# FINANCIAL OPTIONS RESEARCH CENTRE

**University of Warwick**

# Pricing By Arbitrage Under Arbitrary Information

**Simon H Babbs**
**and**
**Michael J P Selby**

June 1998

*Financial Options Research Centre*
*Warwick Business School*
*University of Warwick*
*Coventry*
*CV4 7AL*
*Phone: (01203) 524118*

**FORC Preprint:1998/90**

*Mathematical Finance*, Vol. 8, No. 2 (April 1998), 163–168

# PRICING BY ARBITRAGE UNDER ARBITRARY INFORMATION

SIMON H. BABBS

*First Chicago NBD and University of Warwick*

MICHAEL J. P. SELBY

*Centre for Quantitative Finance, Imperial College, and University of Warwick*

A substantial applications literature on pricing by arbitrage has effectively restricted information to that arising solely from securities markets; return distributions are then governed solely by past prices. We reconsider pricing by arbitrage in markets rendered incomplete by arbitrary information, which, moreover, may influence required returns. We show that contingent claims depending solely on securities' normalized price histories are priced by arbitrage if and only if all risk-adjusted probabilities agree upon the law of primitive securities' normalized prices. We show how existing diffusion-based results can be preserved, and resolve an issue relating to Merton's (1973) stochastic interest rate model.

KEY WORDS: contingent claims analysis, incomplete markets, information filtration, pricing by arbitrage

## 1. INTRODUCTION

In applying their general theoretical framework to diffusion models, Harrison and Kreps (1979, p. 388)[1] restricted the information filtration to that generated by the Brownian motions driving the local martingale components of returns. They then imposed an invertibility condition to ensure that the full Brownian information could be recovered from normalized securities price histories. Analogous information specifications have been the norm in subsequent applications of arbitrage pricing. Such specifications imply that economic agents neither know nor care about anything other than securities markets. This is patently unrealistic and, moreover, fails to accommodate evidence that information outside securities market histories has a bearing on prospective returns (see, e.g., Fama 1991). This raises the question: Can existing arbitrage-based results be preserved if restrictions on the information filtration are dropped—rendering markets incomplete—and required returns depend on arbitrary information?[2]

In Section 2, working with arbitrary adapted price processes, we consider claims whose payoffs (after normalization by the value of some numéraire security) depend only on the subfiltration generated by normalized securities prices. Note that, under the restricted information specifications common in previous analyses, only such contingent claims exist!

This paper is based on results obtained in certain sections of Babbs and Selby (1993).
*Manuscript received January 1996; final revision received July 1996.*

Address correspondence to S. H. Babbs at First National Bank of Chicago, 1 Triton Square, London NW1 3FN, England; e-mail: sbabbs@uk.fcnbd.com

[1] See also Kreps (1981, p. 31) and Harrison and Pliska (1981, pp. 244–250). Harrison and Pliska suggest (p. 248) that their results on diffusion models could be generalized, allowing, inter alia, required returns to depend on more information, but they do not justify this.

[2] The recognition that information not included in securities market histories must be allowed to influence expected returns, and thus become impounded in the market history, precludes a trivial embedding argument.

We show that these claims can be priced by arbitrage if and only if all equivalent martingale measures agree on the risk-adjusted probability law of the normalized security price processes.

In Section 3, we use this result to show how existing pricing results can be preserved in popular diffusion-based arbitrage pricing models. In particular, European call options are priced by arbitrage in Merton's (1973) stochastic interest rate extension of the Black and Scholes (1973) model—a result that eluded Harrison and Kreps (1979) (HK).

Proofs are relegated to the Appendix.

## 2. FULL PRICING BY ARBITRAGE

Presuppose an arbitrary filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P)$, over a fixed finite interval $[0, T]$, satisfying the "usual conditions," and with $\mathcal{F} = \mathcal{F}_T$. We assume that $n + 1$ securities are traded, with adapted price processes (with any intermediate cash flows reinvested) $S_0, \ldots, S_n$, of which at least $S_0$ remains strictly positive. Adopting the zeroth security as numéraire, we define the vector $S^*$ of normalized price processes, $S_k^* \equiv S_k/S_0$.

We leave more detailed specification aside, for all we require here is that any equilibrium pricing operator for contingent claims (defined as $\mathcal{F}_T$-measurable random variables) take the form: $V = S_0(0)\mathrm{E}^*[X/S_0(T)]$, where $V$ is the value of the claim $X$, and $\mathrm{E}^*$ denotes expectations under an equivalent martingale measure (EMM) $P^*$ (i.e., a probability measure equivalent to $P$, and under which $S^*$ becomes a martingale).

DEFINITION 2.1. The (normalized) "securities market history," $\{\mathcal{G}_t : t \in [0, T]\}$ (briefly $\{\mathcal{G}_t\}$) is the completion of the subfiltration generated by the normalized securities price processes, $S^*$.

After effectively restricting the information filtration to $\{\mathcal{G}_t\}$, existing analyses are able to price, by arbitrage considerations alone, all contingent claims (whose normalized payoffs are then, of course, $\mathcal{G}_T$-measurable). Call this "*full pricing by arbitrage*." Under *arbitrary* information, markets will, in general, be incomplete; nevertheless, we find:

THEOREM 2.2. *The following are equivalent, under an arbitrary information filtration:*

(a) *all EMMs yield the same finite-dimensional distributions for* $S^*$;
(b) *all EMMs coincide on* $\mathcal{G}_T$;
(c) *full pricing by arbitrage.*

## 3. SECURITY PRICE DIFFUSION MODELS

The following definition embraces the popular diffusion-based derivatives pricing models.

DEFINITION 3.1. A "security price diffusion" model is one in which the normalized price dynamics can be expressed in the form:

$$S_k^* = A_k + \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk}(S^*(u), u)\, dZ_j(u) \tag{3.1}$$

where: each $A_k$ is a predictable finite variation process of locally integrable variation; the argument list of each $\sigma_{jk}$ signifies dependence upon the stated variables only; and $(Z_1, \ldots, Z_n)$ is a vector standard Brownian motion under $P$.

We first analyze candidate risk-adjusted dynamics of normalized prices:

PROPOSITION 3.2. *Suppose that there exists an "equivalent local martingale measure" (ELMM) $P^*$, (i.e., a probability measure equivalent to $P$, and under which $S^*$ becomes a* local *martingale), and that* $\mathrm{E}[dP^*/dP \mid \mathcal{F}_t]$ *is locally square integrable under $P$.*[3] *Then we can rewrite (3.1) as:*

$$S_k^* = S_k^*(0) + \sum_{j=1}^{n} \int_0^{\cdot} \sigma_{jk}(S^*(u), u)\, dZ_j^*(u), \tag{3.2}$$

*where $(Z_1^*, \ldots, Z_n^*)$ is a vector standard Brownian motion under $P^*$.*

Note that the form of (3.2) is independent of the choice of $P^*$, and of the filtration employed. These facts provide the keys for further analysis.

THEOREM 3.3. *If the conditions of Proposition 3.2 hold, and (3.2) has a unique weak solution*[4] *in which $S^*$ is a martingale, then:*

(a) *every ELMM is an EMM; and*
(b) *full pricing by arbitrage holds.*

Because the usual information specification forces all contingent claims to depend solely on the market history, it is natural to say that existing pricing results are *preserved*, if full pricing by arbitrage holds under an arbitrary filtration. Theorem 3.3 immediately yields:

COROLLARY 3.4. *Suppose that in the existing model, (3.2) has a unique weak solution in which $S^*$ is a martingale. Then existing pricing results are preserved.*

The requirement of a unique weak solution seems invariably to be met in existing models, forming part of their analysis under a restricted filtration. The HK treatment of diffusion models is a case in point: the requirement is explicitly assumed (p. 395).

One application of our results is to Merton's (1973) stochastic interest rate extension of the Black–Scholes model. In applying their general framework to the diffusion case, HK had effectively assumed that the information filtration was that generated by normalized securities price processes. For Merton's model, this meant that the filtration could support only a single Brownian motion,[5] contradicting the fact that it clearly supports at least the

[3] Harrison and Kreps (1979) included a square integrability property in their definition of EMMs. We keep our local property separate.

[4] See, for example, Karatzas and Shreve (1988, pp. 300–301). Because $S^*$ satisfies (3.2) on our original probability space, it provides a strong solution (see Karatzas and Shreve, pp. 284–286). Our focus on weak solutions directs attention to the sufficient conditions for a unique weak solution, which are less onerous than for a strong solution.

[5] This follows from the fact that the normalized stock price can be rewritten in terms of a single Brownian motion that has deterministic coefficients under any ELMM.

two imperfectly correlated motions driving the stock and the bond. HK concluded that their results "do not enable us to claim that, say, European call options can be priced by arbitrage." Merton's model satisfies the conditions of Proposition 3.2 and Theorem 3.3. Moreover, the European call option expiring when the bond matures is measurable under the securities market history. By Theorem 3.3, therefore, it is priced by arbitrage—even though markets are incomplete (there being only two securities, but at least two sources of uncertainty).

## APPENDIX

*Proof of Theorem 2.2.* (c) $\Rightarrow$ (b) Suppose that the EMMs do not all coincide on $\mathcal{G}_T$. Then there exist $A \in \mathcal{G}_T$ and EMMs $P_1, P_2$, such that $P_1(A) \neq P_2(A)$. Consider the contingent claim $X = 1_A S_0(T)$, where $1_A$ denotes the indicator function of $A$. By construction, the normalized payoff of $X$ is $\mathcal{G}_T$-measurable. The prices assigned to $X$ under $P_j$, $j = 1, 2$, are given (with obvious notation) by: $V_j = S_0(0)\mathrm{E}^{(j)}[1_A] = P_j(A)$ (where $\mathrm{E}^{(j)}$ denotes the expectation operator under $P_j$), which differ according to the choice of $j$. The result follows by contradiction.

(b) $\Rightarrow$ (c) Suppose all EMMs coincide on $\mathcal{G}_T$. Let $X$ be a contingent claim whose normalized payoff is $\mathcal{G}_T$-measurable, and $P^*$ any EMM. The value of $X$ under $P^*$ is:

$$V = S_0(0)\mathrm{E}^*[X/S_0(T)], \tag{A.1}$$

where $\mathrm{E}^*$ is the expectation operator for $P^*$. Now, ex hypothesi, $X/S_0(T)$ is $\mathcal{G}_T$-measurable, and all the EMMs coincide on $\mathcal{G}_T$. Thus the right-hand side of (A.1) is independent of the choice of $P^*$.

(b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (b) is an immediate application of the following lemma. □

TECHNICAL LEMMA. *Let $S^*$ be any $\mathbb{R}^n$-valued process defined on a measurable space $(\Omega, \mathcal{F})$. Suppose that $\mathcal{P}$ is a family of probability measures on $(\Omega, \mathcal{F})$ for which the finite-dimensional distributions of $S^*$ coincide. Then the members of $\mathcal{P}$ coincide on $\{\mathcal{G}_t\}$, the filtration generated by $S^*$.*

*Proof of Technical Lemma.* Let $\mathcal{J}^{n\times k}$ denote the $(n \times k)$-fold product of $\mathcal{J}$, the collection of subsets of $\mathbb{R}$ of the forms: $\varnothing$, $(-\infty, b]$, $(a, b]$ $(a, \infty)$, $\mathbb{R}$, where $a, b \in \mathbb{R}$. Define $\mathcal{D}$ as the collection of elements of $\mathcal{G}_T$ of the form:

$$\{\omega \in \Omega : (S^*(t_1), \ldots, S^*(t_k)) \in J\}, \tag{A.2}$$

where $t_1 < \cdots < t_k$ and $J \in \mathcal{J}^{n\times k}$. It is elementary to verify that $\mathcal{D}$ is a semi-algebra (i.e., a family of subsets of $\Omega$, containing $\Omega$, closed under finite intersections, and such that: $D \in \mathcal{D} \Rightarrow \Omega \backslash D$ is expressible as a finite disjoint union of members of $\mathcal{D}$).

Ex hypothesi, the members of $\mathcal{P}$ coincide on $\mathcal{D}$. Let $\mu$ denote their common restriction to $\mathcal{D}$. Clearly $\mu$ is countably additive and $\mu(\Omega) = 1$. Hence, by Caratheodory's Extension

Theorem, $\mu$ has a unique extension to $\sigma(\mathcal{D})$, the sigma-algebra generated by $\mathcal{D}$. So the members of $\mathcal{P}$ agree on $\sigma(\mathcal{D})$.

But, for any $t$, $S^*(t)$ is $\sigma(\mathcal{D})$-measurable,[6] whence $\mathcal{G}_T = \sigma(\mathcal{D})$. The result follows. □

*Proof of Proposition 3.2.* By slight adaptation of Propositions 4 and 5 of Schweizer (1992),[7] we can write $A_k$ in the form:

$$A_k = S_k^*(0) + \sum_{j=1}^{n} \int_0^{\cdot} \theta_j \sigma_{jk}\, dt,$$

where $\theta_1, \ldots, \theta_n$ are predictable "market price of risk" processes, and we can determine that

$$\frac{dP^*}{dP} = \mathcal{E}\left\{ N(T) - \sum_{j=1}^{n} \int_0^T \theta_j\, dZ_j \right\}$$

for some $N \in \mathcal{H}^2_{0,loc}$ orthogonal to each $Z_j$, where $\mathcal{E}\{\quad\}$ denotes the exponential semimartingale.

We now apply Girsanov's Theorem to deduce that, for $j = 1, \ldots, n$,

$$Z_j^* \equiv \int_0^{\cdot} \theta_j\, du + Z_j$$

is a continuous local martingale under $P^*$. Moreover the predictable quadratic (co)variation processes $\langle Z_i^*, Z_j^* \rangle^*$ under $P^*$, coincide with $\langle Z_i, Z_j \rangle$ under $P$, whence $Z^*$ is a vector standard Brownian motion under $P^*$. □

*Proof of Theorem 3.3.* Obviously, any EMM is an ELMM. If (3.2) has a unique weak solution (i.e., determines the probability law of each $S_k^*$) then the finite-dimensional distributions of $S^*$ must coincide under all ELMMs. Combining these observations, $S^*$ has the same law under all EMMs. If, then, $S^*$ is a martingale under this law, we can use Theorem 2.2 to obtain full pricing by arbitrage. □

## REFERENCES

BABBS, S. H., and M. J. P. SELBY (1993): Pricing by Arbitrage in Incomplete Markets, FORC preprint, University of Warwick.

BLACK, F., and M. S. SCHOLES (1973): The Pricing of Options and Corporate Liabilities, *J. Political Economy* 81, 637–59.

FAMA, E. F. (1991): Efficient Capital Markets: II, *J. Finance* 46, 1575–617.

HARRISON, J. M., and D. M. KREPS (1979): Martingales and Arbitrage in Multiperiod Securities Markets, *J. Econ. Theory* 20, 381–408.

[6] Set $k = 1$ in (A.2) and recall that $\sigma(\mathcal{J}^n)$ is the Borel sigma-algebra on $\mathbb{R}^n$.

[7] Similar, but directly applicable results, obtained independently of Schweizer, were obtained as Theorems 4.10 and 4.11 by Babbs and Selby (1993).

HARRISON, J. M., and S. R. PLISKA (1981): Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stoch. Process. Appl.* 11, 215–60.

KARATZAS, I., and S. E. SHREVE (1987): *Brownian Motion and Stochastic Calculus*; in *Graduate Texts in Mathematics* 113. Berlin: Springer Verlag.

KREPS, D. M. (1981): Arbitrage and Equilibrium in Economies with Infinitely Many Commodities, *J. Math. Econ.* 8, 15–35.

MERTON, R. C. (1973): Theory of Rational Option Pricing, *Bell J. Econ. Mgt. Sci.* 4, 141–83.

SCHWEIZER, M. (1992): Martingale Densities for General Asset Processes, *J. Math. Econ.* 21, 363–78.