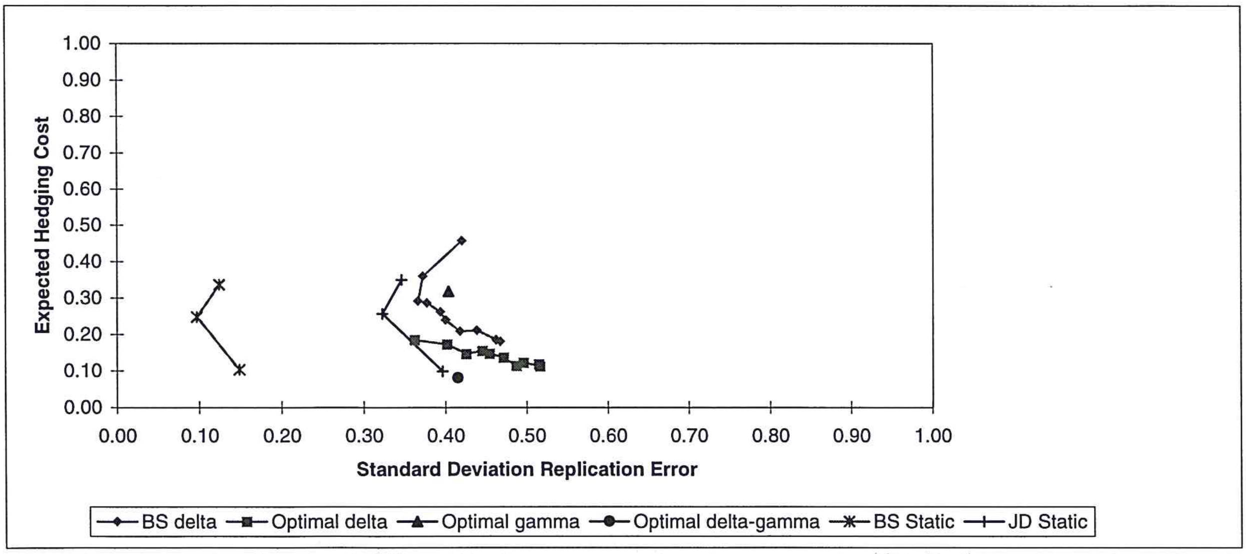

Figure 15: Example Myopic Strategy Gamma-Speed Rebalances

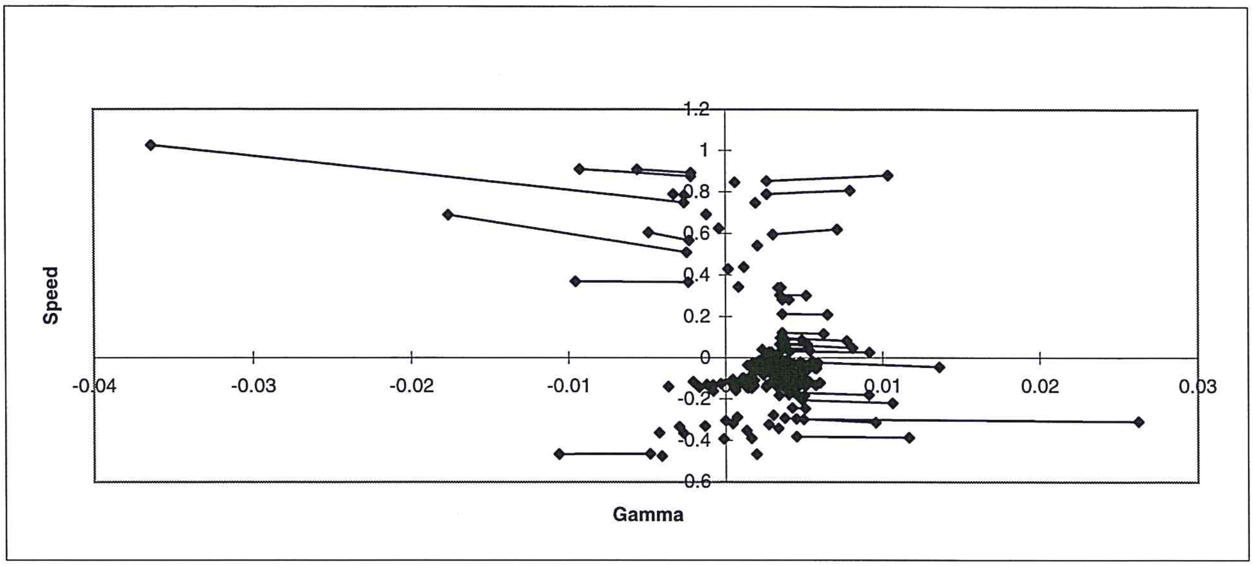

Figure 16: Example of Myopic Strategy Holdings
(Hx(K) is holding in shortest maturity call of strike K)

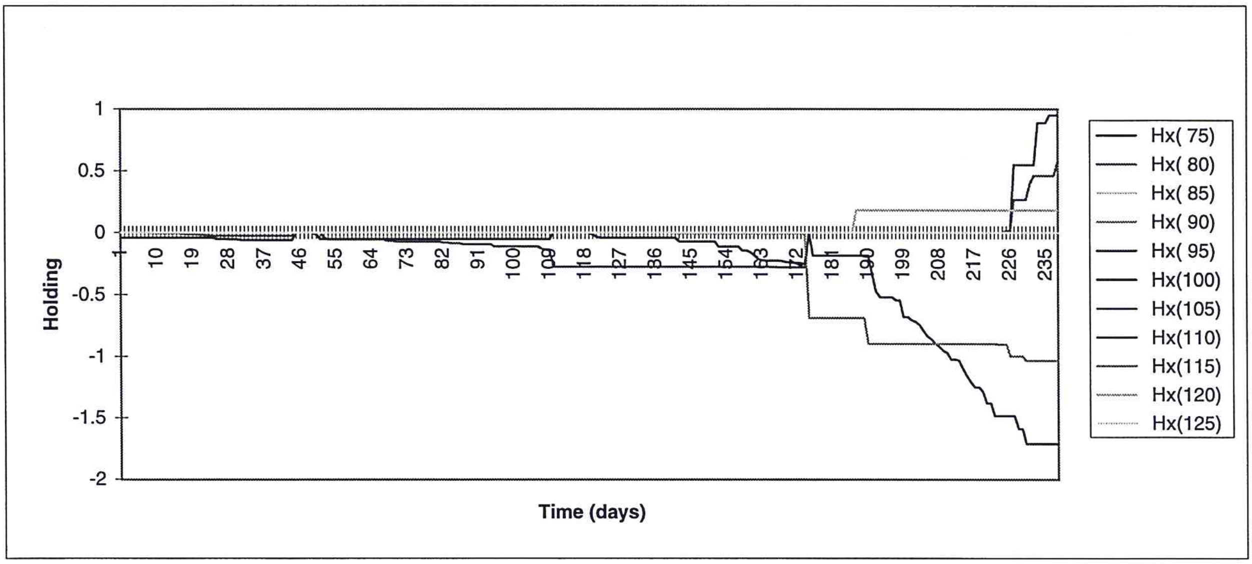

Table 1: Static Hedge in a Black-Scholes World

	Portfolio 1	Portfolio 2	Portfolio 3
Portfolio Value	5.06	5.06	5.06
Expected hedging error	1.40	1.23	1.22
Standard deviation	0.07	0.03	0.01
No. of options:	15	48	72
Barrier Option value:	5.06		

Table 2: Static Hedge in a Jump-Diffusion World

	Portfolio 4	Portfolio 5	Portfolio 6
Portfolio Value	5.82	5.87	5.86
Expected hedging error	3.27	2.06	2.06
Standard deviation	0.45	0.15	0.12
No. of options:	15	48	72
Barrier Option value:	5.87		

Jump-Diffusion model parameters: Diffusion Volatility 13%, Jump size volatility 6%, Poisson mean parameter 5.

Table 3: Portfolio 1 Standard options positions

	Maturity		
Strike	0.08	0.17	0.25
90	0.07	-0.19	0.25
95	-0.64	-0.03	0.17
100	-0.54	0.64	-2.17
105	-4.44	3.31	-4.68
110	2.81	-14.40	25.36

Table 4: Portfolio 2 Standard options positions

	Maturity					
Strike	0.04	0.08	0.13	0.17	0.21	0.25
90	0.00	0.00	0.00	0.00	0.06	-20.00
95	319.12	-20.00	0.00	0.00	0.00	0.00
100	0.01	0.16	1.71	18.78	0.00	0.00
105	0.00	0.00	0.00	0.13	0.58	-5.43
110	0.00	0.00	0.00	0.02	0.25	0.98
115	-0.85	0.26	0.00	0.02	-0.05	0.39
120	2.82	-3.16	2.35	-1.39	1.00	-0.01
125	0.21	-2.07	-10.35	13.35	-3.45	-2.17

Table 5: Portfolio 3 Standard options positions

	Maturity								
Strike	0.03	0.06	0.08	0.11	0.14	0.17	0.19	0.22	0.25
90	0.00	0.00	0.00	0.03	-2.33	18.81	-20.00	-20.00	0.00
95	-0.01	0.00	0.00	-0.01	1.62	-9.28	-20.00	0.00	0.00
100	0.00	0.01	0.01	1.28	-2.78	-3.44	0.00	-0.01	0.01
105	0.00	0.00	-0.13	0.69	-9.27	0.00	0.00	0.00	0.01
110	-0.03	0.37	-2.04	-7.44	0.00	0.00	0.00	0.01	-0.12
115	-0.15	2.13	1.92	0.00	0.00	-0.01	0.01	0.24	3.55
120	-1.04	3.13	-0.01	0.03	-0.06	0.33	7.07	0.93	-2.29
125	-4.72	1.00	-0.01	0.14	-1.68	-12.95	4.05	-0.30	-0.44

Table 6: Distributional Properties of Simulated Asset Returns

Black-Scholes World

Horizon	SD (%)	Skewness	Excess Kurtosis
1d	1.25	0.04	0.05
1w	2.76	0.06	0.10
1m	5.80	0.19	0.48
1y	23.16	0.37	0.09

Jump Diffusion World

Horizon	SD (%)	Skewness	Excess Kurtosis
1d	1.42	0.15	8.06
1w	3.15	0.11	1.42
1m	6.61	0.14	0.28
1y	26.92	0.34	0.49

Stochastic Volatility World

Horizon	SD (%)	Skewness	Excess Kurtosis
1d	1.23	0.00089	1.6086
1w	2.79	0.0129	1.1273
1m	5.74	0.0426	2.3044
1y	22.43	2.0852	6.1318

SD = Standard Deviation, 1d = 1 day, 1w = 1 week, 1m = 1 month, 1y = 1 year.