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First Draft: November 1997
This version: December 2000

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FORC Preprint: 98/92

The Theory of Good-Deal Pricing in Financial Markets*

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Abstract

In this paper the term 'no-good-deal pricing' stands for any pricing technique based on the absence of attractive investment opportunities – good deals – in equilibrium. The theory presented here shows that any such technique can be seen as a generalization of no-arbitrage pricing and that, with a little bit of care, it will contain the no-arbitrage and the representative agent equilibrium as the two opposite ends of a spectrum of possible no-good-deal equilibrium restrictions. We formulate the Extension and the Pricing Theorem in no-good-deal framework and establish general properties of no-good-deal price bounds determined by von Neumann-Morgenstern preferences. Our theory provides common footing to a range of applications, such as Bernardo and Ledoit (2000), Černý (1999), Cochrane and Saá-Requejo (2000), and Hodges (1998).

JEL classification code: G12, D40

Keywords: arbitrage, good deal, virtually desirable claim, state price functional, incomplete market, reward for risk measure, Generalized Sharpe Ratio, admissible price region, equivalent martingale measure

*We are grateful to Antonio Bernardo, John Cochrane, Martin Cripps, Marcus Miller, David Oakes and the participants at the 1st World Congress of Bachelier Finance Society for helpful comments. All remaining errors are our own.

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Introduction

The term 'no-good-deal pricing' in this paper encompasses pricing techniques based on the absence of attractive investment opportunities – good deals – in equilibrium. We borrowed the term from Cochrane and Saá-Requejo (2000) who pioneered the calculation of price bands conditional on the absence of high Sharpe Ratios. Alternative methodologies for calculating tighter-than-no-arbitrage price bounds have been suggested by Bernardo and Ledoit (2000), Černý (1999), Hodges (1998). The theory presented here shows that any of these techniques can be seen as a generalization of no-arbitrage pricing. The common structure is provided by the Extension and Pricing Theorems, already well known from no-arbitrage pricing, see Kreps (1981). We derive these theorems in no-good-deal framework and establish general properties of no-good-deal prices. These abstract results are then applied to no-good-deal bounds determined by von Neumann-Morgenstern preferences in a finite state model[1]. One important result is that *no-good-deal bounds generated by an unbounded utility function are always strictly tighter than the no-arbitrage bounds.* The same is *not true for bounded utility functions.* For smooth utility functions we show that one will obtain the no-arbitrage and the representative agent equilibrium as the two opposite ends of a spectrum of no-good-deal equilibrium restrictions indexed by the maximum attainable certainty equivalent gains.

A sizeable part of finance theory is concerned with the valuation of risky income streams. In many cases this valuation is performed against the backdrop of a frictionless market of basis assets. Whenever the payoff of the focus asset can be synthesized from the payoffs of basis assets the value of the focus asset is uniquely determined and this valuation process is preference-free – any other price of the focus asset would lead to an arbitrage opportunity. In reality, however, the perfect replication is an unattainable ideal, partly due to market frictions and partly due to genuine sources of unhedgeable risk presenting themselves, for example, as stochastic volatility. When perfect replication is not possible – a situation synonymous with an 'incomplete

[1] Each of the no-good-deal restrictions mentioned above is in fact derived from a utility function: for Bernardo and Ledoit it is the Domar-Musgrave utility, for Cochrane and Saa-Requejo it is the truncated quadratic utility, and for Hodges it is the negative exponential utility, see Černý (1999).

market'– the standard Black-Scholes pricing methodology fails because the price of the focus asset is no longer unique.

One way to overcome this difficulty is to single out one price of the focus asset consistent with the price of basis assets. This can be achieved via the representative agent equilibrium, where the 'special' pricing functional is obtained from the marginal utility of the optimized representative agent's consumption, see Rubinstein (1976).

A valid objection against the representative agent equilibrium is that it imposes very strong assumptions about the way the equilibrium is generated. Alternative route is to look for preference-free price bounds, in the spirit of Merton (1973), which leads to the calculation of super-replication bounds[2]. However, these bounds have a practical shortcoming in that they tend to be rather wide and hence not very informative.

Recently a new approach has emerged whereby it is accepted that the price of a non-redundant contingent claim is not unique, but an attempt is made to render the price bound more informative by restricting equilibrium outcomes beyond no arbitrage. Typically, one tries to hedge the focus asset with a self-financed portfolio of basis assets to maximize a given 'reward for risk' measure and rules out those focus asset prices that lead to a highly desirable hedging strategy. Such a procedure gives a price interval for every contingent claim where the interval is the wider the more attractive investment one allows to exist in equilibrium.

The idea of good deals as an analogy of arbitrage comes naturally at this point. Recall that arbitrage is an opportunity to purchase an unambiguously positive claim, that is a claim that pays strictly positive amount in some states and non-negative amounts in all other states, at no cost. While the absence of arbitrage is surely a necessary condition for the existence of a market equilibrium, it is still a rather weak requirement. Considering a claim with zero price that either earns \$1000 or loses \$1 with equal probability, one feels that, although not an arbitrage, such investment opportunity still should not exist in equilibrium. One can then define 'approximate' arbitrage, or as we say here 'good deal', as an opportunity to buy a *desirable claim* at no cost.

[2]See Ritchken (1985) for a one-period finite state setting and El Karoui and Quenez (1995) for a continuous time model.

Historically, good deals have been associated with high Sharpe Ratios. The Arbitrage Pricing Theory of Ross (1976) is a prime example of ruling out high Sharpe Ratios. Further breakthrough came with the work of Hansen and Jagannathan (1991) who established a duality link equating the maximum Sharpe Ratio available in the market and the minimum volatility of discount factors consistent with all prices. While Hansen and Jagannathan use this result to construct an empirical lower bound on discount factor volatility, Cochrane and Saá-Requejo (2000) realize that it can be used in the opposite direction, namely to limit the discount factor volatility and thus to infer the no-good-deal prices conditional on the absence of high Sharpe Ratios.

It is well known that outside the elliptic world the absence of high Sharpe Ratios does not generally imply the absence of arbitrage. Other researchers therefore tried to come up with reward for risk measures that would automatically capture all arbitrage opportunities. Bernardo and Ledoit (2000) base the definition of good deals on the gain-loss ratio and Hodges (1998) uses a generalized Sharpe Ratio derived from the negative exponential utility function. Černý (1999) calculates the Hansen-Jagannathan duality link for good deals defined by an arbitrary smooth utility function and proposes a reward for risk measure generated by the CRRA utility class.

In this paper we point out that, regardless of the specific definition of good deals, the nature of the duality restrictions is formalized in the *extension theorem*, already well known from the no-arbitrage theory[3]. The extension theorem states that any incomplete market without good deals can be augmented by adding new securities in such a way that the resulting complete market has no good deals. The important point is that the set of complete market state prices which do not allow good deals is independent both of the basis and the focus assets. The pricing theorem uses the above fact to assert that any no-good-deal price of a focus asset must be supported by a complete market no-good-deal pricing functional. These results are crucial both for establishing the theoretical properties of no-good-deal prices, which will be discussed here, and for practical applications, see for example Černý (1999).

[3]Incidentally, it is Ross again (we already mentioned his APT contribution) who has introduced the extension theorem to finance in his 1978 paper on the valuation of risky streams. The extension theorem in no-arbitrage setting has been studied extensively in the realm of mathematical finance, starting with Kreps (1981).

The paper is organized as follows: The first section reviews the essentials of no-arbitrage theory and builds the no-good-deal theory in analogous way. The second section derives abstract versions of the Extension Theorem and the Pricing Theorem in no-good-deal framework. Section three applies the theory to desirable claims defined by von Neumann-Morgenstern preferences in finite state space. The results of this section are summarized in the no-good-deal pricing theorem of section four, where we also discuss the similarities and differences between the finite and infinite state space. The fifth section gives a geometric illustration of the theory by taking an example from the literature – desirable claims determined by Sharpe ratio as in Cochrane and Saá-Requejo (2000). Finally, section six concludes.

1 Arbitrage and good deals

In this section we briefly describe the axiomatic theory of no-arbitrage pricing[4] and show how it can be analogously used to define good deals and no-good-deal prices. The model of security market is an abstract one, the application to a multiperiod security market is spelled out in Clark and in the references mentioned in the introduction. One should bear in mind that in this section the origin 0 is not to be taken literally as a position with zero wealth, rather it is the position relative to an initial endowment.

We will have a topological vector space X of all contingent claims. The space of all continuous linear forms on X (strong dual) is denoted X^*. The vector space X will be endowed with a natural ordering $\geq$ which defines the *positive cone*[5] $X_+ \equiv \{x \in X : x \geq 0\}$. The cone of *strictly positive claims* is

$$X_{++} \equiv \{x \in X : x > 0\} = X_+ \backslash \{0\}.$$

Suppose we have a collection of claims with predetermined prices, so called *basis assets*. These claims generate the marketed subspace M and

[4] This section is based on Clark (1993).

[5] In no-arbitrage pricing one works with natural (canonical) ordering. Thus, for example, positive cone in $\mathbb{R}^n$ is formed by n-tuples with non-negative coordinates, positive cone in L^p by non-negative random variables etc. Note that a claim $(1,0,0) \in \mathbb{R}^3$ is strictly positive in the canonical ordering on $\mathbb{R}^3$ but at the same time it is equal to zero with positive probability, therefore 'strictly positive in X' is not to be confused with strictly positive with probability one. The term 'strictly positive' will only be used when we have in mind the canonical ordering on X.

their prices define a price correspondence p on this subspace. The cone of strictly positive claims has the following role:

Definition 1 *A strictly positive claim with zero or negative price is called* arbitrage.

Definition 2 *Let M be a linear subspace of X. A linear functional $p : M \to \mathbb{R}$ is* positive *if $p(m) \geq 0$ for all $m \in M \cap X_+$. We say that p is* strictly positive *if $p(m) > 0$ for all $m \in M \cap X_{++}$.*

Standing assumption 1 There is a strictly positive marketed claim.

Clark shows that under this assumption no arbitrage implies that the price correspondence p is in fact a strictly positive linear functional. This result guarantees, among others, unique price for each marketed claim.

Now we move on to define generalized arbitrage opportunities – good deals. Suppose we have a convex set K disjoint from the origin which we interpret as the set of all desirable claims. At the moment we do not specify how the set of desirable claims is obtained or what are its additional properties. The relationship between arbitrage and strictly positive claims is generalized as follows:

Definition 3 *A desirable claim with zero or negative price is called a* good deal.

Frictionless trading leads to the following definition:

Definition 4 *A claim is* virtually desirable *if some positive scalar multiple of it is desirable. The set of all virtually desirable claims is denoted C_{++},*

$$C_{++} = \bigcup_{\lambda > 0} \lambda K$$

A virtually desirable claim at zero or negative price constitutes a *virtually good deal.* When markets are frictionless the presence of a virtually good deal implies the existence of a good deal, simply by re-scaling the portfolio which gives the virtually good deal. Thus the absence of good deals implies absence of virtually good deals and vice versa.

Proposition 1 *There are no good deals if and only if there are no virtually good deals.*

Geometrically the set of all virtually desirable claims is the convex cone with vertex at 0 generated as a convex hull of 0 and the set of desirable claims K.

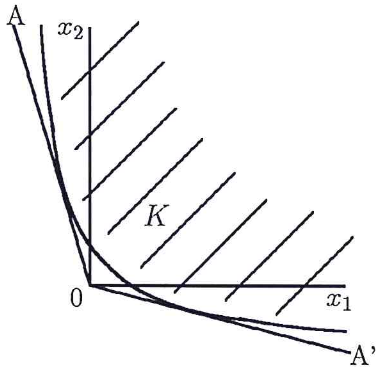

Figure 1: The cone of virtually good deals C (AOA') generated by the set of good deals K

To benefit fully from the analogy between arbitrage and good deals we have to realize that, similarly to X_+, the cone $C \equiv C_{++} \cup \{0\}$ defines ordering on the space of all contingent claims by putting $x_1 \succeq x_2$ when $x_1 - x_2 \in C$ and $x_1 \succ x_2$ when $x_1 - x_2 \in C_{++}$. Similarly as in Definition 2 we can speak of C-positive functionals and C-strictly positive functionals. The key point is that the link between no arbitrage and strictly positive pricing rule carries over to good deals.

Theorem 1 *Suppose that there is a (virtually) desirable marketed claim and the price correspondence p on the marketed subspace M gives no good deals. Then $p : M \to \mathbb{R}$ is a C-strictly positive linear functional, i.e. $p(m)$ is unique for all $m \in M$, $p(m_1+m_2) = p(m_1)+p(m_2)$ for all $m_1, m_2 \in M$ and p assigns strictly positive price to all (virtually) desirable marketed claims.*

Proof. The proof follows from the proof of Theorem 1 in Clark (1993) when X_+ is replaced with C, or equivalently $>$ with $\succ$. ∎

Since we are guaranteed that p is a linear functional we can define a subspace $M_0(p)$ of all claims with zero price which plays essential role in the extension theorem.

2 Extension theorem

2.1 The idea

The extension theorem states that an incomplete market without good deals can be augmented by adding new securities in such a way that the resulting complete market has no good deals. The important point is that the set of complete market state prices which provide no good deals is independent both of the basis and the focus assets present in the market. The pricing theorem uses the above fact to provide a complete characterization of the no-good-deal price region since any no-good-deal price of a focus asset must be supported by a complete market no-good-deal pricing functional.

The pricing function on the marketed subspace defines a yet smaller subspace of marketed claims with zero price, denoted $M_0(p)$.

Definition 5 *For a given strictly positive pricing functional p on M we say that*

$$M_0(p) = \{m \in M : p(m) = 0\}$$

is a zero investment marketed subspace[6].

In the absence of good deals this subspace must be disjoint from the set of good deals K,

$$M_0(p) \cap K = \emptyset.$$

As the figure 2 suggests it is quite natural to expect that if $M_0(p)$ is disjoint from K then there is a hyperplane H containing $M_0(p)$ and still disjoint from K. The separating hyperplane H is interpreted as the zero investment subspace of the completed market. The fact that H is disjoint from K guarantees that there are no good deals in the completed market.

2.2 Technicalities

Mathematicians distinguish among three types of separation of two convex sets. Weak separation means the separating hyperplane may touch both

[6]The term 'zero investment portfolio' was introduced by Ingersoll (1987), alternatively one could use the term 'zero cost marketed subspace'.

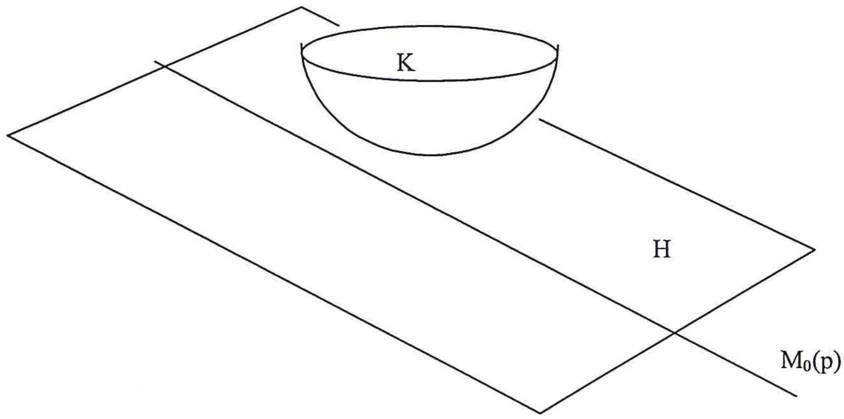

Figure 2: Illustration to the Extension Theorem

sets. Strict separation signifies that the separating hyperplane does not touch either of the convex bodies but can come arbitrarily close to each of them. Strong separation occurs when there is a uniform gap between the separating hyperplane and both of the convex sets. It is hard to find references to semistrict separation, which is what we need here, because the separating hyperplane will touch $M_0(p)$ but we would like it to be disjoint from K.

By drawing pictures in $\mathbb{R}^2$ one is tempted to conjecture that semistrict separation is always possible in finite dimension. However, this conjecture is false, as a three dimensional example in Lemma A.1 shows. Thus the situation in finite dimension is quite clear: K and $M_0(p)$ can always be weakly separated, and it follows from Lemma A.1 that in general one cannot expect more. When K is closed and bounded the two sets can be strongly separated, and when K is open the two sets can be separated semistrictly, see for example Beavis and Dobbs (1990).

In infinite dimension not even weak separation is available automatically, for a nice counterexample see Schachermayer (1994). Weak separation is available when K has non-empty interior and semistrict separation is possible when K is open. Strong separation becomes possible when K is compact. These facts motivate the following definition.

Definition 6 *We say that the set of desirable claims K is* boundedly generated *if there is a closed bounded set $B \subset K$ such that any desirable claim in K can be obtained as a scalar multiple of a desirable claim in B.*

For boundedly generated sets of desirable claims we obtain a clear-cut result both for the extension and pricing theorem thanks to weak compactness of bounded sets in standard probability spaces.

Theorem 2 (Extension Theorem) *Suppose $X = L^q(\Omega, \mathcal{F}, P), 1 < q < +\infty$, the set of good deals K is closed and boundedly generated and the zero investment marketed subspace is closed. Then there is a C-strictly positive continuous linear extension of the pricing rule p on the marketed subspace to the whole market if and only if there is no good deal.*

Proof. See Appendix A. ■

Definition 7 *Suppose we fix a set of desirable claims K with the implied cone C of virtually desirable claims. A continuous C-strictly positive functional on X is called a* no-good-deal pricing functional[7]. *The set of all such functionals is denoted C^*_{++}*

$$C^*_{++} = \{\varphi \in X^* : \varphi(x) > 0 \text{ for all } x \in K\}.$$

Making use of the Extension Theorem we can completely characterize the no-good-deal price region for several focus assets jointly[8].

Theorem 3 (Pricing Theorem) *Suppose X is an L^p space, $1 < p < +\infty$, and the set of desirable claims is closed and boundedly generated. Let us have a closed marketed subspace M in which prices are given by a C-strictly positive and continuous linear functional ϕ. Let there be further m focus assets with payoffs $y_1, y_2, \dots, y_m$, no-good-deal price of which we want to find. Then*

a) the no good deal price region P for these claims is given as

$$P = \left\{(\varphi(y_1), \dots, \varphi(y_m)) \in \mathbb{R}^m : \varphi \in C^*_{++} \text{ and } \varphi|M = \phi\right\},$$

where $\varphi|M$ is the restriction of φ to M,

b) P is a convex set in $\mathbb{R}^m$,

[7] In practical applications it is unusual to work with abstract linear functionals. For different representations of complete market pricing rules see e.g. Dybvig and Ross (1987), page 104.

[8] For discussion of these results see Theorem 5.

c) defining $N \equiv \mathrm{span}(M, y_1, y_2, \dots, y_m)$ *the dimension of the price region* P *satisfies*

$$\dim P = \mathrm{codim}_N M \tag{1}$$

which is the codimension of the marketed subspace in the enlarged marketed subspace N.

d) the no-good-deal price of y_i *is unique if and only if* y_i *is redundant, that is* $y_i \in M$

e) let K_1 *and* K_2 *be two boundedly generated sets of desirable claims,* $K_1 + \varepsilon B_1 \subset K_2$ *for some* $\varepsilon > 0$*, where* B_1 *is a unit ball in* X *in strong topology. Let* P_1 *and* P_2 *be the corresponding no-good-deal price regions. Then*

$$\mathrm{cl}P_2 \subset \mathrm{rel} - \mathrm{int}P_1.$$

Proof. See Appendix A ■

3 Desirable claims and agent preferences

The results derived so far were concerned with an abstract set of good deals. In this section we will show how desirable claims can be determined by agent preferences, in particular by expected utility, and examine when good deals defined in this way include all arbitrage opportunities. This approach allows to formulate a whole *range of equilibrium restrictions* as we choose K smaller or larger. We discuss two limiting cases of no-good-deal pricing – the no-arbitrage pricing and representative agent equilibrium. We only have a complete answer for X finite dimensional, so we stick to this case from the beginning, leaving the technical issues related to infinite dimension to section 4.2.

Consider a preference relation $\succeq^*$ which is a) convex, in the sense that the level set $\{x \in X : x \succeq^* y\}$ is convex for all $y \in X$; b) X_{++} strictly increasing, i.e. $x - y \in X_{++}$ implies $x \succ^* y$; c) continuous, i.e. both sets $\{x \in X : x \prec^* y\}, \{x \in X : x \succ^* y\}$ are open.

For any strictly increasing utility function U the preference relation

$$x \succ^* y \Leftrightarrow \mathrm{E}U(x + w_r) > \mathrm{E}U(y + w_r) \tag{2}$$

satisfies the conditions a), b) and c). The reference point w_r is very often taken as wealth resulting from the risk-free investment, with x and y being excess returns. The analysis remains valid, however, even when reference wealth level w_r is stochastic. We may want to think of w_r, for example, as the representative agent's optimal wealth derived from investing into basis assets only.

Let **1** be a claim that pays 1 unit of the numeraire in each state of the world. Let us take a non-negative number a and define $K(a)$ as the upper level set

$$K(a) = \{x \in X : x \succeq^* a\mathbf{1}\}.$$

Thus we obtain a family of sets of desirable claims indexed by the desirability level a[9]. The quantity a is interpreted as the certainty equivalent gain over and above the reference wealth level w_r. Monotonic transformations of a define various, but in essence equivalent, reward for risk measures[10].

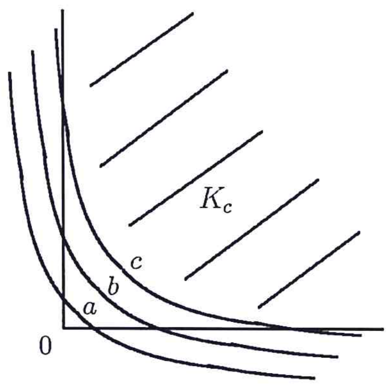

Figure 3: Sets of desirable claims $K_a \supset K_b \supset K_c$ indexed by desirability levels $a < b < c$

Note that if a claim x is desirable then all claims $x + X_+$ are desirable

[9]This works well for smooth utility functions. Bernardo and Ledoit use Domar-Musgrave (piecewise linear) utility function which gives $K(a) = K(b)$ for all $a > 0, b > 0$. The widening of the set of desirable claims is not achieved by changing the parameter a but rather by changing the shape of the utility function, that is by varying the gain-loss ratio – the ratio of slopes of the two linear parts of the function.

[10]Not to be confused with 'coherent risk measures', of Artzner et al. (1999). As noted in Hodges (1998), the lower good-deal bound is a coherent risk measure in the sense of Arztner et al., whereby the set of 'acceptable risks' is identified with the set of desirable claims. See also Jaschke and Kuechler (2001)

too, which is a natural property that all 'good' sets of desirable claims should satisfy.

The key question is, whether, or under what assumptions, the set of desirable claims is boundedly generated. First, let us discuss situations when it is not.

Definition 8 *The set $K(a)$ has an* asymptote $x \in X$ *if* $\{\lambda x | \lambda \in \mathbb{R}\} \cap K(a) = \emptyset$ *and for any* $\varepsilon > 0$ $\{\lambda x | \lambda \in \mathbb{R}\} \cap K(a - \varepsilon) \neq \emptyset$.

Clearly, unless $K(a)$ is asymptote-free one cannot hope, in general, that it will be boundedly generated. With this observation in mind we proceed to examine sets of desirable claims generated by Von Neumann-Morgenstern preferences.

3.1 Arbitrage subsumed by good deals

In order for no-good-deal pricing to be economically meaningful the absence of good deals must imply the absence of arbitrage. For this to be true each strictly positive claim must be virtually desirable, mathematically $C_{++}(a) \supseteq X_{++}$. In general not all arbitrage opportunities will be covered by virtually good deals. This leads us to the following definition:

Definition 9 *We say that the preference relation* $\succeq^*$ *is* arbitrage-sensitive *if and only if for any desirability level a and any strictly positive*[11] *claim x a sufficiently large scalar multiple of x is preferred to the claim $a\mathbf{1}$.*

In other words arbitrage sensitivity requires that a sufficiently high position in any arbitrage opportunity gives an (arbitrarily) good deal. A simple example of strictly increasing preferences that do not satisfy this requirement is given below.

The indifference curve has two asymptotes, one vertical and one horizontal. In such a case the set of virtually good deals will contain the interior of the positive quadrant but not the axes x and y.

In the case of von Neumann-Morgenstern preferences the arbitrage sensitivity condition is met by unbounded utility functions (Lemma B.3), but

[11] Note again that strictly positive does not mean strictly positive with probability one, but rather non-negative and different from zero. See also the footnote in section 1.

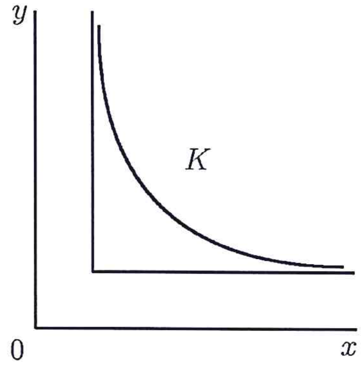

Figure 4: Set of desirable claims K such axes x and y are not virtually desirable

it is *violated by all bounded utility functions*, because strictly positive claims which pay nothing with sufficiently high probability do not constitute virtually desirable claims, see Lemma B.2. At the same time the 'inside' of positive orthant (that is all claims which are strictly positive with probability 1) is virtually desirable. Thus to prevent arbitrage opportunities one must take

$$C_+ = \operatorname{cl} \bigcup_{\lambda>0} \lambda K$$

instead of

$$C_+ = \bigcup_{\lambda>0} \lambda K.$$

Although this strengthening of no-good-deal equilibrium is purely cosmetic from the practical point of view, it highlights a different problem. Since one is not guaranteed that $C_+ \supsetneqq X_+$, the no-good-deal price bounds generated by bounded utility functions are not necessarily tighter than the no-arbitrage bounds. This problem is pointed out in Bernardo and Ledoit (2000) and it is present equally in Sharpe Ratio restrictions of Cochrane and Saá-Requejo (2000) and generalized Sharpe Ratio analysis of Hodges (1998).

3.2 Arbitrage as a limiting case of good deals

Suppose now that the preferences are arbitrage sensitive, i.e. $C_{++}(a) \supseteq X_{++}$ for all $a \in \mathbb{R}$. At the same time the sets $C_{++}(a)$ become progressively smaller as the desirability level a increases. It is interesting to see under what conditions good deals reduce to arbitrage in the limit, that is under what circumstances do we have

$$\bigcap_{a\geq 0} C_{++}(a) = X_{++}.$$

Definition 10 *We say that the preference relation $\succeq^*$ is* downside-sensitive *if each ray generated by a non-positive claim is dominated by a claim $a\mathbf{1}$ where a is a sufficiently large positive number.*

As an immediate consequence we have

Proposition 2 *For preferences which are arbitrage-sensitive no arbitrage is a limiting case of no good deal equilibria as $a \to \infty$ if and only if the preference relation is downside-sensitive.*

For von Neumann-Morgenstern preferences to be downside-sensitive the generating utility function must discount negative outcomes sufficiently heavily,

$$\lim_{x\to-\infty} \frac{x}{U(x)} = 0 \tag{3}$$

as demonstrated in Lemma B.4. This condition is satisfied by all frequently used utility functions, except for the Domar-Musgrave utility, see footnote 9.

More importantly, by virtue of Lemma B.1 the downside sensitivity property (3) guarantees that the sets of desirable claims are boundedly generated.

Theorem 4 *For any unbounded utility function satisfying*

$$\lim_{x\to-\infty} \frac{x}{U(x)} = 0$$

the set of desirable claims $K(a)$ is boundedly generated for any $a \in \mathbb{R}$.

Proof. See Appendix B. ■

3.3 Representative agent equilibrium as a limiting case

Suppose for simplicity that the reference wealth level is $w_r = 0$. As $a \to 0$ the cone of virtually good deals is getting wider and eventually becomes a hyperplane provided that the indifference surface is sufficiently smooth. At

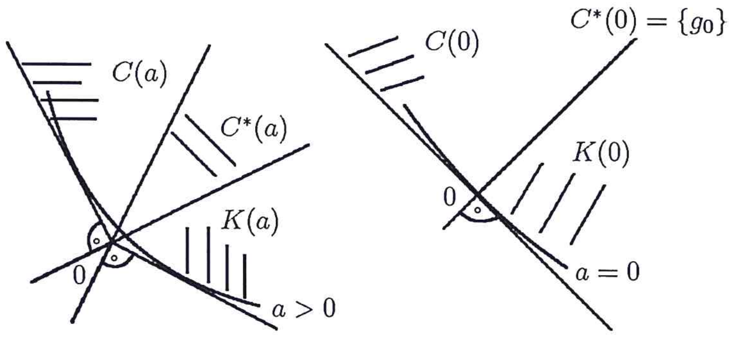

Figure 5: Set of desirable claims $K(a)$ for $a \to 0$

the same time the cone of complete market state prices becomes narrower until it finally collapses into the gradient of indifference surfaces.

In the presence of basis assets we first find the market portfolio w_M that achieves the maximum certainty equivalent gain a_M. It is clear that if we add more assets then the attainable certainty equivalent gain will be at least a_M. The condition $a \leq a_M$ is now equivalent to requiring that the new asset does not shift the efficient frontier

$$\mathrm{E}U(Z + w_M) - \mathrm{E}U(w_M) \leq 0, \tag{4}$$

where Z is the excess return of the new asset with respect to the market portfolio. For Z sufficiently small and U sufficiently smooth we have $\mathrm{E}U(Z + w_M) - \mathrm{E}U(w_M) \approx \mathrm{E}U'(w_M)Z = 0$, the last equality being the consequence of the no-good-deal condition (4). This implies that the new claim must be priced with the change of measure proportional to the marginal utility of the representative agent. When there are no basis assets this amounts to risk-neutral pricing[12] because $U'(R^f w_0)$ is constant. We formalize this intuition in the following section.

[12] Not to be confused with the pricing under risk-neutral probabilities. Here we mean the risk-neutral valuation under objective probabilities, one which is frequently used in macroeconomics.

4 No-good-deal pricing theorem for utility-based equilibrium restrictions

4.1 Finite state space

In a finite state model with equilibrium restrictions generated by an unbounded utility function one obtains a clear-cut characterization of no-good-deal price bounds.

Theorem 5 *Suppose* $\dim X < +\infty$. *Let us have a downnside-sensitive unbounded (arbitrage-sensitive) utility function U, which is once differentiable and strictly concave. Denote the set of all claims with desirability level a or higher as $K(a)$. Let there be a marketed subspace M in which there is no arbitrage and denote M_0 the set of marketed claims with zero price. Assume that there is a risk-free security with return R^f. Let there be further m focus assets with payoffs $y_1, y_2, \dots, y_m$, and let us denote $P(a) \subset \mathbb{R}^m$ the region of prices of the focus assets such that no claim in the extended market has desirability level exceeding a. Then*

a)

$$\sup_{w \in M_0} U(w) \equiv a_M < +\infty$$

is achieved in M_0. Let us denote the unique argmax w_M – the market portfolio.

b) $P(a)$ is empty for $a < a_M$ and it is a non-empty convex set for $a \geq a_M$,

c) defining $N \equiv \mathrm{span}(M, y_1, y_2, \dots, y_m)$ the dimension of the price region $P(a)$ satisfies

$$\dim P(a) = \mathrm{codim}_N M \tag{5}$$

for $a > a_M$, whereas $P(a_M)$ is a singleton

$$P(a_M) = \{\left(\frac{\mathrm{E}m_M y_1}{R^f}, \frac{\mathrm{E}m_M y_2}{R^f}, \dots, \frac{\mathrm{E}m_M y_m}{R^f}\right)\} \tag{6}$$

with $m_M = \frac{U'(w_M)}{\mathrm{E}U'(w_M)}$

d) if $\mathrm{codim}_N M > 0$, that is if at least one focus asset is non-redundant, then for all a and b such that $a_M \leq a < b$

$$\mathrm{cl}P(a) \subset \mathrm{rel}-\mathrm{int}P(b),$$

that is for $a < b$ the no-good-deal price region $P(a)$ is strictly smaller than $P(b)$

e) denoting P_{NA} the no-arbitrage price region for the focus assets we have

$$\lim_{a \to +\infty} P(a) = P_{NA}$$

that is $\cup_{a \in \mathbb{R}} P(a) = P_{NA}$.

f) if $\operatorname{codim}_N M > 0$ then for any desirability level a the no-good-deal price region is strictly smaller than the no-arbitrage price region.

Remark 1 1. *One needs an unbounded utility function to make sure that good deals include all arbitrage opportunities. A bounded utility function leaves out strictly positive claims which are equal to zero with sufficiently high probability. For the same reason the set of good deals defined by a bounded utility function need not be asymptote-free, in which case one cannot expect property d) to hold.*

2. *The condition*

$$\lim_{x \to -\infty} \frac{x}{U(x)} = 0$$

is necessary (and sufficient) for no-good-deal restrictions to reduce to no-arbitrage restrictions in the limit. For unbounded utility functions this condition implies that the sets of good deals are asymptote-free. In finite dimension asymptote-free set is always boundedly generated (see the proof of Theorem 5).

3. *Existence of a risk-free security simplifies the pricing formula (6) but this assumption is not necessary. It suffices to have a marketed claim x_0 which is strictly positive, in fact desirable would suffice, see the proof of Theorem 2. The pricing formula has to be adjusted accordingly, replacing R^f with $\frac{\mathrm{E}U'(w_M)\frac{x_0}{p_0}}{\mathrm{E}U'(w_M)}$, where $p_0 > 0$ is the price of x_0.*

4. *Smoothness of U is necessary (and sufficient) to obtain the singleton property of $P(a_M)$. Strict concavity is sufficient but not necessary, in addition it implies uniqueness of the market portfolio w_M. The smoothness assumption is relaxed in Bellini and Fritelli (2000), whose results imply that in general $P(a_M)$ is non-empty but not necessarily a singleton when $X = L^\infty$.*

4.2 Infinite state space

We do not know how to rephrase Theorem 5 in an infinitely dimensional state space. Let us at least summarise some of the important differences that make the problem in infinite dimension harder and more interesting.

1. **Continuity:** Expected utility in finite dimension is automatically continuous. In infinite dimension continuity is determined by the left tail of the utility function and the topology. For example, for a utility function with

$$\lim_{x \to -\infty} \frac{U'(x)}{|x|^{\delta-1}} = const$$

the expected utility is continuous in $L^p, p \geq \delta \geq 1$. With continuous expected utility Extension Theorem is available via Hahn-Banach theorem. Similarly, with U defined on the whole real line the expected utility is continuous in L^∞,this fact is used in Bellini and Fritelli (2000).

2. **No arbitrage vs. bounded attractivness:** In finite dimension no arbitrage implies that maximum certainty equivalent gain attainable in the marketed subspace is always finite. In infinite dimension this is no longer true, one can have no arbitrage but yet the attractiveness of self-financed investment opportunities may be unbounded. Consider, for example, a complete market where the state of the world is determined by the random variable X with $\chi^2(1)$ distribution. Assume that the risk-free rate is 0 and suppose the state prices are given by the following state price functional (change of measure)

$$m(X) = const \frac{e^{\frac{X}{2}}}{1+X}.$$

Since $\mathrm{E}\frac{e^{\frac{X}{2}}}{1+X}$ is finite the constant above can be set to satisfy

$$\mathrm{E}m = 1.$$

It is known, see for example Černý (1999), that the certainty equivalent gain for the negative exponential utility in a complete market is

$$a_M = \mathrm{E}m \ln m.$$

However in this case $a_M = \infty$ as the integral

$$\mathrm{E}m \ln m = \int_0^\infty \frac{e^{\frac{x}{2}}}{1+x}\left[x - \ln(1+x)\right]\frac{e^{-\frac{x}{2}}}{\sqrt{x}}dx$$

diverges at the upper bound. As we pointed out above, expected utility is continuous in this case.

3. **Asymptotes:** As in finite dimension, asymptotes can only be strictly positive, and such asymptotes can exist only when utility is bounded. For unbounded utility satisfying $\lim_{x \to -\infty} \frac{x}{U(x)} = 0$ any set of desirable claims is asymptote-free. However, unlike in finite dimension, this does not imply that the set of desirable claims is boundedly generated. One can easily see this by examining a sequence of strictly positive rays which have non-zero payoff with increasingly smaller probabilities. In other words, the positive cone in infinite dimension is not boundedly generated by von Neumann-Morgenstern preferences. This is really caused by the upper tail of the utility function, thus it has nothing to do with continuity of preferences.

5 Geometric illustration - Sharpe ratio restrictions

The simplest illustration of the duality between the set of desirable claims and the set of no-good-deal complete market pricing functionals comes from the mean-variance framework. The term 'good deal' was introduced by Cochrane and Saá-Requejo (2000) in a specific situation where desirable claims are those with high Sharpe ratio of the excess return. This particular application of no-good-deal equilibrium provides a very nice geometric illustration of the theory developed in sections 1 and 2.

It is convenient to have $X \equiv L^2$. Denoting h the bound on Sharpe ratios which are acceptable in equilibrium the set of desirable claims is given as

$$K(h) = \{x \in X : \frac{\mathrm{E}x}{\sqrt{\mathrm{E}x^2 - (\mathrm{E}x)^2}} \geq h\}.$$

We note that the cone of virtually desirable claims $C_{++}(h)$ is identical to $K(h)$ and it can be rewritten more conveniently as

$$C(h) = K(h) = \{x \in X : \frac{\mathrm{E}x}{||x||} \geq \frac{h}{\sqrt{1+h^2}}\},$$

where $||\cdot||$ is the L^2 norm. The geometry of the cone of desirable claims is simple – it is a circular cone with the axis formed by vector $\mathbf{1} \in L^2$ and the angle at the vertex is α, such that $\cos\alpha = \frac{h}{\sqrt{1+h^2}}$, and consequently $\cot\alpha = h$, see Figure 2.

Recall that no-good-deal price functionals φ must satisfy $\varphi(K) > 0$ and that each continuous linear functional φ on L^2 is uniquely represented by a random variable $m \in L^2$ as follows

$$\varphi(x) = \mathrm{E}mx.$$

Thus the cone of no-good-deal pricing rules can be identified with the cone of discount factors

$$\tilde{C}^*_{++}(h) = \{m \in L^2 : \mathrm{E}mx > 0 \text{ for all } x \in C_{++}(h)\}.$$

Note that every no-good-deal discount factor must be *at sharp angle* with every desirable claim. But since the shape of $C_{++}(h)$ is very simple we can characterize $\tilde{C}^*_{++}(h)$ explicitly.

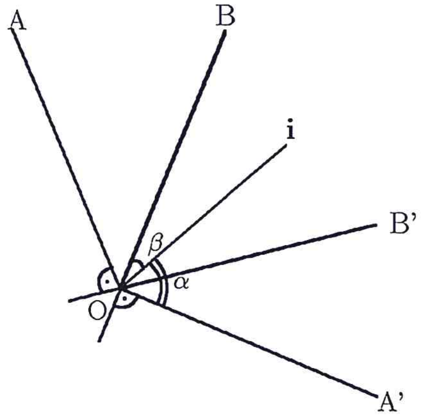

Figure 6: Cone of good deals K (AOA') and the cone of discount factors $\tilde{C}^*$ (BOB') determined by maximum attainable Sharpe ratio $h = \cot\alpha$.

As the picture shows the cone of no-good-deal discount factors is again a circular cone with the axis $\mathbf{1} \in L^2$ and with the angle at the vertex $\beta = \frac{\pi}{2} - \alpha$, that is

$$\tilde{C}^*(h) = \{m \in L^2 : \frac{\mathrm{E}m}{\sqrt{\mathrm{E}m^2 - (\mathrm{E}m)^2}} > \cot(\frac{\pi}{2} - \alpha) = \tan\alpha = \frac{1}{\cot\alpha} = \frac{1}{h}\}.$$

In other words any discount factor $m \in L^2$ that prevents Sharpe Ratios higher than h must satisfy $\frac{\mathrm{E}m}{\mathrm{V}(m)} \geq \frac{1}{h}$ which is the condition obtained by Hansen and Jagannathan (1991).

5.1 Preventing arbitrage

The above relationship describes the duality between $C_{++}(h)$ and $C^*_{++}(h)$ but it does not guarantee that the functionals in $C^*(h)$ are strictly positive. To fix this problem one has to rule out both high Sharpe ratios and all arbitrage opportunities. However, one cannot take $C_{++}(h) \cup X_{++}$ as the set of desirable claims because this set is not convex and the extension property would be immediately lost. Cochrane and Saá -Requejo therefore take convex hull of $C_{++}(h) \cup X_{++}$ as the set of desirable claims, which means they are ruling out not only high Sharpe ratios and arbitrage opportunities but also all convex combinations of the two, that are generally neither arbitrage opportunities nor high Sharpe ratios. It can be shown, however, that this set of desirable claims is generated by a truncated quadratic utility function, and that it can be associated to a level of a *generalized* Sharpe ratio, see Černý (1999).

6 Conclusions

The theory presented here shows that pricing techniques which impose equilibrium restrictions stronger than no arbitrage can be seen as a generalization of no-arbitrage pricing. We derived the Extension and Pricing Theorem in no-good-deal framework and showed that the Extension Theorem captures the trade-off between equilibrium outcomes and discount factor restrictions. We have shown that equilibrium restrictions implied by von Neumann-Morgenstern preferences contain no-arbitrage and representative agent equilibrium as the two opposite ends of a spectrum of possible restrictions. In finite state models we have settled the question of how tight are the no-good-deal price bounds generated by a utility function. It is somewhat surprising that price bounds implied by strictly increasing utility functions are not always tighter than the no-arbitrage bounds. At the same time our results moderate the Bernardo-Ledoit critique of CRRA bounds – in finite state models these are always tighter than the no-arbitrage bounds.

Appendix A

Proof of Theorem 2

Since K is boundedly generated, there is a closed bounded set $B \subset K$ such that $K \subset \bigcup_{\lambda>0} \lambda B$. Therefore, it is enough to strictly separate B and $M_0(p)$.

With $1 < p < \infty$ L^p is a reflexive space. By Theorem 19 C in Holmes[13] $M_0(p)$ and B can be strictly separated by a continuous linear functional, i.e. there is $\varphi \in X^*$ such that $\varphi(M_0(p)) = 0$ and $\varphi(B) > 0$. However, this implies $\varphi(C_{++}) > 0$.

By standing assumption 1 there is a marketed strictly positive claim x_0. Define $H \triangleq M_0(\varphi)$. Because H does not intersect C_{++} , and therefore X_{++}, we have $x_0 \notin H$. Finally, because H is a hyperplane we have the spanning property $X = H \oplus Span[x_0]$, so that each claim $y \in X$ has a unique decomposition $y = y_H + \lambda_y x_0$, where $y_H \in H$. By construction $p(y) \triangleq \lambda_y p(x_0)$ is a no-good-deal price of claim y. It is easily seen now that $\tilde{p} = \frac{p(x_0)}{\varphi(x_0)}\varphi$ is an extension of the original pricing rule p. Since φ is C-strictly positive and continuous $\tilde{p}$ must be C-strictly positive and continuous which completes the proof.

Proof of Theorem 3

a) By Theorem 2 there is no good deal in N if and only if there is a $C \cap N$-strictly positive continuous pricing functional φ in N. It is the continuity that we are worried about. We will show that no good deal in N implies that $N_0(\varphi)$ is closed and disjoint from K. Then the assertion follows from the Extension Theorem.

Functional φ has to price correctly all claims in M, $\varphi|M = \phi$. This implies $(N \supset) N_0(\varphi) \supset M_0(\phi)$. Note, however, that $\mathrm{codim}_N M_0(\phi) \leq m + 1$ and hence $\mathrm{codim}_{N_0(\varphi)} M_0(\phi) \leq m + 1$. In other words $N_0(\varphi) = M_0(\phi) \oplus L$ where $\dim L$ is finite. Since ϕ is continuous and M is closed, the zero investment

[13] For completeness we provide the proof of that part of the theorem which is relevant to us and which Holmes leaves as an exercise: It is known that the unit ball $U(X)$ in a normed reflexive space is weakly compact (Theorem 16 F). Furthermore for convex sets 'closed' is equivalent to 'weakly closed' (Corollary 12 A). J is closed, convex and bounded, therefore weakly compact. N is convex, closed and therefore weakly closed. The separation theorem for one closed and one compact convex set (Corollary 11 F) asserts that N and J can be strictly separated by a weakly continuous functional ψ, however such functional is continuous in the original topology on X as well (Theorem 12 A).

marketed subspace $M_0(\psi)$ is closed in X. Then also $N_0(\varphi) = M_0(\psi) \oplus L$ is closed because L is finite dimensional. Now φ is $K \cap N$-strictly positive and therefore $N_0(\varphi) \cap K = \emptyset$.

b) The convexity of the no-good-deal price region follows from the convexity of complete market no-good-deal state prices and the part a). Namely, if $p_1, p_2 \in P$ then by assertion a) there exist functionals $\varphi_1, \varphi_2 \in C^*_{++}, \varphi_i(y) = p_i$, that price correctly all claims in M. Of course, the functional $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in C^*_{++}$ prices claims in M correctly, too, and therefore by assertion a)

$$(\lambda\varphi_1 + (1-\lambda)\varphi_2)(y) = \lambda p_1 + (1-\lambda)p_2 \in P.$$

c) To prove the last statement we will first demonstrate that the cone of no-good-deal pricing functionals C^*_{++} is open. Let us take B as in the proof of Theorem 2 and denote $\kappa = \sup_{x \in B} ||x||$.

i) Take an arbitrary $\varphi \in C^*_{++}$ and denote $\varepsilon(\varphi) = \inf_{x \in B} \varphi(x)$. We claim that $\varepsilon > 0$. For the purpose of contradiction suppose that $\varepsilon = 0$. Then there is a sequence $x_n \in B$ such that $\lim \varphi(x_n) = 0$. Since B is closed, convex and bounded, from the reflexivity of X follows that B is weakly sequentially compact (Holmes, Theorem 16F and Corollary 18 A). Hence there is a subsequence x_k converging weakly to $x \in B$ implying $\varphi(x) = 0$. However, $x \in B \subset C_{++}$ contradicts $\varphi \in C^*_{++}$.

Thus $\varphi(K) \geq \varepsilon > 0$. Taking an arbitrary functional $\psi \in X^*$ such that $||\psi|| < \frac{\varepsilon}{2\kappa}$ and $x \in X_{++}$ we have

$$(\varphi + \psi)(x) > \varphi(x) - ||\psi||\,||x|| > \varepsilon - \frac{\varepsilon}{2\kappa}\kappa > 0$$

which means that $\varphi + \psi \in X^*_{++}$ whenever $||\psi|| < \frac{\varepsilon}{2\kappa}$. Since φ is arbitrary this means that X^*_{++} is open in the norm topology on X^*.

ii) Let us first assume that $\text{codim}_N M = m$. Applying Hahn-Banach theorem to the subspace $\text{span}(M \cup \{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m\})$ and the point y_j one can find linear functionals $\psi_j, j = 1, \ldots, m$ such that $\psi_j(M) = 0$ for all j and $\psi_j(y_i) = \delta_{ij}$ (Kronecker's delta). By the Extension Theorem the pricing rule on M can be extended to a strictly positive functional φ_0 that correctly prices securities in M. Note that functionals $\varphi_0 + \lambda\psi_j$ too price these securities correctly and moreover for $|\lambda|$ sufficiently small $\varphi + \lambda\psi_j$ will be a strictly positive functional by the result in i). Thus the price vectors $\varphi_0(y) + \lambda\psi_j(y)$ give no-good-deal prices for securities $y = (y_1, \ldots, y_m)$ consistent with the predetermined prices of securities in M. By construction $\psi_j(y)$ are linearly independent vectors in $\mathbb{R}^m$ and recall that $\dim P$ is defined as the dimension of the affine hull of $P-\varphi_0(y)$ (which is a linear subspace) thus the dimension of the price region P is at least $\operatorname{rank}(\psi_1(y),\dots,\psi_m(y))=m$, and of course it cannot be more than m.

iii) In a general case a certain number of vectors y_i, say $m-l$, will lie in the marketed subspace M. However, for any $\tilde{\varphi}$ that prices correctly claims in M the difference $\tilde{\varphi}(y_i)-\varphi_0(y_i)$ will be zero, hence if $y_i\in M$ it will not contribute to the dimensionality of the price region.

That leaves l claims that do not belong to the marketed subspace and these we partition into two groups – the first $\tilde{m}$ claims $c_1'=(y_1,\dots,y_{\tilde{m}})$ that are linearly independent and the remaining $l-\tilde{m}$ claims $c_2'=(y_{\tilde{m}+1},\dots,y_l)$ that can be expressed as a linear combination of the first $\tilde{m}$ claims, $c_2=Dc_1$ with $D\in\mathbb{R}^{(l-\tilde{m})\times\tilde{m}}$.

First of all it is clear that there cannot be more than $\tilde{m}$ linearly independent vectors of the type $\psi(y_1),\dots,\psi(y_l)$. If there were more, one could find a non-trivial linear combination of these vectors that annuls the first $\tilde{m}$ coordinates, $\sum_i\lambda_i\psi_i(c_1)=0$. However, such a linear combination annuls the remaining $l-\tilde{m}$ coordinates as well since

$$\sum_i\lambda_i\psi_i(c_2)=\sum_i\lambda_i\psi_i(Dc_1)=\sum_i\lambda_iD\psi_i(c_1)=D\left(\sum_i\lambda_i\psi_i(c_1)\right)=0.$$

On the other hand one can find $\tilde{m}$ linearly independent prices of the desired form by the procedure described in ii). Thus $\dim P=\tilde{m}$ and by construction $\tilde{m}=\operatorname{codim}_N M$.

d) Suppose there is just one security to be priced, say security y_i and denote its no-good-deal price region $P\subset\mathbb{R}^1$. We set $N=\operatorname{span}(M\cup\{y_i\})$ and have $\operatorname{codim}_N M=0$ if and only if $y_i\in M$. By definition the no-good-deal price of y_i is unique if and only if $\dim P=0$. Then the assertion c) implies that y is uniquely priced if and only if y is redundant.

e) i) Define B_1, B_2 and κ_1, κ_2 in analogy to B and κ in c). We want to show that for all $\varphi_2\in C^*_{2++}$ and for all $\psi\in X^*$ such that $||\psi||<\frac{\varepsilon\nu}{2\kappa_1}$ we have

$$\varphi_2+\psi\in C^*_{1++}.$$

Let us take an arbitrary $\varphi_2 \in C^*_{2++}$ which implies $\varphi_2(J_1 + \varepsilon Ball(0,1)) > 0$ and consequently

$$\varphi_2(J_1) - \varepsilon||\varphi_2|| > 0. \tag{A.1}$$

Furthermore, if φ_2 prices correctly all the basis assets then $||\varphi_2|| \geq ||\phi|| = \nu$. The standing assumption 1 implies $\nu > 0$. Taking an arbitrary $\psi \in X^*$ such that $||\psi|| < \frac{\varepsilon\nu}{2\kappa}$ and making use of (A.1) we have

$$(\varphi_2 + \psi)(J_1) \geq \varepsilon||\varphi_2|| - ||\psi||\kappa \geq \varepsilon\nu - \frac{\varepsilon\nu}{2\kappa}\kappa > 0,$$

QED.

ii) We can now construct linear functionals ψ_i as in c) ii) and denote $c \equiv \max_{i=1,\dots,m} ||\psi_i||$. Let $\tilde{B}_\delta$ be a δ-ball in $\mathbb{R}^m$ with L^2 norm. By construction of functionals ψ_i we have

$$\{\sum_{i=1}^{m} \theta_i \psi_i(y) | (\theta_1, \dots, \theta_m) \in \tilde{B}_\delta\} = \tilde{B}_\delta \tag{A.2}$$

On the other hand

$$\left\| \sum_{i=1}^{m} \theta_i \psi_i \right\| \leq \sum_{i=1}^{m} |\theta_i| \, ||\psi_i|| = c \sum_{i=1}^{m} |\theta_i| = c||\theta||_1 \leq c\sqrt{m}||\theta||_2.$$

With $\delta = \frac{\varepsilon\nu}{2\kappa_1 c\sqrt{m}}$ therefore $\|\sum_{i=1}^{m} \theta_i \psi_i\| \leq \frac{\varepsilon\nu}{2\kappa_1}$ for $\theta \in \tilde{B}_\delta$. Combining this result with e) i) we have for an arbitrary $\varphi_2 \in C^*_{2++}$

$$\left(\varphi_2 + \sum_{i=1}^{m} \theta_i \psi_i\right)(y) \in P_1 \text{ for } \theta \in \tilde{B}_\delta. \tag{A.3}$$

Let $p_2 \in \text{cl}P_2$ then there is $\varphi_2 \in C^*_{2++}$ such that $||\varphi_2(y) - p_2||_2 < \frac{\delta}{2}$. By virtue of (A.2) and (A.3) we then have

$$p_2 + \tilde{B}_{\frac{\delta}{2}} \subset P_1.$$

QED.

Lemma A.1 *Let*

$$\begin{aligned} A &= \{(x,y,z) | z \geq \frac{1}{x + \sqrt{y}}, x + \sqrt{y} > 0, y \geq 0, x \leq 1\} \\ B &= \{(0,0,z) | z \in \mathbb{R}\} \end{aligned}$$

Then A and B are two disjoint closed convex sets and they cannot be semistrictly separated, e.i. there does not exist a vector $(n_1, n_2, 0)$ such that for all $(x,y,z) \in A$ we have $n_1 x + n_2 y > 0$.

Proof. Available from authors on request. ■

Appendix B

Proof of Theorem 4 Suppose to the contrary that the set of desirable claims $K(a)$ is not boundedly generated. Then there must be a sequence $\{x_n\}$, $||x_n|| \to +\infty$, $x_n \in K(a)$ with the property that $\lambda x_n \notin K(a)$ for $\lambda < 1$. Nonetheless, X being finite dimensional the sequence $\frac{x_n}{||x_n||}$ must converge (if necessary by passing to a subsequence). Let us denote the limit z. From Lemma B.1 we know that $P(z < 0) = 0$. By virtue of Lemma B.3 for sufficiently large constant κ the claim $\kappa\alpha \in K(2a)$ and therefore, $\kappa\alpha + \delta B_1 \in K(a)$ for δ sufficiently small, because the expected utility is a continuous function on X. Furthermore, $\frac{x_n}{||x_n||} \to \alpha$ implies that there is n_0 such that for all $n > n_0$ $\frac{x_n}{||x_n||} \in \alpha + \frac{\delta}{\kappa} B_1$ and therefore

$$\frac{\kappa x_n}{||x_n||} \in \kappa\alpha + \delta B_1 \in K(a).$$

Since $||x_n|| \to +\infty$ there is n_1 such that for all $n > n_1$ $||x_n|| > \kappa$.Thus for $n > n_1$

$$\frac{\kappa}{||x_n||} x_n \in K(a) \text{ and } \frac{\kappa}{||x_n||} < 1$$

which contradicts our assumption that $\lambda x_n \notin K(a)$ for $\lambda < 1$. QED.

Proof of Theorem 5:

a) i) Suppose, to the contrary, that $\sup_{w \in M_0} \mathrm{E}U(w) = +\infty$. Then there is a sequence $\{w_n\} \in M_0$ such that $\{\mathrm{E}U(w_n)\}$ is increasing and unbounded from above. If $\{||w_n||\}$ were bounded then there would be a convergent subsequence $\{w_{n_k}\} \to w \in X$ and $\mathrm{E}U$ would not be continuous at w. Thus it must be that $\{||w_n||\}$ is unbounded. In that case, however, $\{w_n\}$ is an unbounded sequence of desirable claims and because X is finite-dimensional we can find a common direction (if necessary passing to a subsequence) $\left\{\frac{w_n}{||w_n||}\right\} \to z \in X$, in fact $z \in M_0$ because M_0 is closed. By Lemma B.1 z is strictly positive and $z \in M_0$ contradicts the assumption of no arbitrage. Q.E.D.

ii) We have shown $\sup_{w \in M_0} \mathrm{E}U(w_r + w) \triangleq \mathrm{E}U(w_r + a_M) < +\infty$. By definition of supremum there is a sequence $\{w_n\} \in M_0$ such that $\{\mathrm{E}U(w_r + w_n)\} \to \mathrm{E}U(w_r + a_M)$. Now because the sets of desirable claims are boundedly generated and M_0 is a linear subspace, we can always choose $\{w_n\}$ bounded. This implies $\{w_n\} \to w_M \in M_0$ (again using a subsequence if necessary) and by continuity of expected utility $EU(w_r + w_M) = EU(w_r + a_M)$. The uniqueness of w_M follows from the strict concavity of the (expected) utility function. Q.E.D.

b) The first part follows from a). For the second part of the statement it is enough to consider $a = a_M$. By Hahn-Banach Theorem one can strongly separate M_0 and the interior of $K(a_M)$ by a hyperplane N_0, and because $K(a_M)$ is closed the same hyperplane separates $K(a_M)$ and M_0 weakly. By standing assumption 1 there is a marketed strictly positive claim x with positive price $p(x) > 0$. Because N_0 does not contain internal points of $K(a_M)$ we must have $\sup_{w \in N_0} EU(w_r + w) = EU(w_r + a_M)$. For the same reason $N_0 \cap X_{++} = \emptyset$ and therefore $x \notin N_0$. Finally, because N_0 is a hyperplane we have the spanning property $X = N_0 \oplus Span[x]$, so that each claim $y \in X$ has a unique decomposition $y = y_0 + \lambda_y x$, where $y_0 \in N_0$. By construction $p(y) \triangleq \lambda_y p(x)$ is a no-good-deal price of claim y. Convexity of the price region follows from the argument presented in the proof of Theorem 3, part b).

c) The first part follows from Theorem 3. For the second part, it is clear from a) that the separating hyperplane has to cross $K(a_M)$ at w_M. It follows from Theorem 1.29 in Beavis and Dobbs (1990) that EU has a unique supergradient at w_M, by direct calculation this supergradient is

$$\zeta = (\Pr(\omega_1)U'\left[w_r(\omega_1) + w_M(\omega_1)\right], \dots, \Pr(\omega_{\dim X})U'\left[w_r(\omega_k) + w_M(\omega_k)\right]),$$

where $k = \dim X$. By definition the supergradient has the property

$$EU(w_r + w_M + \Delta w) \leq EU(w_r + w_M) + \zeta\Delta w = EU(w_r + w_M) + EU'(w_r + w_M)\Delta w.$$

As long as $EU'(w_r + w_M)\Delta w = 0$ for all $\Delta w \in N_0$ we have

$$EU(w_r + w_M + N_0) \leq EU(w_r + w_M)$$

and there is no good deal in the completed market. The normalisation discussed in the proof of Theorem 2 shows that the pricing functional is

$$p(y) = p(x_0)\frac{EU'(w_r + w_M)y}{EU'(w_r + w_M)x_0}.$$

When x_0 is a risk-free asset this formula simplifies to

$$p(y) = \frac{E\frac{U'(w_r + w_M)}{EU'(w_r + w_M)}y}{R}.$$

To show uniqueness realize that by Theorem 1.30 in Beavis and Dobbs EU is continuously differentiable at $w_r + w_M$. The Taylor expansion of the form $f(x) = f(x_0) + f_x(\lambda x_0 + (1-\lambda)x)(x - x_0)$ for some $0 < \lambda < 1$ with $f = EU$ and $x_0 = w_r + w_M$ implies that the hyperplane defined by ζ is the only hyperplane passing through w_M that does not intersect the interior of $K(a_M)$.

d), e), f) $K(a)$ is boundedly generated by bounded closed set $B_a \subset K(a)$, likewise $K(b)$ is generated by $B_b \subset K(b)$. Because B_b is compact, EU is uniformly continuous on B_b. Therefore there is $\varepsilon > 0$ such that $B_b + Ball(0, \varepsilon) \subset K(a)$. If $B_b + Ball(0, \varepsilon) \not\subseteq B_a$ then we can always redefine B_a as the closed convex hull of $B_a \cup [B_b + Ball(0, \varepsilon)]$. Then the assumption of Theorem 3 e) is satisfied and the rest follows.

Lemma B.1 *Suppose X is a probability space, $\dim X < \infty$,.and U is a downside-sensitive utility. If an unbounded sequence of desirable claims has a common direction, then this direction is strictly positive. Mathematically, if , $||x_n|| \to \infty$, $\frac{||x_n||}{x_n} \to z$ and $EU(w_r + x_n) \geq EU(w_r + a)$ for a fixed $a \in \mathbb{R}$, then $z \geq 0$, $\Pr(z > 0) > 0$.*

Proof

Let us define $w_{\min} = \min w_r$, $w_{\max} = \max w_r$ and analogously $x_{\min}$, $x_{\max}$. Since we have finitely many states there is a state with the smallest probability $p_{\min}$.

i) For x to be a desirable claim we must have

$$EU(w_r + x) - EU(w_r + a) \geq 0.$$

We can rewrite this statement using conditional distribution

$$\Pr(x < 0)\mathrm{E}\left[U(w_r + x)|x < 0\right] + \Pr(x \geq 0)\mathrm{E}\left[U(w_r + x)|x \geq 0\right] \geq EU(w_r + a). \tag{B.1}$$

Let us appraise the left hand side from above. Denoting ξ the supergradient of U in $w_{\min}$ we can write

$$\begin{aligned} EU(w_r + x|x \geq 0) &\leq U(w_{\max}) + \xi \mathrm{E}[x|x \geq 0] = \\ &= U(w_{\max}) + \xi \frac{\mathrm{E}x^+}{\Pr(x \geq 0)} \end{aligned}$$

Assuming that $x_{\min} < 0$ we obtain

$$\Pr(x < 0)\mathrm{E}\left[U(w_r + x)|x < 0\right] \leq p_{\min} U(w_{\max} + x_{\min}) + \left[\Pr(x < 0) - p_{\min}\right] U(w_{\max}).$$

Plugging the last two expressions in the equation (B.1) we obtain

$$p_{\min}U(w_{\max}+x_{\min})+\xi\mathrm{E}x^{+}\geq\gamma$$
$$\gamma=\mathrm{E}U(w_r+a)-\Pr(x\geq 0)U(w_{\max})-[\Pr(x<0)-p_{\min}]\,U(w_{\max})$$
$$\gamma\geq\mathrm{E}U(w_r+a)-U(w_{\max})\equiv c(a)$$

Thus for x to be desirable we must have

$$p_{\min}U(w_{\max}+x_{\min})+\xi\mathrm{E}x^{+}\geq c(a) \tag{B.2}$$

where $p_{\min}$, $w_{\max}$, ξ, and $c(a)$ do not depend on x.

ii) Let us now take a sequence of desirable claims $||x_n||\to\infty$, $\frac{||x_n||}{x_n}\to z$. If $z_{\min}$ were negative then by continuity $\frac{(x_n)_{\min}}{||x_n||_1}\to\alpha_{\min}$ and hence $(x_n)_{\min}\to-\infty$. At the same time $x_n\in K(a)$ and (B.2) implies that

$$p_{\min}U(w_{\max}+(x_n)_{\min})+\xi\mathrm{E}x_n^{+}\geq c(a).$$

where $c(a)<0$ without loss of generality. After rearranging the terms we arrive at

$$\frac{w_{\max}+(x_n)_{\min}}{U(w_{\max}+(x_n)_{\min})}\frac{(x_n)_{\min}}{(w_{\max}+(x_n)_{\min})\,p_{\min}}\geq\frac{-(x_n)_{\min}}{\xi\mathrm{E}x_n^{+}-c(a)}.$$

The right hand side can be appraised from below

$$\frac{-(x_n)_{\min}}{\xi\mathrm{E}x_n^{+}-c(a)}\geq\frac{\mathrm{E}x_n^{-}}{\xi\mathrm{E}x_n^{+}-c(a)}.$$

The limit of the left hand side is, by the dominance condition (3), equal to zero. Because the numerator on the right hand side goes to $+\infty$, it must be that $\mathrm{E}x_n^{+}\to+\infty$. This however implies that

$$\begin{aligned}0&=\lim_{n\to+\infty}\frac{\mathrm{E}x_n^{-}}{\xi\mathrm{E}x_n^{+}-c(a)}=\lim_{n\to+\infty}\frac{\mathrm{E}x_n^{-}}{\mathrm{E}x_n^{+}}\frac{1}{\xi-\frac{c(a)}{\mathrm{E}x_n^{+}}}=\frac{1}{\xi}\lim_{n\to+\infty}\frac{\mathrm{E}x_n^{-}}{\mathrm{E}x_n^{+}}\\0&=\lim_{n\to+\infty}\frac{\mathrm{E}x_n^{-}}{\mathrm{E}x_n^{+}}=\frac{\mathrm{E}\alpha^{-}}{\mathrm{E}\alpha^{+}}\end{aligned}$$

which contradicts $P(\alpha<0)>0$.

Lemma B.2 *Suppose that $U:\mathbb{R}\to\mathbb{R}$ is a strictly increasing concave function such that $u=\sup_{x\in\mathbb{R}}U(x)<\infty$. Suppose further that $x\in L^p$ is strictly positive. Then*

$$\lim_{\lambda\to\infty}\mathrm{E}U(x+\lambda y)=u$$

if and only if $y\in L^p$ is strictly positive with probability 1.

Proof. i) Firstly, let us take $y \in L^p$ such that $P(y \leq 0) = \pi > 0$. Then we have

$$\mathrm{E}U(x + \lambda y) \leq (1 - \pi)u + \pi\left(\mathrm{E}U(x)|y \leq 0\right).$$

Since U is strictly increasing, we must have $U(t) < u$ for all $t \in \mathbb{R}$ and hence also $\mathrm{E}\left[U(x)|y \leq 0\right] = \tilde{u} < u$. Consequently

$$\lim_{\lambda \to \infty} \mathrm{E}U(x + \lambda y) \leq (1 - \pi)u + \pi\tilde{u} < u$$

ii) Now consider $y \in L^p$ such that $P(y \leq 0) = 0$. Define

$$\begin{aligned} \pi_0 &= P(y \geq 1) \\ \pi_n &= P(\frac{1}{n} > y \geq \frac{1}{n+1}) \text{ for } n = 1, 2, \ldots \\ p_n &= \sum_0^n \pi_k \end{aligned}$$

By assumption $\lim_{n \to \infty} p_n = 1$. Now we have

$$\mathrm{E}U(x + \lambda y) \geq p_n U(\frac{\lambda}{n+1})$$

and therefore

$$\lim_{\lambda \to \infty} \mathrm{E}U(x + \lambda y) \geq p_n u \text{ for all } n$$

Since $\lim_{n \to \infty} p_n = 1$ it must be true that $\lim_{\lambda \to \infty} \mathrm{E}U(x + \lambda y) \geq u$. ■

Lemma B.3 *Suppose that $U : \mathbb{R} \to \mathbb{R}$ is strictly increasing, concave and unbounded from above. Suppose further that $x \in L^p$ is bounded below. Then*

$$\lim_{\lambda \to \infty} \mathrm{E}U(x + \lambda y) = \infty$$

for all strictly positive $y \in L^p$.

Proof. Define π_k as above and set $x_{\min} = \operatorname{ess\,inf} x$. Since by assumption $P(y > 0) > 0$ there must be $k \in \mathbb{N}$ such that $\pi_k > 0$. Then

$$\mathrm{E}U(x + \lambda y) \geq p_k U(x_{\min} + \frac{\lambda}{k+1})$$

and letting $\lambda \to \infty$ we have

$$\lim_{\lambda \to \infty} \mathrm{E}U(x + \lambda y) = \infty.$$

■

Lemma B.4 *Von Neumann-Morgenstern preferences are downside-sensitive if and only if the generating utility function satisfies*

$$\lim_{x \to -\infty} \frac{x}{U(x)} = 0.$$

Proof. First we show the 'if' part. Let us take $y \in L^p$ such that $y \notin L^p_+$, i.e. $P(y < 0) = \varepsilon > 0$. Then one of the numbers

$$\pi_0 = P(y < -1)$$
$$\pi_n = P(-\frac{1}{n} \le y < -\frac{1}{n+1}) \text{ for } n = 1, 2, \ldots$$

has to be positive, otherwise $P(y < 0) = 0$.

Without loss of generality we can assume that $\pi_k > 0$. Denoting ξ the left hand side derivative of U in zero and taking $\lambda > 0$ we obtain

$$\begin{aligned} \mathrm{E}U(\lambda y) &= \mathrm{E}\left[U(\lambda y)|y < 0\right] P(y < 0) + \mathrm{E}\left[U(\lambda y)|y \ge 0\right] P(y \ge 0) = \\ &\le \pi_k U(-\frac{\lambda}{k+1}) + (1 - \pi_k)U(0) + \xi\lambda \mathrm{E}y_+ \equiv v(\lambda). \end{aligned}$$

Note that $v(\lambda)$ is a continuous function on $\mathbb{R}_+$ and thus $\sup_{\lambda \ge 0} v(\lambda) = \infty$ if and only if $\lim_{\lambda \to \infty} v(\lambda) = \infty$. However, instead we have

$$\lim_{\lambda \to \infty} \frac{v(\lambda)}{\lambda} = \xi \mathrm{E}y_+ - \lim_{x \to -\infty} \frac{\pi_k}{k+1} \frac{U(x)}{x} = -\infty$$

and hence $\lim_{\lambda \to \infty} v(\lambda) < \infty$, $\sup_{\lambda \ge 0} v(\lambda) < \infty$, and consequently $\sup_{\lambda \ge 0} \mathrm{E}U(\lambda y) < \infty$ for all $y \notin L^p_+$ which completes the proof.

The 'only if' part is shown easily once we realize that $\frac{U(x)}{x}$ is a decreasing function of x. We can take a random variable with two atoms $P(y = -1) = \pi$ and $P(y = 1) = 1 - \pi$. Then

$$\mathrm{E}U(\lambda y) = \pi U(-\lambda) + (1 - \pi)U(\lambda)$$

and

$$\lim_{\lambda \to \infty} \frac{\mathrm{E}U(\lambda y)}{\lambda} = -\pi \lim_{x \to -\infty} \frac{U(x)}{x} + (1 - \pi) \lim_{x \to \infty} \frac{U(x)}{x}. \tag{B.3}$$

Now $\lim_{x \to -\infty} \frac{U(x)}{x}$ is finite and if $\lim_{x \to \infty} \frac{U(x)}{x}$ is positive one can always take π small enough so that the limit (B.3) is positive. But then $\lim_{\lambda \to \infty} \mathrm{E}U(\lambda y) = +\infty$. ■

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