Implied Risk-Neutral Distribution: A Comparison of Estimation Methods

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Abstract

This paper examines two alternative approaches to recovering the risk-neutral density function from contemporaneous option prices. First, we propose to recover the risk-neutral probabilities through a parameterization of the equivalent martingale measure using the Generalised Beta distribution. Then, we use a non-parametric method to approximate the volatility smile using B-splines approximating functions and use the chain rule of differentiation to recover the implied distribution. We end the paper with a comparison of the two estimation methods using a sample of daily closing prices of the Chicago Mercantile Exchange options on the S&P 500 index future to assess the quality and stability of the implied distributions through statistical analysis.

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1. INTRODUCTION

The absence of arbitrage opportunities in a complete market implies the existence of a unique probability measure Q under which all discounted prices are martingales. Using the martingale approach and having a bank account $B_t = e^{rt}$ as a numeraire, we have that the price of a general European contingent claim V with maturity at the time T and general payoff function $g(S)$ is given by:

$$V(S_t, t) = e^{-r(T-t)} E^Q\left[g(S_T)\right] \tag{1}$$

$$V(S_t, t) = e^{-r(T-t)} \int g(S_T) dQ(S_T, T / S_t, t) \tag{2}$$

From (2) it is evident that if we know the cumulative risk-neutral distribution of the stock process we can value the contingent claim. Conversely, in the case of European options, given the option prices there exists an implied cumulative distribution Q such that (2) holds. In other words, the option valuation problem is equivalent to the problem of determining the distribution of the asset variable S_T under the martingale measure.

We review the main approaches used in the literature to estimate the implied risk-neutral distribution function from option prices. We begin Section 2 with a description of the Breeden and Litzenberger (1978) approach to recover risk-neutral probabilities from portfolios of standard call options that replicate elementary securities payoff. Their results lead to a vast group of techniques to recover distributional information from option prices. We then present a proposition stating the convexity and non-arbitrage conditions that the contingent claim pricing function must satisfies in order to generalise Breeden and Litzenberger (1978) results. This proposition underlies a non-parametric estimation method that we propose in the following section. Section 3 describes three different approaches to estimate the risk-neutral distribution and reviews relevant studies on parametric and non-parametric estimations. We then present our own methods to estimate the implied risk-neutral distribution function from option prices. In Section 4, we propose to recover the risk-neutral probabilities through a parameterization of the equivalent martingale measure using the Generalised Beta distribution. This distribution has enough flexibility to produce satisfactory results when fitting the observed volatility smiles. Section 5 presents our own non-parametric method to approximate the volatility smile using B-splines approximating functions. We use the chain rule of differentiation to recover the implied distribution (in the same way as presented independently by Shimko (1993)). We describe our method and examples are given of the estimates using this non-parametric approach.

There are many papers in the academic literature that examine the estimation of the risk-neutral distribution from option prices but we are not aware of published papers that examine their results in terms of the plausibility of the resulting estimates. The purpose of Section 6 is to assess the stability of the implied risk-neutral distribution for a sample of daily closing prices of the Chicago Mercantile Exchange (CME) options on the Standard & Poor's 500

(S&P 500) index future. We evaluate the performance of our nonparametric and parametric models through the examination of the estimation residuals and the implied probability time series. This analysis allows us to compare the two approaches in terms of their fitting performance and their time series stability. Last section summarises the paper.

2. CONVEXITY IN ARBITRAGE-FREE OPTIONS PRICES AND THE RIS-NEUTRAL DISTRIBUTION FUNCTION

Breeden and Litzenberger (1978) were pioneers in the estimation of probability distributions from contemporaneous option prices. They derived the price of a primitive security from the cost of a portfolio of European call options with different strikes and the same maturity. They assume there exists an asset *S* with a finite number of future states and consider a butterfly spread portfolio of European call options with maturity at time *T* constructed by shorting two options with strike price *K* and buying one option with strike $K+\Delta K$ and one with strike $K-\Delta K$:

$$\Pi(K,T)=\frac{\left[C(K+\Delta K,T)-C(K,T)\right]-\left[C(K,T)-C(K-\Delta K,T)\right]}{\Delta K} \tag{3}$$

The payoff of portfolio Π is the same as the one of a pure security that pays off one unit of cash if and only if S_T is equal to *K*. Thus, it turns out that a complete set of options at all exercise prices is equivalent to a complete set of Arrow-Debreu securities where the probability of reaching any of the states of the asset domain is: [1]

[1] An Arrow-Debreu security is also known as a pure security or elementary claim. It pays one unit of cash in one specific state of the world and nothing in any other state. See Arrow (1964) and Debreu (1959).

$$q_i(S_i) = \frac{1}{B_T} \Pi(K,T) \Big|_{K=S_i} \tag{4}$$

Let us now consider $\frac{1}{\Delta K}$ shares of the latter portfolio:

$$\frac{1}{\Delta K} \Pi(K,T) = \frac{[C(K+\Delta K,T) - C(K,T)] - [C(K,T) - C(K-\Delta K,T)]}{(\Delta K)^2} \tag{5}$$

Assuming that the asset S has a continuous payoff and taking the limit of expression (5) as ΔK goes to zero, it is clear that the expression approaches to the second derivative of $C(K,T)$[2]:

$$\lim_{\Delta K \to 0} \frac{1}{\Delta K} \Pi(K,T) = \frac{\partial^2}{\partial K^2} C(K,T) \tag{6}$$

Thus, we can consider the second derivative of the call prices formula with respect to the strike price to be a continuous state price function that measures the value of one unit of cash to be delivered at the time T in the event that the value of the asset happens to be K at that time. Therefore, this pricing function for elementary claims can be used to value any derivative security on the asset S with payoff function $g(S_T)$ at the time T:

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[2] Indeed, the payoff function of the portfolio Π tends to a Dirac delta function with mass at K.

$$V(S,T)=\int_0^{+\infty} g(S)\frac{\partial^2}{\partial K^2}C(K=S,T)dS \tag{7}$$

One important characteristic of Breeden and Litzenberger (1978) approach is that no assumptions are made about the underlying asset price dynamics and market participants preferences are not restricted as they are reflected in the call option prices. Their work is the starting point of a line of research addressed to the recovery of relevant aspects of the underlying asset distribution from option market data.

The derivation of Breeden and Litzenberger (1978) results shows intuitively how the first and second derivatives of the option valuation formula can be considered good estimates of the risk-neutral distribution and density functions. The following proposition modestly generalises their results:

Proposition 1: If call prices satisfy the following arbitrage restrictions related to the actual underlying security price S_0 and strike price K:

a) $S_0 \geq C(K) \geq MAX\left[0, S_0 - e^{-rT}K\right]$ (8)

b) $\frac{\partial C(K)}{\partial K} \geq -e^{-rT}$ and $\lim_{K\to+\infty}\frac{\partial C(K)}{\partial K}=0$ (9)

c) $C(K)$ is a monotone, convex and differentiable function on a continuous set $\Gamma \subset \Re$

then there exists a risk-neutral probability whose cumulative distribution function is given by:

$$Q(S_T) = 1 + e^{rT} \left. \frac{\partial C(K)}{\partial K} \right|_{K=S_T} \tag{10}$$

and when C is twice differentiable the density function is given by:

$$dQ(S_T) = e^{rT} \left. \frac{\partial^2 C(K)}{\partial K^2} \right|_{K=S_T} \tag{11}$$

Proof.

Let us first explain the kind of arbitrage opportunities that could arise if any of the arbitrage relationships are violated. Firstly, restriction (a) sets the boundaries for the call prices with $C = S_0$ for $K = 0$. If the call has a negative price, i.e., $C(K) < 0$, then a riskless profit could be made by buying the call (receiving an instant positive profit equal to the value of the call) and holding it until expiration to make a non-negative income equal to the value of the call at expiration. If $C(K) \leq S_0 - e^{-rT}K$, once again we can make an instant profit by buying the call and selling the portfolio $S_0 - e^{-rT}K$. Then at expiration date we receive a non-negative payoff equal to $MAX[0, K - S_T]$. Finally, if $C(K) > S_0$ buying the stock and selling the call would create an instant profit of $C(K) - S_0$ and generate a non-negative amount at expiration equal to $S_T - MAX[0, K - S_T]$. Second, restriction (b) determines the asymptotes of the call option prices. It is clear than when the strike is very low relative to the value of the underlying security then the option is very likely to be exercised and a one-unit increase on the strike price would imply a reduction of one-unit in the final payoff at the expiration date. In the same way if the strike is very high relative to the underlying security then the option is worthless and a marginal increase in the strike would have little effect on the price of the option. Restriction (b) also determines the monotonicity of the call option prices, i.e., it establishes that

$$C(K_1) - C(K_2) \leq e^{-rT}(K_2 - K_1), \qquad \text{for } K_2 > K_1 \tag{12}$$

A violation of this condition would create a vertical spread arbitrage opportunity by selling $C(K_1)$, buying $C(K_2)$ and keeping the amount $e^{-rT}(K_2 - K_1)$ in a bank account. This operation produces an instant profit and a non-negative amount on the expiration date. We shall see later that monotonicity on the call prices guarantees a non-decreasing risk-neutral distribution function. Finally, the third restriction (c) further constrains the general shape of the call price curve and implies a non-negative risk-neutral density function when it exists.

The current value of an European call option is determined as the discounted present value of its expected payoff at the maturity date:

$$C(K) = e^{-rT} \int_K^{+\infty} (S_T - K) dQ(S_T) \tag{13}$$

hence:

$$\frac{\partial C}{\partial K} = -e^{-rT} \int_K^{+\infty} dQ(S_T)$$

$$\frac{\partial C}{\partial K} = e^{-rT}\left(Q(K) - 1\right)$$

$$Q(K) = 1 + e^{rT}\frac{\partial C(K)}{\partial K} \tag{14}$$

or equivalently

$$Q(S_T) = 1 + e^{rT}\left.\frac{\partial C(K)}{\partial K}\right|_{K=S_T} \tag{15}$$

Thus, the risk-neutral distribution is given as a function of the first derivative of the call option pricing formula. Here the monotonicity assumption with respect to the strike price is equivalent to the function Q being non-decreasing. Note that if,

$$\lim_{K\to 0}\frac{\partial C(K)}{\partial K} = -e^{-aT} > -e^{-rT} \quad \text{with } a > r \tag{16}$$

then there is a probability mass at $S_T = 0$ equal to:

$$Q(S_T = 0) = 1 - e^{(r-a)T} \tag{17}$$

If C is twice differentiable the risk-neutral density function is given by the following expression:

$$dQ(S_T) = e^{rT} \frac{\partial^2 C(K)}{\partial K^2}\bigg|_{K=S_T} \tag{18}$$

where given restrictions b) and c) we confirm that $dQ(S_T)$ satisfies the following properties:

$$dQ(S_T) = e^{rT} \frac{\partial^2 C(K = S_T)}{\partial K^2} \geq 0 \tag{19}$$

a direct implication of the convexity assumption and

$$\begin{aligned} \int_0^{+\infty} dQ(K = S_T) dK &= \int_0^{+\infty} e^{rT} \frac{\partial^2 C(K = S_T)}{\partial K^2} dK \\ &= e^{rT} \left[\lim_{K \to \infty} \frac{\partial C(K = S_T)}{\partial K} - \lim_{K \to 0} \frac{\partial C(K = S_T)}{\partial K} \right] \\ &= 1 \end{aligned} \tag{20}$$

Also note that these assumptions are sufficient to ensure that the mean of the distribution is the forward price since:

$$\begin{aligned} E_Q &= \int_0^{\infty} S_T dQ(S_T) \\ &= -\frac{1}{e^{-rT}} \int_0^{\infty} \frac{\partial C(K = S_T)}{\partial K} dS_T \\ &= \frac{1}{e^{-rT}} S \text{ , i.e., the underlying forward price.} \end{aligned} \tag{21}$$

end of proof.

Proposition 1 provides the basic relationships to determine probabilities from no-arbitrage prices. In a no-arbitrage world the first derivative of the price function produces the cumulative distribution function and the second derivative produces the probability density function. In the following section we review alternative methods to estimate the risk-neutral density function, some of them are a direct application of the above results.

3. RISK-NEUTRAL DISTRIBUTION: ESTIMATION METHODS

There are three common approaches to the estimation of the risk-neutral distribution function in the finance literature. The first begins by describing the dynamics of the underlying security and then obtaining an implicit characterisation (but not necessarily in a closed form) of the risk-neutral density function. The Black-Scholes (1973) assumption that the underlying security follows a Brownian motion process implies a lognormal distribution for the security returns. Cox and Ross (1976) constant elasticity variance model, Hull and White (1987) stochastic volatility model and Scott (1997) jumps and stochastic volatility model are also examples where the dynamics assumed for the underlying implicitly determine its distribution. The drawback of this approach is that for general underlying security processes, i.e., processes with jumps and non-stationary volatility, there is not a closed form solution for the risk-neutral density function and numerical methods have to be used to recover it. The second approach is called the parametric approach. It assumes that the risk-neutral distribution belongs to a general distribution family whose vector of unknown parameters values Θ must be estimated from the asset or option data. This parameterization of the equivalent martingale measure allows the development of general non-arbitrage option pricing models placing minimal structure on the process of the underlying security. The third is the non-parametric approach. This is the most general of the approaches, where no assumptions are made about the underlying security dynamics nor about the probability measure. The non-parametric methods usually approximate or interpolate the call price curve or implied volatility smile by means of polynomial approximations or optimisation techniques to recover the risk-neutral density function.

The key advantage of the parameterization of the equivalent martingale measure is its generality. Jumps and stochastic volatility in the underlying security are reflected in the relevant moments of the terminal distribution, thus it is always possible to find a flexible function with a wide skewness-kurtosis range that captures the distributional properties of the underlying security independently of its process dynamics.

Several papers have addressed the problem of general no-arbitrage option pricing models by assuming a particular functional form for the density function. Jarrow and Rudd (1982) use the generalised Edgeworth series expansion for a more general no-arbitrage option pricing model. They approximate the risk-neutral density function by a lognormal distribution in terms of a series expansion involving second and higher moments. The resulting option price is expressed in terms of the Black-Scholes formula plus three adjustment terms to take into account discrepancies between the variance, skewness and kurtosis of the lognormal distribution and the true underlying security distribution. The drawback of using Edgeworth or another probabilistic expansion is that it does not always represent a proper probability density function because there are many intervals for which it could take negative values and this substantially reduces the flexibility of this approximation in terms of its skewness-kurtosis range.[3]

Others attempts to use option price information to recover the risk-neutral distribution function include Fackler and King (1990) using closing price option data to examine the calibration of no-arbitrage implied price distributions for four agricultural commodity and Sherrick et al. (1992) using a three parameters Burr distribution to study the non-stationarity of expected S&P 500 futures price distribution. Sherrick et al. (1996) use the Burr III to estimate the risk-neutral distribution using daily data on options on soybean futures. The Burr family distribution may take on a wide range of skewness and kurtosis including all the region covered by the commonly used distributions such as the gamma family, the Weibull family, the lognormal family, the normal distribution, etc.[4]

Malz (1996) assumes the exchange rate follows a jump-diffusion process and uses over the counter currency option data to estimate the risk-neutral probability distribution of the pound sterling exchange rate against the mark. In a recent paper, Melick & Thomas (1997) express the bounds of the American option price in terms of the risk-neutral distribution. They use these bounds together with observed option prices on crude oil futures to solve for the parameters of a mixture of three lognormal distributions.

In contrast to the parametric method outlined in the previous section, the non-parametric approach does not assume any particular distribution family for the risk-neutral measure or for the dynamics of the underlying security. Instead, a given call option market price curve or

[3] See Johnson and Kotz (1970) for a detailed analysis of Series Expansions.
[4] See Tadikamalla (1980) for a detail exposition of the Burr family distribution.

implied volatility smile is modelled by means of polynomial approximations or optimisation techniques to recover the risk-neutral probabilities from option prices.

We know from Cox and Ross (1976) and Breeden and Litzenberger (1978) that given the risk-neutral measure we can calculate options prices, and conversely, we can identify the risk-neutral probabilities from a given set of option prices. Two related approaches have been used in the literature to estimate the risk-neutral probabilities by means of non-parametric methods. In the first, optimisation techniques are used to solve for the risk-neutral probabilities under different pricing constraints and prior or starting distributions assumptions. Rubinstein (1994) proposes a quadratic optimisation that results in a discrete estimate of the risk-neutral density probabilities defined on a set of stock prices corresponding to the terminal node of a binomial tree. He specifies a prior parametric distribution, typically the lognormal and the risk-neutral probabilities are then estimated by minimising its distance to the prior under the constraints that it correctly prices a selected set of derivatives securities. In a later paper, Jackwerth and Rubinstein (1997) work in an improved version of Rubinstein (1994) estimation technique. They apply optimisation to solve for the probabilities but trying to minimise the roughness of the resulting distribution measured by the integral of its squared second derivative. Kelly and Buchen (1995) propose the use of maximum entropy as a criterion for choosing among alternative probability distributions in the solution space of the risk-neutral distribution estimates.

The second approach is to fit a curve to call prices (either directly or indirectly via implied volatilities) and then obtain the risk-neutral density as proportional to its second derivative. This approach is a direct implementation of the results presented in Proposition 1, i.e., if European call prices are continuous on the underlying security with the same maturity date then the terminal risk-neutral distribution can be determined as the second derivative of the call pricing function with respect to its strike price. In order to implement this formula, different approximations on the call price curve or implied volatility smile have been suggested in the literature.

The problem with these estimates arises when we want to implement this idea for a discrete set of market option prices and strikes that sparsely cover a small interval in the domain of the underlying asset distribution. At first instance, it seems obvious to interpolate the option prices with a smooth function of the strike. There are several papers in the literature that attempt to approximate the option price curve in order to recover the risk-neutral probability measure by applying Proposition 1 results. Bates (1991) and Mayhew (1995) use natural splines to interpolate the option prices on a partition of the asset price domain. Apart from the pricing restriction, Mayhew (1995) imposes different constraints to identify all the splines parameters and to satisfy the properties of a density function. Abadir and Rockinger (1997) present a nonlinear method to estimated the risk-neutral density function based on the estimation of the parameters of a general function based on a generalisation of Kummer functions. Their method have the advantage of leading to a valid density (positive function that integrates to one) without exogenous constraints.

Another way of dealing with the strike interpolation problem is approximating the implied volatility smile and assuming that the volatility is a deterministic function of the strike price. Shimko (1993) proposes the following methodology. First, he calculates the implied volatilities of a given set of call options with different strikes and the same maturity. Second, he uses a quadratic function to interpolate the implied volatilities between the minimum and the maximum strike. Third, he substitutes the smoothed volatility curve into the Black-

Scholes formula to generate smooth call option prices as a continuous function of the strike. Finally, taking the second derivatives of this function, he determines the risk-neutral density function for the given range of strikes. Outside this range, he uses lognormal distributions to approximate the tails of the density function.

One limitation of Shimko's methodology is the use of quadratic functions as approximation for the observed market smiles. It is well documented that the implied volatility curve presents a smile shape with different levels of skewness depending on the market[5]. If the skew of the smile is too pronounced, then a quadratic function seems not to be suitable for fitting the smile.

Finally, we mention the kernel estimation technique suggested by Ait-Sahalia and Lo (1995). They use a three-dimensional kernel regression of the implied volatilities on asset price, strike and time to maturity to estimate the volatility function. This approach has the disadvantage of needing hundred of data points for reasonable levels of accuracy. Hence, it is not a good method for estimating the volatility function when just a few strikes are available in sets of daily data.

4. IMPLEMENTING THE PARAMETRIC APPROACH

We implement the parametric approach through the use of the Generalised Beta (GB2) distribution proposed by Bookstaber and McDonald (1987) for describing security returns. This is a four parameter distribution that is extremely flexible and it includes a wide range of well-known distributions as limiting and special cases, e.g., lognormal, Burr type 12 and Burr type III and a wide range of mixed distributions. The GB2 density function is defined as follows:

$$GB2(y;a,b,p,q)=\frac{|a|y^{ap-1}}{b^{ap}B(p,q)\left[1+\left(y/b\right)^{a}\right]^{p+q}} \qquad y>0 \tag{23}$$

where B denotes the beta function given by:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}d_t \tag{24}$$

The shape and moments of the distribution are directly specified by its four parameters: b is a scale parameter and it has a direct effect on the mean of the distribution:

$$E(y)=\frac{bB\left(p+1/a,q-1/a\right)}{B(p,q)} \tag{25}$$

The parameter a has effect on the kurtosis of the distribution and with the parameter q determines the existence of higher moments. No moments of order equal to or higher than aq will exist. Finally, the skewness of the distribution is determined by the interaction of the parameters p and q. We refer the reader to McDonald (1984) for a detailed analysis of the GB2 distribution.

[5] See for example Rubinstein (1994).

4.1.1 Parameter Estimation

For a given contract/day and a set of n call options and m put options market prices, with the same maturity T and different strike prices, we estimate the parameters of the model by minimising the sum of squared errors between observed market prices and theoretical prices given by the following equations:

$$C_i(K_i) = e^{-rT} \int_{K}^{+\infty} (S_T - K_i) GB2(S_T; a, b, p, q)\, \mathrm{dS_T} \qquad i = 1 \ldots n \tag{26}$$

$$P_j(K_j) = e^{-rT} \int_{0}^{K} (K_j - S_T) GB2(S_T; a, b, p, q) \mathrm{dS_T} \qquad j = 1 \ldots m \tag{27}$$

$$S_t = e^{-rT} \int_{0}^{\infty} S_T GB2(S_T; a, b, p, q) \mathrm{dS_T} \tag{28}$$

We know already the mean of the distribution is given by the forward price of the underlying security. This additional information must also be used to estimate the implied risk-neutral distribution parameters. Estimation of the parameters is performed by solving a non-linear constrained optimisation problem by means of Sequential Quadratic Programming method routines (See Grace (1995)).

4.2 DATA

We used six contracts from the sample of daily closing prices of the Chicago Mercantile Exchange (CME) options on the Standard & Poor's 500 (S&P 500) index future. The futures contracts have maturities on June and September of 1987 and on March, June, September and December of 1992. They are cash settled on the Thursday prior to the third Friday of the contract month. Riskless interest rates have been calculated from London euro-currency interest rates collected from Datastream.

CME options on the S&P 500 index future are American style, i.e., they can be exercise at any moment prior to maturity and their prices implicitly contain early exercise premiums. We use the Barone-Adesi and Whaley (1987) approximation to recover the implied volatilities from American option prices and used them to recalculate pseudo-European option prices using the Black-Scholes option pricing formula. We used the Barone-Adesi and Whaley model to translate the early exercise premium information into the implied volatility and the Black-Scholes model to express this information into European prices. Thus, the models are used as simple devices to obtain and translate the implied volatility information.

Pricing errors and/or tick-size rounding in the available data are reflected in the smile and particularly in-the-money puts and calls have distorted and jagged smiles. Given that the procedure used for fitting the risk-neutral density function is very sensitive to the estimate of the smile, we decided to work with an extended smile calculated from combined daily data sets of out the money calls and puts. Doing so we include in our estimation only parts of the smile with small cross-sectional noise and we also extend the observed set of strikes prices taking a wider range of available out the money options. Table 1 describes the samples considered in this paper.

4.3 Empirical results

Figures 1 and 2 show the estimates of the risk-neutral density function using the lognormal and the GB2 distributions for two option samples with 90 days to expiration. We can identify a distinct change in the shape of the distributions between the lognormal and the GB2 distributions for the post-crash sample. This is consistent with the empirical evidence that before the crash the S&P 500 index distribution resembled the lognormal distribution and after the crash a more flexible distribution is needed to fit the volatility structure observed. Figure 3 plots a set of implied densities given by the GB2 distribution for options with the same maturity. All distributions are skewed to the left, the opposite of the right skewness associated to the Black-Scholes lognormality assumption.

[Please insert Figures 1-3 here]

We calculated the skewness and kurtosis time patterns of the resulting implied distributions. Figures 4-7 show the skewness and kurtosis for the sample contracts. For the pre-crash sample we find that the implied probability distributions are left skewed for short maturities and positive skewed for long maturities. Although the pre-crash skew sample is very dispersed, for medium-long maturities the level of skewness is close to the lognormal levels. A different pattern is observed for the post-crash contracts where distributions are consistently left skewed and at significant levels with respect to the lognormal estimates. The post-crash skewness also show a decreasing trend across time. Time variability and strongly negative levels of the implicit skewness after the crash has been reported in different studies by Bates (1991), Bates (1994), Rubinstein (1994), Jackwerth and Rubinstein (1996) and others. Both samples show distributions with higher levels of kurtosis than the lognormal estimates. However, the post-crash distributions present higher kurtosis levels that are negative correlated to the maturity of the options. This pattern has been explained by the presence of jumps and stochastic volatility in the underlying security process. Basically, excess kurtosis in short maturity distributions is associated to the presence of jumps in the underlying security process whereas excess kurtosis for longer maturities is usually explained by the presence of stochastic volatility in the dynamics of the underlying. Jumps in the underlying dynamics produce a series of independent variance shocks with an overall effect on the kurtosis of the distribution that decrease with the maturity of the options (a direct implication of the central theorem). Stochastic volatility models are assumed to be instantaneously lognormal with effect on medium/long horizons distributions.

[Please insert Figures 4-7 here]

The parametric estimation of the risk-neutral distribution through the GB2 distribution or another distribution family has several advantages: First, the estimator is itself a probability function, therefore the estimated distributions will satisfy the properties of a density function without additional constraints in the implementation method. We also obtain probability estimates outside the observed strike interval. This avoids an additional extrapolation of the tails of the distribution.[6] Second, this approach is very general allowing the GB2 distribution to be replaced for any other distribution. Third, the implementation is very easy and only requires a standard optimization software. However, we ought to be cautious. Models that

[6] The tails of the density function will depend on the functional form assumed for the distribution. For a rich enough family the fit would not be unique with more than one distribution satisfying the probability mass previously determined for the tails.

are not flexible enough to fit the observed option prices within the arbitrage bounds and asymptotes are likely to produce distorted estimates of the implied densities.

5. IMPLEMENTING THE NON-PARAMETRIC APPROACH

The idea of approximating the volatility curve to recover a continuous formula for the call prices is very convenient. First, it could be compared to the common market practice of estimating implies volatilities to calculate corrected Black-Scholes prices. Second, this method allows arbitrary definitions for the volatility curve, implied or historical volatilities could be use to approximate the volatility function and individual considerations could be incorporated into the model resulting in different option pricing models with deterministic volatility functions.

We propose to estimate the risk-neutral probability distribution using the method suggested by Shimko (1993) but with a more flexible approximation for the volatility. Given a set of observed options prices we want to approximate a continuous volatility function $\sigma(K)$ in order to recover the probability distribution through the first two derivatives of the pricing formula respect to the strike price:

$$\frac{\partial}{\partial K} C(K,\sigma(K)) = \frac{\partial C}{\partial K} + \frac{\partial C}{\partial \sigma}\frac{\partial \sigma}{\partial K} \tag{29}$$

$$\frac{\partial^2}{\partial K^2} C(K,\sigma(K)) = \frac{\partial^2 C}{\partial K^2} + 2\frac{\partial^2 C}{\partial \sigma \partial K}\frac{\partial \sigma}{\partial K} + \frac{\partial C}{\partial \sigma}\frac{\partial^2 \sigma}{\partial K^2} + \frac{\partial^2 C}{\partial \sigma^2}\left[\frac{\partial \sigma}{\partial K}\right]^2 \tag{30}$$

We use a linear combination of cubic B-splines functions as a more flexible approximation for the volatility structure. The estimate of the risk-neutral probability distribution is very sensitive to the changes in the volatility so we look for a spline function that is as smooth as possible but which still fits the observed option values closely enough. Using a smoothing spline curve fitting algorithm proposed by Dierckx (1995), our problem is to estimate a cubic spline *s(K)* such that it minimises the following function

$$\varepsilon = \eta + \rho\delta \tag{31}$$

where

$$\eta = \sum_{i}^{nk}\left(s^{(3)}(\lambda_i^{+}) - s^{(3)}(\lambda^{-}_{i})\right)^2 \quad \text{and} \quad \delta = \sum_{i}^{n}\left(C_i - C(s(K_i))\right) \tag{32}$$

η is a measure of smoothness given by the size of the discontinuity jump of the third derivative of $s(K)$ at the interpolation knots λ_i and δ measures the accuracy of the approximation respect to the observed option prices. ρ represents the penalty imposed on the roughness of the approximation. To avoid the nonlinearity of the problem we use the following approximation for δ:

$$\delta = \sum_{i}^{n}\left(C_i - C(s(K_i))\right)^2$$

$$= \sum_{i}^{n} \frac{\left(\sigma_i - s(K_i)\right)^2 \left(C_i - C(s(K_i))\right)^2}{\left(\sigma_i - s(K_i)\right)^2} \tag{33}$$

$$\cong \sum_{i}^{n} w_i^2 \left(\sigma_i - s(K_i)\right)^2$$

where $w_i = \dfrac{\partial C}{\partial \sigma}$ is the Vega of the option

We require to look for the optimal ρ such that δ is less or equal than a given degree of accuracy. As extreme cases we have that if $\rho = +\infty$, $s(K)$ becomes the cubic interpolating spline and if $\rho = 0$, $s(K)$ becomes the weighted least-squares polynomial of third degree.

5.1 Implied distribution tails

An obvious disadvantage of the spline interpolation technique is that it does not recover the tails of the risk-neutral density function outside the range of available strike prices. Figures 8 and 9 show the graphs of the implied volatility smile and the truncated density function for the CME September 1992 future index contract on the 11th of June 1992. The density is truncated on both tails so they have to be approximated by extrapolating either the volatility smile, the call price curve or the truncated density function itself.

[Please insert Figures 8 and 9 here]

Before reviewing different methods to estimate the risk-neutral distribution tails, it is important to understand how the insufficient number of available strike prices for each maturity option affects the assumed functional form of the density. Option prices only provide information about the conditional mean of the distribution and probability mass (area under the density curve) for the intervals: below the lowest strike (left tail), between each strike, and above the highest strike (right tail).

Any number of distributions could generate the same results for these probabilities and conditional expectations. Therefore, we would expect ambiguity about the best fit, since we can always find an alternative distribution (e.g., a series of uniform densities) that will be observationally equivalent to the given distribution relative to the data considered in the sample. Furthermore, assuming that we obtain a good fit of the distribution between the range of observed strikes, there is still ambiguity about the tails of the distribution for we can change the tails a lot without affecting the predictions we fit with. This freedom for choosing the tails of the distribution allows us to obtain distributions that are right or left skewed and with fat or thin tails. For example, we could approximate the right tail of the distribution in Figure 9, by using a degenerate distribution with probability mass at only one point (creating a distribution with very small kurtosis) or by using a degenerate function with probability mass at two points (one of them far off in the right tail to create infinite kurtosis). Our concern is to fit a functional form for the distribution in order to get a range of non-pathological tails.

Shimko (1993) uses the estimated values of the implied risk-neutral density and distribution functions to solve for the parameters of a lognormal distribution at each tail of the truncated risk-neutral distribution. Thus, he proposes an estimated density function whose tails are given by lognormal distributions. Although his methodology has the advantage of being easy to implement, it does not take into account possible changes of the smile shape beyond the available range of strikes. It is also not clear how convenient is the choice of the lognormal distribution for the density extrapolation given its very limited range of skewness-kurtosis.

We suggest a more flexible version of Shimko technique to approximate the tails of the distribution. Although basically we are following Shimko's extrapolation idea, our methodology differs from his in two aspects: First, we use the more general Ramberg distribution instead of the lognormal distribution originally proposed by Shimko. Second, we use the estimates of the implied risk-neutral distribution and density functions to solve for the parameters of the Ramberg distribution whilst maximising and minimising the kurtosis of the resulting estimate. This gives us a range of plausible tails setting bounds for the distribution in terms of kurtosis levels.

Ramberg et al (1979) proposes a four parameter distribution which allows a wide variety of curve shapes, negative skewness and high levels of kurtosis. The Ramberg distribution is a generalisation of the two parameters Lambda distribution proposed by Tukey (1960) and it is defined by its percentile function R [7]:

$$R(p) = \lambda_1 + \left[p^{\lambda_3} - (1-p)^{\lambda_4}\right] / \lambda_2 \qquad 0 \le p \le 1 \tag{34}$$

and its density function:

$$\begin{aligned} f(x) &= f(R(p)) \\ &= \lambda_2 \left[\lambda_3 p^{\lambda_3 - 1} + \lambda_4 (1-p)^{\lambda_4 - 1}\right]^{-1} \qquad 0 \le p \le 1 \end{aligned} \tag{35}$$

[7] The percentile function is the inverse of the distribution function.

The moments of the distribution are related to its parameters in the following way: λ_1 is a location parameter, λ_2 is a scale parameter and λ_3 and λ_4 define the skewness and kurtosis of the distribution.

When extrapolating the tails, we use the risk-neutral distribution and density estimates to recover the parameters of the Ramberg distribution such that its value at the end of the strikes interval coincides with the value of the truncated distribution. One of the advantages of using the Ramberg distribution is that the density function is given in terms of the percentile function, therefore, matching the value of the density function at any point implicitly would match the value of its distribution function at the same point. Another advantage is that we have now four parameters (instead of two in the case of the lognormal) so we have two additional degrees of freedom that are going to be used to choose the more convenient set of parameters among the space of Ramberg density functions that matches the required values. Ideally, we want a risk-neutral density function with fat tails so we decided to solve for the parameters that maximise the kurtosis of the resulting distribution.

Given x_1, x_2 and x_3 three points at the upper (lower) end of the strike interval (at which we know the values of the risk-neutral density and distribution functions), we solve the following optimisation problem to recover the right (left) tail of the distribution :

Find $\bar{\lambda} \in \Re^4$ such that,

$$h(\bar{\lambda}) = \max_{\theta \in \Re^4} h(\lambda) \tag{36}$$

and

$$g(x_i/\lambda) = f_i \qquad i = 1,2,3.$$

where g is the Ramberg density function, f_1, f_2 and f_3 are the known values of the probability functions at the end (beginning) of the strike price interval and h is a non-decreasing, continuous function of the Ramberg distribution kurtosis. Figure 10 shows the resulting tail approximations for the example considered in Figured 9 using both the lognormal and the Ramberg distributions.

[Please insert Figure 10]

Maximising the kurtosis we get an excess kurtosis equal to 3.99 whereas the excess kurtosis using the lognormal distribution is 2.1. Thus, it is clear that we must be careful when interpreting any estimated distribution, especially in the regions below the lowest strike and above the highest strike where we have information only on the conditional expectations and the probabilities.

It should be mentioned that both approaches, the parametric and the non-parametric, require an approximation of the tails of the distribution for there is no information about the density function outside the range of observed strikes. The ambiguity of the parametric approach is given by the richness of the functional form assumed for the distribution whereas for the non-parametric approach is given by the variety of specifications that can be used to approximate the tails.

5.1.1 Density function shapes

We used our nonparametric method to calculate the implied risk-neutral density functions for the September 1987 and September 1992 CME Future option data. For the purpose of these examples, the volatility functions were extrapolated keeping the curvature of the B-Spline approximation beyond the available set of strikes. Figures 12 and 13 plot the smiles for seven maturities of the September Future option contract in 1987 and 1992. After the 1987 crash, the implied volatility curves are characterised by pronounced smiles and skews compare to the ones obtained during the pre-crash period. It is also evident how the smile and skew become more pronounced as time to maturity decreases. This might be explained by the short term effect of jumps and the long term effect of stochastic volatility in the asset price dynamics.

[Please insert Figures 12 and 13 here]

Figures 14 and 15 show the non-parametric estimates of the risk-neutral density function for the sample dates considered before. Once again there are obvious differences in the shape of the pre-crash and post-crash distributions. Pre-crash distributions look very much like a lognormal distribution while the post-crash distributions are more left skewed and leptokurtic. We can also observe how the distribution becomes more leptokurtic as the option approaches to the expiration date. This corresponds to the pronounced smiles observed for short term options. Tables 2 and 3 contain the summary statistics of the distributions.

[Please insert Figures 14 and 15 here]

The B-splines approximation of the volatility smiles is flexible enough so as to fit the cross-sectional option prices data almost exactly. This is an obvious advantage with respect to other approximations although the risk of overfitting the observed data is high. A drawback of the non-parametric approach is the ambiguity of the method itself given by the extrapolation of the volatility smile (or call prices curve) in order to recover the tails of the distribution. This obviously makes it more difficult to implement when compared to the parametric approach.

5.2 COMPARISON OF THE PARAMETRIC AND NON-PARAMETRIC APPROACHES

Most of recent studies about risk-neutral distribution estimation has been limited to an examination of which alternative distribution would better fit options prices with the lognormal as a benchmark, which is by itself, an odd comparison. Whether the resulting implicit distributions are plausible or not has been less thoroughly examined. As mentioned by Bates (1995a) one of the problems of implicit parameter estimation is that there is not an associated statistical theory, so comparisons are restricted to goodness-of-fit tests.

We have already described and implemented two different methods to recover the risk-neutral distribution. One method is more flexible than the other and both introduce residuals with respect to the observed option values due to market imperfections and model misspecification.

For instance, we know that the flexibility of the non-parametric approach results in a better fit of the observed volatility structures than the GB2 estimates. However, there is also a risk of overfitting. Non-parametric methods, like the B-spline approximation to the volatility smile described in this paper can also reflect noise and pricing errors in the market data producing improbable implied distributions.

We evaluate the performance of the parametric and non-parametric estimations using the following comparisons: First, we provide a comparison of the residuals between the observed implied volatilities and the estimated volatilities implied by the alternative estimates distributions. This would indicate if there is a significant improvement in the goodness-of-fit of the non-parametric approach with respect to the parametric one. Second, we analyse the estimates of the distributions themselves. If prices are exact and continuous, and if the pricing models holds exactly for every single option and for different maturities, time series of implied risk-neutral distributions can be recovered such that they are a martingale under the risk-neutral measure, i.e., if π_t and π_s are two observed distributions under the risk-neutral measure for times t and s, with $t < s$, then $E(\pi_s / F_t) = \pi_t$. Of course, in real situations there are market frictions and pricing errors and the estimated distributions will not give the martingale property exactly. However, time series of plausible implied risk-neutral distributions are expected to present random walk properties with uncorrelated innovations. We set these properties as a benchmark to assess the stability of the implied distributions.

5.2.1 Implied volatility residuals

We calculated the residuals between the observed implied volatilities and the estimated volatilities implied by the estimated distributions. The residuals are given by:

$$e(K_i, t) = \sigma_{obs}(K_i, t) - \sigma_{imp}(K_i, t) \quad (37)$$

where K_i is an observed strike price, σ_{obs} is the Barone-Adesi and Whaley implied volatility calculated from the option data and σ_{imp} is the volatility implied by the three models considered in this comparison: the non-parametric, the GB2 and the lognormal parametric models. We restricted our analysis to the post-crash contracts.

Summary statistics of the percentage implied volatility residuals are reported in Table 4. The results are presented for three intervals of moneyness. The moneyness was calculated as the percentage ratio between strikes and the average of the observed index future prices per future contract. As we expected, the non-parametric method provides the best fit of the observed volatility structures. None of the means of the residuals are significantly different from zero. The magnitudes of the GB2 residuals are satisfactorily small with average residuals statistically significant only for deep out/in-the-money options. The economic significance of the GB2 residuals have yet to be assessed. At-the-money options are the most sensitive to volatility errors so for a given error in the estimated implied volatility, the dollar valuation error is larger for at-the-money options than for out/in-the-money options. Thus, the GB2 residuals may not be economically significant.

The poor performance of the lognormal distribution is again illustrated by the significant errors obtain for all categories. The patterns of the residuals is consistent with the observed volatility smiles in the index markets. Black-Scholes prices appears to be to low for in-the-money calls and for out-the-money puts.

Underfitting is also revealed by autocorrelated residuals, since for the right degree of fit we would expect to obtain uncorrelated residuals. We applied a general test of randomness or white noise based on the first L autocorrelations through the Portmanteau statistic or Q statistic calculated for each strike series as follows[8] [9]:

$$Q(K_i,\tau) = T(T+2)\sum_{\tau=1}^{L}\frac{r(K_i,\tau)^2}{(T-\tau)} \sim \chi_L^2 \tag{38}$$

where $r(K_i,\tau)$ is the τ-lag sample autocorrelations of the residuals:

$$r(K_i,\tau) = \frac{\sum_{t=\tau+1}^{T} e(K_i,t)e(K_i,t-\tau)}{\sum_{t=1}^{T} e(K_i,t)^2} \tag{39}$$

Tables 5-8 show the sample autocorrelations and the Q statistic for each strikes series and for 1, 2 and 3 periods lag. The non-parametric approach passed the test of randomness for all contracts and almost all strikes series. The GB2 model instead presents higher levels of autocorrelation with respect to the non-parametric model, although the levels are not significant for more than a half of the of strikes series per contract. The signs of the autocorrelation coefficients for the GB2 are predominantly positive. This may be interpreted as the times series of the residuals having more random walk properties than white noise

[8] White noise is defined as a sequence of uncorrelated random variables with constant mean and variance.
[9] See Harvey (1993) for a complete analysis of the Portmanteau test.

properties. The lognormal case, as expected, presents high positive autocorrelation levels for all strikes and lags.

We also calculated the autocorrelation coefficients and the Q statistics across time and strikes using the following expressions:

$$r(\tau) = \frac{\sum_i \sum_{t=\tau+1}^{T} e(K_i, t)\, e(K_i, t-\tau)}{\sum_i \sum_{t=1}^{T} e(K_i, t)^2} \tag{40}$$

and

$$Q(\tau) = T(T+2) \sum_{\tau=1}^{L} \frac{r(\tau)^2}{(T-\tau)} \sim \chi_L \tag{41}$$

Qualitatively the results using the more general statistics are the same. The non-parametric model does not present significant levels of autocorrelation and the GB2 and the lognormal models are strongly rejected by the randomness tests.

Summarising, the evidence reported in Tables 5-8 support the notion that models with a very flexible volatility structure will better fit the observed options prices data. The non-parametric model provides the best fit but satisfactory results can also be obtained with the GB2 approximation. The residuals for this model appear not to be economically significant.

5.2.2 Plausible Distribution Functions

When there is no arbitrage and the market is complete, there is an unique equivalent probability measure under which all the asset in the economy are martingales. This equivalent martingale measure satisfies the Kolmogorov's equations and it is itself a martingale. Hence, without market frictions and pricing errors we would expect the time series of a risk-neutral distribution to have random walk properties[10]. This is the argument for our second analysis of the implied distributions. If any of our methods is overfitting the data then it is likely to generate probability distributions time series with negative autocorrelated innovations. This indicates the method is clearly picking up noise in the data and reflecting it in the implied probabilities distributions. This, of course, it is not a desirable property of an estimation method.

We used the estimated risk-neutral distributions to calculate time series of the probability of several strike intervals, i.e., for two given strikes K_i and K_j, with $K_i < K_j$, we calculated the following probability for all dates available in the sample data:

$$F_t\left(K_i, K_j\right) = \int_{K_i}^{K_j} dQ_t \quad t = 1,2...T \tag{42}$$

[10] A random walk characterizes a time series that moves randomly away from its current position: $y_t = y_{t-1} + \varepsilon_t, \quad t = 1,..,T$. A random walk process is not stationary but the mean is constant over the time and it is equal to the initial value y_0. The first difference of a random walk process is white noise, i.e., uncorrelated random variables with constant mean and variance.

Figures 16-19 illustrate our analysis with the time series for the future index and the probability of the strike interval (410,420) for the September 1992 contract implied distributions.

[Please insert Figures 16-19 here]

Notice the highly irregular pattern described by the probability series generated through the non-parametric approach with respect to the GB2 and the lognormal probability time series. We also calculated the innovations or first differences of these time series for each strike interval:

$$\varepsilon_t\left(K_i,K_j\right)=F_t\left(K_i,K_j\right)-F_{t-1}\left(K_i,K_j\right)\ ,\ t=1...T \qquad (43)$$

Figures 20-22 show the innovations time series of the probability time series plotted in the previous figures. The innovations for the non-parametric approach show an autocorrelated pattern and absolute magnitudes significantly larger than the ones obtained through the GB2 and the lognormal distributions. The lognormal distribution innovations are the smoothest and smallest.

[Please insert Figures 20-22 here]

We calculated the autocorrelation coefficients and Q statistics for the probability innovations series. Tables 9-12 summarise the results for each estimation model. The non-parametric approach presents negative and significant first order autocorrelation for all strikes intervals and all contracts. It is also rejected when tested for second and third orders autocorrelations. The GB2 model probability innovation series have significant autocorrelation in no more than three strikes series in the March, June and September contracts. December series are significantly autocorrelated at the 5% level but the null hypothesis of uncorrelated innovations cannot be rejected at the 1% level for the most of the strikes. Still the levels of autocorrelation are significantly lower than the ones presented by the non-parametric model series. The probability innovation series given by the lognormal model present similar characteristics to the GB2 model series.

We conclude from this part of the analysis that the high level of negative autocorrelation of the innovations series obtained through the non-parametric approach may be an indication of overfitting the market data, picking up noise and pricing errors that later are reflected in the distributions. On the other hand, parametric models with flexible distributions like the GB2 model clearly produce plausible distributions with time series properties close to the desirable random walk properties.

5.3 SUMMARY

We presented two alternative parametric and nonparametric methods to recover the risk-neutral distribution from contemporaneous option prices. The parametric method assumes that the future distribution of the underlying asset is a Generalised Beta distribution. We showed this distribution is flexible enough to capture a wide variety of shapes for the implied volatility smiles.

We use the GB2 parametric model to examine six contracts from the sample of daily closing prices of the Chicago Mercantile Exchange (CME) options on the Standard & Poor's 500 (S&P 500) index future. Implied levels of skewness and kurtosis have very different patterns before and after the crash. Before the crash, the skewness of the implied distributions appears to be negative for short maturities and positive for long maturity options. We found the level of excess kurtosis also increases with the maturity of the options. After the crash the pattern is well defined. The skewness for all maturities is predominantly negative and the level of excess kurtosis is negative correlated with the maturity of the options.

We proposed a nonparametric method using B-splines approximations for the volatility smile. The flexibility of the nonparametric method allows an almost exact representation of the observed volatility smiles. This method requires an additional extrapolation for the tails of the risk-neutral distribution. We explained how the enormous freedom for choosing the tails creates ambiguity about the distribution for we can change the tails a lot without affecting the predictions we fit with. We examined alternative specifications for the tails and proposed the flexible Ramberg distribution to approximate the tails of the risk-neutral density function by minimising or maximising the kurtosis of the resulting estimated distribution. This allows us to calculate bounds for the density function in terms of kurtosis levels and used them as a benchmark for any interpretation of the estimated distribution in the regions below (above) the lowest (highest) strike.

We also examined the performance of the parametric and non-parametric methods in terms of the quality of the fit and the stability of the resulting implied distributions. As we expected, as a result of the flexibility of the B-splines to fit the volatility function, the non-parametric model outperformed the GB2 and Lognormal parametric models in fitting the observed option data. However, our results indicate that the GB2 model is flexible enough to capture the market price curves and/or volatility smiles with residuals that appear not to be economically significant.

To assess the stability of the implied distribution we analysed the autocorrelation pattern of the probability time series innovations. The implied distributions recovered using the non-parametric approach generated high negative autocorrelated innovations indicating an overfitting at the estimation stage. On the other hand, the GB2 and lognormal implied distributions presented time series with properties similar to a random walk. We conclude that the implied distributions are less plausible as the estimation model becomes less parsimonious with a high risk of overfitting the data.

The implications of our results depend on the context of the use of the information. For example, if an exact description of the prices is required for a given day then the non-parametric approach would provide the better fit. However, for valuation and risk management analysis where a time series of plausible implied distributions is required, we recommend the use of the parametric approach with a flexible distribution like the one proposed in this paper.

Table 1: Descriptive statistics of the sample data
CME Future on S&P 500 Index Options

Contract	Total of days	Total of options	No. of strikes per day		Average index future	Max strike	Min strike
			Min	Max			
Jun-1987	103	1155	6	20	286.45	220	325
Sep-1987	139	1410	6	18	302.11	235	355
Mar-1992	142	1849	8	18	399.14	340	450
Jun-1992	189	2220	7	20	405.62	340	450
Sep-1992	148	2000	6	21	413.14	340	460
Dec-1992	222	2786	5	20	415.66	350	460

Table 2: S&P 500 Index Future Implied Distributions
September 1987 contract

	Maturity (days)						
	98	84	70	56	42	28	14
Forward	301.30	311.05	308.75	309.30	323.40	336.80	320.00
Mean	301.75	311.20	308.86	309.44	323.43	336.84	319.98
Std. Deviation	30.93	27.37	23.74	18.67	16.97	17.58	12.75
Skewness	0.07	0.05	-0.21	-0.38	-0.49	-0.19	-0.02
Kurtosis	2.87	3.11	3.71	2.98	3.81	3.27	3.17

Table 3: S&P 500 Index Future Implied Distributions
September 1992 contract

	Maturity						
	98	84	70	56	42	28	14
Forward	410.30	403.20	415.00	412.15	420.65	418.50	418.45
Mean	410.27	403.14	415.50	411.95	421.40	418.47	418.54
Std. Deviation	31.32	29.15	23.80	20.82	17.30	15.57	9.26
Skewness	-1.07	-0.94	-1.07	-1.21	-0.50	-0.93	-1.09
Kurtosis	3.92	4.12	4.32	5.06	4.15	5.81	5.87

Table 4: Summary statistics of implied volatility residuals

Average Moneyness	B-Splines			GB2			Lognormal			No. of Obs.
	Mean	Std. Dev.	t statistic	Mean	Std. Dev.	t statistic	Mean	Std. Dev.	t statistic	
[-inf, -3.5]	-0.0024	0.2692	-0.4603	0.1198	0.6153	10.0327	2.6346	1.6580	81.8914	2656
[-3.5,3.5]	-0.0009	0.3630	-0.1347	-0.0023	0.5759	-0.2190	-0.2021	6.2730	-1.7796	3051
[3.5, inf]	0.0043	0.2291	0.8866	0.0591	0.6494	4.3087	-2.4846	7.7275	-15.2143	2239
Total	0.0001	0.2996	0.0174	0.0560	0.6127	8.1531	0.1029	6.0761	1.5098	7946

Summary statistics for the percentage differences between the observed implied volatilities and the estimated volatilities implied by the estimated distributions. Average moneyness is calculated as the percentage ratio between strikes and the average S&P 500 index future per contract.

Table 5: Autocorrelations of the implied volatility residuals March 1992 contract

Autocorrelations

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	-0.0490	-0.0252	-0.1350	0.0451	0.1403	-0.0441	0.8370	0.7776	0.7191
375	-0.1225	0.0307	0.2315	0.4490	0.4535	0.4099	0.9123	0.8605	0.8025
380	-0.0294	0.1100	-0.1136	0.4481	0.4681	0.3110	0.9063	0.8622	0.8230
385	-0.1387	-0.0909	-0.1132	0.3283	0.1885	0.2258	0.9316	0.8935	0.8552
390	0.1247	-0.0671	0.1179	0.2644	0.0428	0.1936	0.8974	0.8397	0.8105
395	-0.0314	0.2423	-0.0112	0.0205	0.2053	-0.0961	0.8591	0.8462	0.7784
400	0.0689	0.1395	0.1208	0.0852	0.2152	0.0809	0.8389	0.8158	0.7714
405	-0.0812	-0.1117	0.0663	0.1378	0.0114	0.0878	0.8770	0.8051	0.7652
410	0.0137	-0.0235	0.1443	0.0864	0.1144	0.1042	0.9043	0.8665	0.8375
415	0.0780	0.0091	-0.0944	0.0855	0.1461	0.0192	0.8814	0.8425	0.8099
420	0.0095	0.0103	0.0637	0.1749	0.2347	-0.0225	0.9444	0.9290	0.9148
425	0.1052	0.0077	-0.0607	0.0060	0.3026	-0.0943	0.9174	0.8637	0.8148
430	0.0883	0.2603	0.0731	0.2548	0.2770	0.0941	0.9597	0.9451	0.9333
435	0.0474	0.0635	0.0804	0.2361	0.1209	0.1318	0.9151	0.9084	0.8797
440	0.0490	0.1673	0.0182	0.4474	0.4183	0.2622	0.9817	0.9694	0.9618

Q-statistic

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	0.219	0.277	1.976	0.193	2.085	2.274	66.577	124.683	174.928
375	1.202	1.278	5.685	16.135	32.817	46.632	67.442	128.237	181.817
380	0.100	1.504	3.015	23.694	49.781	61.398	97.773	187.037	269.089
385	1.847	2.649	3.906	10.566	14.085	19.190	85.948	165.846	239.828
390	2.100	2.712	4.618	9.647	9.902	15.154	111.953	210.717	303.418
395	0.107	6.513	6.527	0.046	4.730	5.765	82.676	163.650	232.807
400	0.650	3.336	5.368	1.032	7.660	8.602	101.348	197.880	284.831
405	0.647	1.884	2.325	1.919	1.932	2.728	78.479	145.303	206.283
410	0.026	0.102	3.000	1.060	2.931	4.496	117.764	226.667	329.160
415	0.530	0.537	1.333	0.643	2.544	2.577	69.939	134.602	195.067
420	0.012	0.025	0.558	4.068	11.454	11.522	122.208	241.366	357.780
425	0.654	0.658	0.883	0.002	5.785	6.356	53.068	100.914	144.237
430	0.904	8.839	9.470	7.795	17.086	18.168	112.400	222.319	330.436
435	0.095	0.269	0.556	2.457	3.118	3.924	37.745	75.872	112.538
440	0.230	2.948	2.981	20.625	38.836	46.066	101.214	200.892	300.004

All Sample

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
	1	2	3	1	2	3	1	2	3
r	0.004	0.017	0.013	0.243	0.240	0.151	0.915	0.880	0.845
Q	0.040	0.621	0.996	90.507	178.893	213.791	1302.153	2506.209	3618.952

	Q statistic critical values		
	Lag		
	1	2	3
1%	6.630	9.210	11.340
5%	3.840	5.990	7.820

Table 6: Autocorrelations of the implied volatility residuals
June 1992 contract

Autocorrelations

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	-0.1179	-0.0711	0.1609	-0.0538	0.1306	-0.0057	0.8826	0.8549	0.8368
375	0.0292	-0.2617	-0.0237	0.4634	0.2256	0.2714	0.9390	0.9055	0.8659
380	-0.1168	-0.1053	-0.0064	0.2679	0.2603	0.1628	0.9038	0.8791	0.8464
385	-0.2125	-0.0179	-0.1271	0.4765	0.4060	0.4337	0.9481	0.9240	0.9066
390	-0.1298	0.1793	-0.0574	0.1478	0.1119	0.2271	0.8816	0.8515	0.8349
395	0.0442	0.0290	0.0485	0.1971	0.1899	0.1391	0.9265	0.8987	0.8818
400	0.1081	-0.0458	0.0615	0.1215	0.1505	0.1504	0.7998	0.7634	0.7382
405	-0.0910	0.0466	0.0072	0.0506	0.1436	0.0843	0.5855	0.5653	0.5323
410	-0.0271	0.1721	-0.0908	0.1622	0.2008	0.1658	0.7448	0.7233	0.7027
415	0.0512	0.1020	0.1394	0.1337	0.2355	0.1117	0.6283	0.5835	0.4908
420	-0.0304	0.1938	0.0149	0.1553	0.2330	0.0922	0.9035	0.9022	0.8796
425	0.0679	0.0635	0.1261	0.0699	-0.0560	0.0231	0.9152	0.8861	0.8653
430	0.1345	-0.0252	0.0785	0.2944	0.2802	0.1483	0.9494	0.9414	0.9312
435	0.2575	0.1383	0.0727	0.2828	0.1423	0.1640	0.9515	0.9385	0.9321
440	0.1195	-0.0201	0.0679	0.2510	0.1208	0.0341	0.9594	0.9509	0.9409

Q-statistic

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	1.237	1.692	4.052	0.307	2.135	2.138	82.602	160.869	236.600
375	0.036	2.996	3.021	9.465	11.766	15.183	38.859	75.923	110.711
380	1.759	3.201	3.206	10.267	20.030	23.878	118.460	231.335	336.715
385	3.387	3.412	4.660	17.716	30.754	45.842	71.054	139.451	206.193
390	2.495	7.286	7.781	3.647	5.749	14.471	130.599	253.181	371.757
395	0.190	0.272	0.506	3.964	7.683	9.698	87.589	170.840	251.829
400	2.010	2.372	3.030	2.835	7.208	11.597	121.542	232.874	337.548
405	0.836	1.058	1.063	0.274	2.503	3.278	36.685	71.219	102.145
410	0.126	5.252	6.688	5.027	12.773	18.082	105.400	205.345	300.186
415	0.263	1.314	3.299	1.842	7.614	8.925	40.672	76.104	101.438
420	0.153	6.424	6.461	4.511	14.719	16.326	153.469	307.354	454.426
425	0.475	0.895	2.569	0.548	0.902	0.963	93.836	182.613	268.085
430	2.586	2.678	3.572	13.785	26.347	29.889	145.135	288.760	430.190
435	6.104	7.885	8.382	7.843	9.848	12.541	91.470	181.380	271.001
440	1.715	1.764	2.327	9.012	11.115	11.285	133.482	265.553	395.784

All sample

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
	1	2	3	1	2	3	1	2	3
r	-0.004	0.018	0.007	0.120	0.123	0.102	0.870	0.849	0.829
Q	0.037	0.816	0.928	35.979	73.408	99.520	1893.520	3697.783	5419.045

Table 7: Autocorrelations of the implied volatility residuals
September 1992 contract

Autocorrelations

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	0.3102	0.0082	-0.0783	-0.0253	-0.0311	0.1023	0.9358	0.9087	0.9011
375	0.0862	-0.2958	-0.1458	0.4005	0.3128	0.3500	0.9208	0.8888	0.8634
380	0.0053	-0.0263	0.0021	0.2522	0.2601	0.2645	0.9390	0.9084	0.8785
385	-0.2189	0.0408	0.2037	0.5140	0.5052	0.4197	0.9193	0.8984	0.8710
390	0.0385	-0.0010	-0.0986	0.3003	0.3003	0.2160	0.9067	0.8906	0.8719
395	-0.1347	-0.1229	-0.0028	0.2482	0.3503	0.1322	0.9099	0.8867	0.8354
400	0.0111	0.1865	-0.0984	0.1832	0.1454	0.1593	0.8862	0.8465	0.8109
405	0.2728	0.1690	0.1770	0.1024	0.3491	0.1648	0.7511	0.7137	0.6868
410	0.2295	0.1993	0.1319	0.0955	-0.0130	-0.1666	0.4513	0.3547	0.2223
415	-0.0969	0.1356	-0.0373	0.1258	0.1064	-0.0901	0.5724	0.4913	0.2939
420	0.0705	0.1224	0.0704	0.0948	0.0275	0.0471	0.7118	0.6633	0.6216
425	-0.1767	0.0851	0.1053	0.2697	0.1300	0.1914	0.8910	0.8590	0.8242
430	0.1325	0.1011	0.1610	0.2771	0.1721	0.1257	0.9059	0.8829	0.8599
435	0.1674	0.0981	0.0462	0.2811	0.5277	0.1302	0.9479	0.9288	0.9137
440	0.0441	0.0405	-0.0893	0.1148	0.1619	0.0331	0.9525	0.9498	0.9335

Q-statistic

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	5.971	5.976	6.370	0.047	0.120	0.917	64.847	126.867	188.753
375	0.290	3.812	4.693	6.109	9.947	14.906	33.137	64.920	95.825
380	0.003	0.073	0.073	6.810	14.119	21.758	96.133	186.960	272.736
385	3.834	3.969	7.380	21.938	43.400	58.406	70.172	138.059	202.684
390	0.189	0.189	1.444	12.628	25.349	31.979	115.938	228.626	337.421
395	1.688	3.110	3.111	6.099	18.382	20.151	81.982	160.676	231.274
400	0.017	4.819	6.165	4.899	8.007	11.765	116.245	223.046	321.748
405	7.594	10.537	13.802	1.102	14.027	16.939	60.385	115.438	166.923
410	7.481	13.164	15.671	1.358	1.384	5.578	30.346	49.227	56.693
415	1.005	2.994	3.145	1.710	2.944	3.837	36.371	63.423	73.196
420	0.707	2.851	3.565	1.330	1.442	1.775	76.525	143.419	202.580
425	3.246	4.006	5.184	8.003	9.880	13.989	88.147	170.851	247.713
430	2.354	3.735	7.265	11.134	15.461	17.786	122.301	239.266	351.007
435	2.691	3.626	3.835	7.747	35.338	37.037	88.988	175.324	259.774
440	0.243	0.449	1.463	1.739	5.227	5.374	122.500	245.253	364.741

All sample

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
	1	2	3	1	2	3	1	2	3
r	0.019	0.029	0.014	0.159	0.194	0.087	0.871	0.846	0.818
Q	0.804	2.594	3.008	56.754	141.212	158.327	1717.866	3339.868	4857.427

Table 8: Autocorrelations of the implied volatility residuals
December 1992 contract

Autocorrelations

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	0.0466	-0.2861	-0.0314	0.0855	-0.1862	0.0974	0.9510	0.9238	0.9211
375	0.0655	0.1758	-0.1736	0.3680	0.3422	0.2746	0.9605	0.9194	0.8785
380	0.0265	0.0998	0.0124	-0.0329	0.0384	0.0016	0.9439	0.9324	0.9138
385	-0.0402	0.2681	0.0388	0.0458	0.1972	0.2734	0.9353	0.9247	0.9064
390	-0.0199	-0.2725	0.0729	0.0763	-0.0660	0.1228	0.9239	0.8940	0.8869
395	-0.2060	-0.2213	0.0663	0.0937	0.1552	0.2110	0.9023	0.8743	0.8497
400	-0.1213	-0.1714	0.2813	0.0476	0.0307	0.2464	0.8310	0.7950	0.7749
405	-0.1111	-0.0057	0.0604	0.1683	0.2287	0.2291	0.8134	0.7491	0.7128
410	0.0094	-0.1563	0.0283	0.0295	-0.0438	0.0584	0.5678	0.5053	0.4234
415	-0.1397	0.1867	-0.0245	0.0639	0.1771	0.0659	0.6248	0.5337	0.4690
420	0.0254	-0.0248	-0.1339	0.0817	-0.0549	0.0033	0.5982	0.4852	0.4447
425	-0.0745	0.0840	0.0507	-0.0224	-0.0070	0.0126	0.8564	0.7647	0.7049
430	0.0620	-0.0708	-0.0293	-0.0532	-0.1564	-0.0072	0.8804	0.8533	0.8551
435	-0.0909	0.0818	0.0051	0.0946	0.2651	0.0469	0.9413	0.9312	0.9133
440	-0.0987	0.0624	-0.0727	0.0332	0.0527	0.0776	0.9438	0.9388	0.9369

Q-statistic

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Strike	1	2	3	1	2	3	1	2	3
370	0.219	8.578	8.679	0.906	5.240	6.437	113.986	222.434	331.146
375	0.168	1.412	2.662	5.428	10.256	13.456	36.977	71.827	104.581
380	0.109	1.665	1.689	0.191	0.451	0.452	159.478	315.998	467.201
385	0.128	5.884	6.006	0.170	3.361	9.579	72.634	144.544	214.542
390	0.069	13.063	13.999	1.095	1.920	4.786	160.493	311.575	461.105
395	4.158	9.012	9.452	0.860	3.248	7.707	81.446	158.716	232.481
400	2.958	8.897	24.962	0.478	0.678	13.613	147.098	282.371	411.521
405	1.346	1.349	1.754	3.145	9.005	14.945	73.461	136.360	193.848
410	0.018	5.049	5.214	0.187	0.601	1.340	69.651	125.057	164.147
415	2.107	5.910	5.976	0.457	4.004	4.499	43.733	75.943	101.048
420	0.137	0.268	4.106	1.449	2.107	2.109	79.451	131.953	176.265
425	0.582	1.330	1.606	0.052	0.057	0.074	79.968	144.331	199.561
430	0.799	1.846	2.026	0.610	5.893	5.905	173.631	337.501	502.803
435	0.901	1.636	1.639	0.922	8.235	8.466	99.261	197.310	292.520
440	1.977	2.772	3.857	0.235	0.833	2.134	199.524	397.874	596.316

All sample

	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
	1	2	3	1	2	3	1	2	3
r	-0.013	-0.023	0.008	0.035	0.010	0.065	0.868	0.839	0.820
Q	0.452	1.859	2.015	3.396	3.651	15.211	2123.890	4109.928	6007.315

Table 9: Autocorrelations of the implied probability series innovations March 1992 contract

Autocorrelations

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
360-350	-0.4185	0.0539	-0.0797	-0.3155	0.1191	-0.1051	-0.0951	0.0656	-0.0778
370-360	-0.5160	0.2434	-0.1621	-0.2287	-0.0019	-0.1041	0.0075	0.0365	-0.0829
380-370	-0.3355	-0.1625	0.0157	0.0593	-0.0223	-0.0659	-0.0136	-0.0049	-0.0212
390-380	-0.3790	-0.0998	0.1169	0.0408	-0.0055	-0.0418	-0.2410	-0.0457	-0.0570
400-390	-0.3037	-0.1597	0.0105	-0.0853	0.0154	-0.1150	-0.1328	0.0536	-0.0969
410-400	-0.4004	0.0617	0.1214	0.0679	0.2585	0.0635	0.0620	0.1240	0.3174
420-410	-0.3976	-0.0860	0.1741	0.1654	-0.0721	-0.0613	-0.2708	-0.0425	-0.0349
430-420	-0.3940	-0.0099	-0.0857	-0.1425	-0.0208	-0.1353	-0.4050	-0.0374	-0.0213

Q-statistic

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
360-350	7.894	8.028	8.330	4.686	5.370	5.916	0.426	0.633	0.932
370-360	16.523	20.263	21.952	3.298	3.298	4.005	0.003	0.083	0.505
380-370	10.023	12.403	12.426	0.314	0.358	0.754	0.016	0.019	0.060
390-380	15.808	16.914	18.448	0.185	0.189	0.386	6.567	6.804	7.178
400-390	12.267	15.684	15.699	0.975	1.007	2.807	2.328	2.710	3.969
410-400	22.443	22.980	25.074	0.655	10.213	10.794	0.547	2.747	17.267
420-410	20.395	21.356	25.331	3.557	4.238	4.735	9.683	9.923	10.087
430-420	17.856	17.867	18.727	2.355	2.406	4.569	19.361	19.527	19.582

All sample

	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
r	-0.387	-0.032	0.044	-0.009	0.046	-0.060	-0.252	-0.008	-0.008
Q	120.109	120.939	122.498	0.059	1.761	4.712	51.408	51.465	51.511

Table 10: Autocorrelations of the implied probability series innovations June 1992 contract

Autocorrelations

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
360-350	-0.3808	-0.1625	0.0501	-0.0278	-0.2139	0.0395	-0.0742	-0.1470	-0.0857
370-360	-0.3972	0.0192	-0.1871	-0.0887	-0.0484	-0.0510	-0.0837	-0.2795	-0.0190
380-370	-0.5062	0.1221	-0.1283	-0.1013	-0.0330	-0.0198	-0.4327	-0.0862	0.2101
390-380	-0.3428	-0.2508	0.1907	-0.3662	0.0082	0.2952	0.0573	-0.1114	-0.0232
400-390	-0.4362	-0.1706	0.2413	-0.4861	0.1072	0.0332	-0.1826	-0.2777	0.0037
410-400	-0.3153	-0.1576	0.0479	0.0501	-0.0721	-0.0025	0.0197	0.2167	0.0176
420-410	-0.4824	0.0903	-0.0544	-0.0692	-0.1138	0.1917	-0.1458	-0.1092	0.0540
430-420	-0.4605	0.0149	-0.0432	-0.2823	0.0270	-0.0348	-0.2745	0.1130	-0.1313

Q-statistic

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
360-350	8.707	10.320	10.477	0.051	3.124	3.230	0.369	1.840	2.348
370-360	10.107	10.131	12.450	0.566	0.738	0.930	0.512	6.304	6.331
380-370	22.560	23.889	25.372	0.749	0.830	0.859	18.357	19.093	23.516
390-380	12.576	19.376	23.343	13.546	13.552	22.544	0.378	1.818	1.881
400-390	28.739	33.165	42.077	38.278	40.152	40.332	5.234	17.423	17.425
410-400	17.401	21.776	22.183	0.471	1.453	1.455	0.073	8.856	8.914
420-410	39.570	40.964	41.474	0.876	3.261	10.062	3.888	6.081	6.622
430-420	30.967	30.999	31.276	12.437	12.552	12.743	11.157	13.059	15.649

All sample

	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
r	-0.417	-0.054	0.017	-0.193	-0.014	0.049	-0.150	-0.014	-0.009
Q	163.711	166.402	166.667	36.555	36.743	39.093	22.612	22.821	22.894

Table 11: Autocorrelations of the implied probability series innovations September 1992 contract

Autocorrelations

	B-splines model			GB2 model			Lognormal model		
Strike		Lag			Lag			Lag	
Interval	1	2	3	1	2	3	1	2	3
360-350	0.0067	-0.3557	-0.1338	-0.2103	-0.1107	0.2213	-0.0041	0.2004	0.0451
370-360	-0.2512	-0.1908	-0.0080	-0.2799	-0.0406	-0.0154	-0.2506	-0.0128	-0.0345
380-370	-0.4800	0.2404	-0.2359	-0.0836	-0.2466	-0.0547	-0.0182	-0.1411	-0.0956
390-380	-0.5175	0.0781	0.0197	-0.0775	-0.1641	-0.1303	-0.1045	-0.1746	0.0170
400-390	-0.5095	0.0435	-0.0167	0.0062	-0.0087	-0.1030	0.2204	0.0280	-0.1577
410-400	-0.4080	0.1182	-0.1744	0.0151	-0.0564	-0.1376	-0.0071	-0.2171	-0.1646
420-410	-0.2627	-0.1070	-0.0393	0.2036	0.0243	-0.0421	0.0781	0.2738	0.1410
430-420	-0.3544	-0.1637	0.0334	0.1469	-0.1898	-0.1811	0.2198	-0.0740	-0.1579

Q-statistic

	B-splines model			GB2 model			Lognormal model		
Strike		Lag			Lag			Lag	
Interval	1	2	3	1	2	3	1	2	3
360-350	0.001	3.217	3.697	1.068	1.379	2.694	0.000	1.021	1.076
370-360	1.586	2.547	2.549	1.892	1.934	1.940	1.579	1.583	1.616
380-370	14.525	18.231	21.862	0.441	4.340	4.536	0.021	1.297	1.894
390-380	26.251	26.856	26.894	0.613	3.391	5.158	1.071	4.093	4.122
400-390	33.228	33.473	33.509	0.005	0.015	1.459	6.414	6.519	9.854
410-400	23.306	25.278	29.600	0.033	0.504	3.328	0.007	6.800	10.732
420-410	9.874	11.524	11.748	6.178	6.267	6.535	0.891	11.918	14.864
430-420	17.588	21.367	21.525	3.151	8.450	13.309	7.053	7.858	11.549

All sample

	B-splines model			GB2 model			Lognormal model		
		Lag			Lag			Lag	
	1	2	3	1	2	3	1	2	3
r	-0.404	-0.015	-0.041	0.110	-0.074	-0.115	0.085	-0.045	-0.086
Q	120.708	120.867	122.131	9.325	13.578	23.793	5.491	7.040	12.704

Table 12: Autocorrelations of the implied probability series innovations December 1992 contract

Autocorrelations

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
380-370	-0.3750	-0.2427	0.1208	-0.3091	0.0028	0.0581	-0.2559	0.1423	-0.1055
390-380	-0.3213	-0.3204	0.1324	-0.1908	-0.0052	0.1002	0.0120	-0.1118	0.0085
400-390	-0.4801	-0.0131	-0.0298	-0.1523	0.0396	0.0004	-0.1021	0.0563	-0.1318
410-400	-0.5474	0.1154	-0.1183	-0.1600	0.0468	-0.0234	-0.0470	0.1752	-0.1103
420-410	-0.3761	-0.0540	0.0780	-0.1699	0.1272	-0.1170	-0.0026	0.2172	0.0560
430-420	-0.3987	0.0113	-0.1432	-0.1593	0.1672	-0.0717	0.1662	-0.0897	-0.0971
440-430	-0.3732	0.1133	0.0259	-0.1951	0.2339	-0.0316	0.2913	0.1796	-0.0792
450-440	-0.3810	-0.0846	-0.0174	-0.2254	-0.0199	0.0691	-0.1030	0.0727	0.0148
460-450	-0.4846	0.0698	-0.0667	-0.0690	-0.0703	-0.0443	0.1480	-0.1462	-0.1386

Q-statistic

Strike Interval	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
380-370	15.614	22.215	23.865	10.705	10.706	11.090	7.337	9.627	10.897
390-380	14.866	29.760	32.321	5.245	5.248	6.716	0.021	1.847	1.858
400-390	41.491	41.522	41.683	4.223	4.510	4.510	1.866	2.436	5.581
410-400	60.525	63.229	66.085	5.246	5.698	5.812	0.449	6.712	9.209
420-410	29.425	30.036	31.313	6.181	9.660	12.616	0.001	10.000	10.668
430-420	34.025	34.053	38.486	5.256	11.075	12.149	5.991	7.747	9.812
440-430	29.814	32.575	32.720	8.104	19.815	20.031	18.579	25.680	27.068
450-440	28.308	29.710	29.769	10.268	10.348	11.324	2.102	3.155	3.199
460-450	38.990	39.804	40.552	0.804	1.644	1.980	3.700	7.336	10.623

All sample

	B-splines model Lag 1	2	3	GB2 model Lag 1	2	3	Lognormal model Lag 1	2	3
r	-0.414	-0.010	-0.018	-0.178	0.090	-0.018	0.132	0.095	-0.065
Q	275.963	276.130	276.675	51.553	64.745	65.249	28.545	43.335	50.217

Table 13: Random walk Lo & MacKinlay Test for the implied probability series innovations

March 1992 contract

Strike	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Interval	2	3	4	2	3	4	2	3	4
360-350	-10.929	-10.100	-10.229	-12.452	-8.678	-8.096	-5.762	-4.841	-5.831
370-360	-22.740	-16.860	-16.696	-10.782	-10.049	-10.768	0.377	1.599	-0.078
380-370	-19.963	-23.235	-23.030	3.233	2.509	1.100	-1.309	-1.833	-2.556
390-380	-31.401	-32.783	-28.599	3.121	2.804	1.627	-14.925	-15.857	-16.749
400-390	-31.922	-36.554	-35.366	-7.173	-6.411	-8.467	-10.123	-8.467	-9.785
410-400	-45.294	-39.070	-31.392	2.680	11.171	14.134	0.546	2.506	7.769
420-410	-46.626	-45.982	-37.299	9.998	6.338	3.314	-8.686	-9.382	-10.016
430-420	-36.709	-34.657	-34.675	-11.714	-11.377	-13.577	-12.544	-13.122	-13.517

June 1992 contract

Strike	B-splines model			GB2 model			Lognormal model		
	Lag			Lag			Lag		
Interval	2	3	4	2	3	4	2	3	4
360-350	-19.964	-21.838	-20.189	-2.031	-8.664	-9.079	-4.959	-8.372	-10.165
370-360	-21.607	-19.552	-20.727	-7.675	-8.429	-9.068	-8.473	-15.365	-16.599
380-370	-37.319	-30.623	-29.947	-14.375	-13.459	-11.246	-11.437	-12.241	-10.625
390-380	-33.896	-41.536	-35.815	-16.929	-16.494	-11.214	4.143	0.090	-1.534
400-390	-43.631	-48.615	-40.565	-37.503	-32.032	-28.108	-12.096	-19.180	-19.874
410-400	-49.233	-56.153	-52.788	2.619	-0.308	-1.375	-1.294	5.168	6.508
420-410	-61.584	-52.499	-49.630	-7.942	-12.478	-9.618	-13.401	-15.360	-14.224
430-420	-45.766	-41.501	-39.363	-23.353	-19.614	-17.995	-19.443	-13.626	-13.463

Figure 1: S&P 500 Index Future implied risk-neutral density function
90 days to maturity
03/06/87

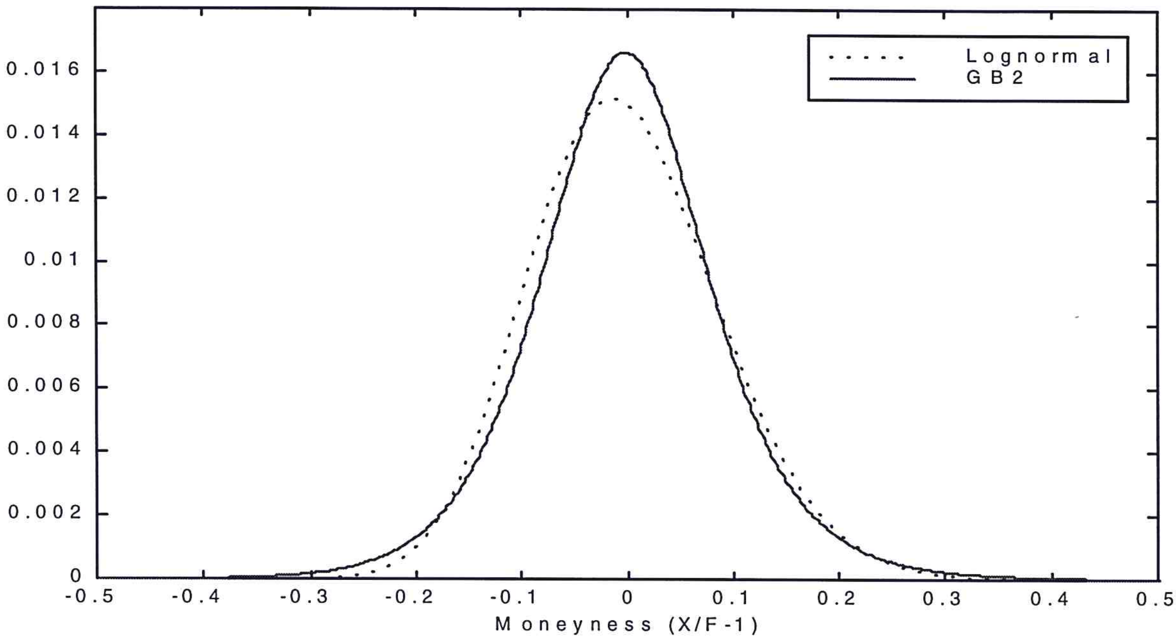


Figure 2: S&P 500 Index Future implied risk-neutral density function
90 days to maturity
03/06/92

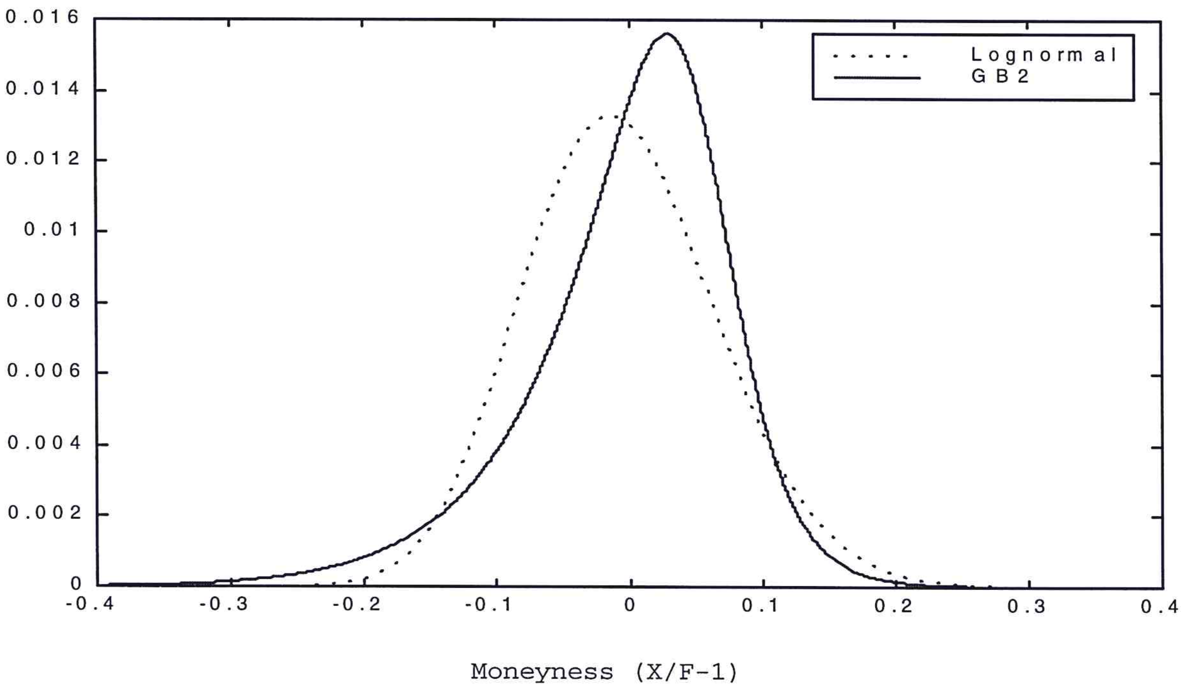


Table 14: Random walk Lo & MacKinlay Test for the implied probability series innovations

September 1992

Strike Interval	B-splines model Lag 2	3	4	GB2 model Lag 2	3	4	Lognormal model Lag 2	3	4
360-350	-0.407	-3.145	-4.655	-2.622	-3.183	-2.420	-0.079	1.760	2.351
370-360	-5.402	-6.617	-6.663	-4.368	-4.614	-4.751	-5.973	-6.354	-6.754
380-370	-15.397	-11.177	-11.940	-3.172	-7.384	-9.080	-2.709	-7.716	-10.761
390-380	-34.618	-29.840	-26.750	-5.470	-10.449	-14.033	-7.489	-11.463	-11.348
400-390	-45.040	-40.457	-37.994	0.473	-0.097	-3.422	12.341	11.863	8.252
410-400	-46.954	-37.474	-38.740	0.932	-1.083	-4.670	-0.559	-5.055	-8.412
420-410	-30.484	-34.490	-34.306	8.989	8.447	6.791	0.165	6.606	9.741
430-420	-36.217	-40.845	-39.007	7.489	0.795	-4.593	15.968	11.089	5.400

December 1992

Strike Interval	B-splines model Lag 2	3	4	GB2 model Lag 2	3	4	Lognormal model Lag 2	3	4
380-370	-26.087	-32.694	-31.063	-18.330	-16.670	-14.463	-12.784	-8.520	-8.385
390-380	-31.916	-44.573	-41.956	-18.894	-17.551	-13.425	1.415	-5.215	-6.418
400-390	-71.505	-66.979	-63.095	-22.021	-16.902	-14.318	-13.299	-8.677	-11.195
410-400	-82.115	-68.799	-66.646	-25.645	-18.869	-16.664	-5.319	3.263	2.157
420-410	-55.326	-55.420	-49.183	-20.041	-10.898	-11.222	-0.492	7.662	10.559
430-420	-66.941	-60.323	-61.452	-17.751	-9.411	-9.119	10.234	6.523	2.904
440-430	-62.681	-50.389	-42.852	-26.935	-11.026	-7.736	17.265	20.518	18.642
450-440	-54.280	-56.907	-55.825	-25.568	-24.337	-20.360	-8.749	-5.240	-3.696
460-450	-50.593	-44.897	-43.819	-7.292	-10.604	-12.715	14.600	7.016	0.299

Figure 3: S&P 500 Index Future implied risk-neutral density functions
30 days to maturity

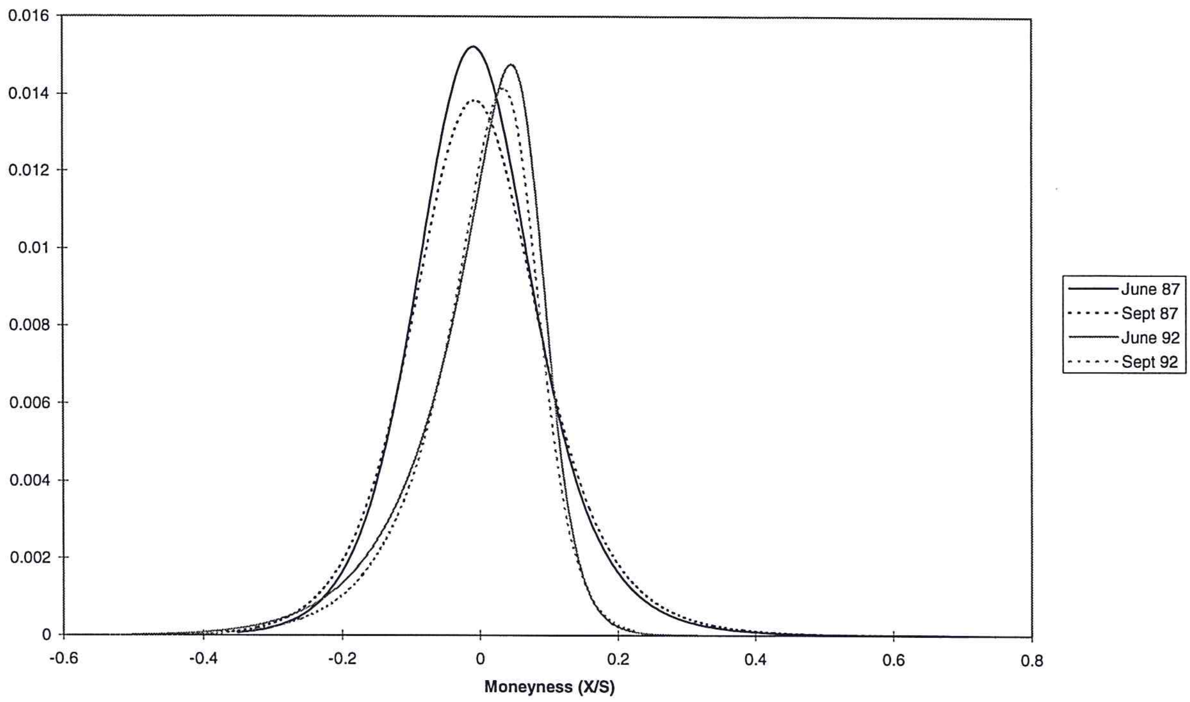


Figure 4: Skewness of the implied probabilities distributions
June and September 1987 contracts

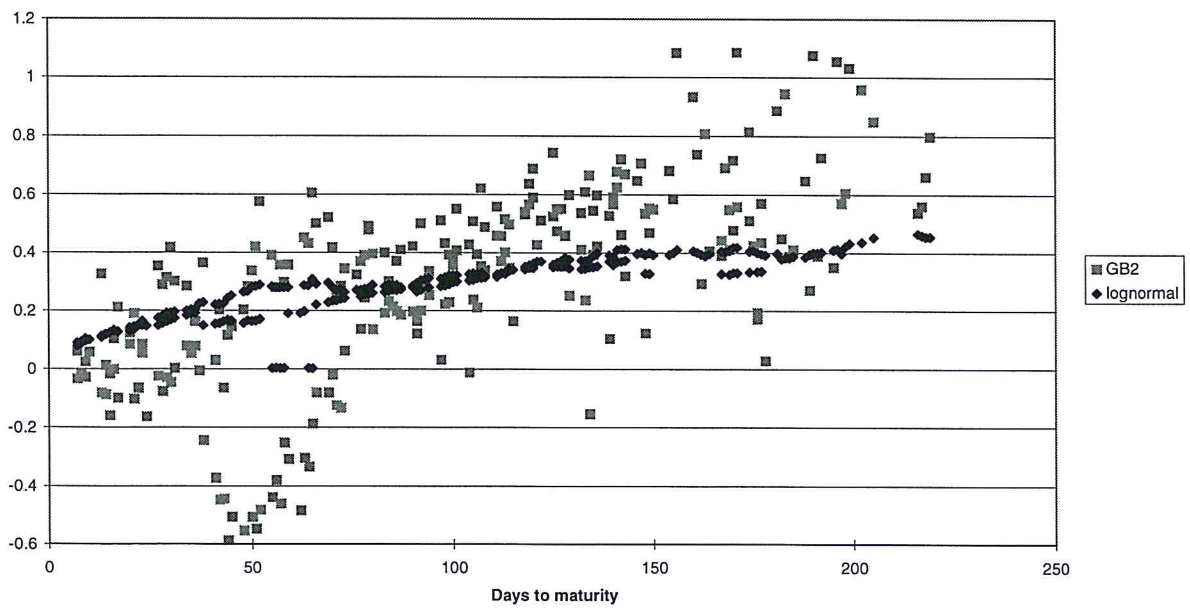


Figure 5: Excess kurtosis of the implied probabilities distributions June and September 1987 contracts

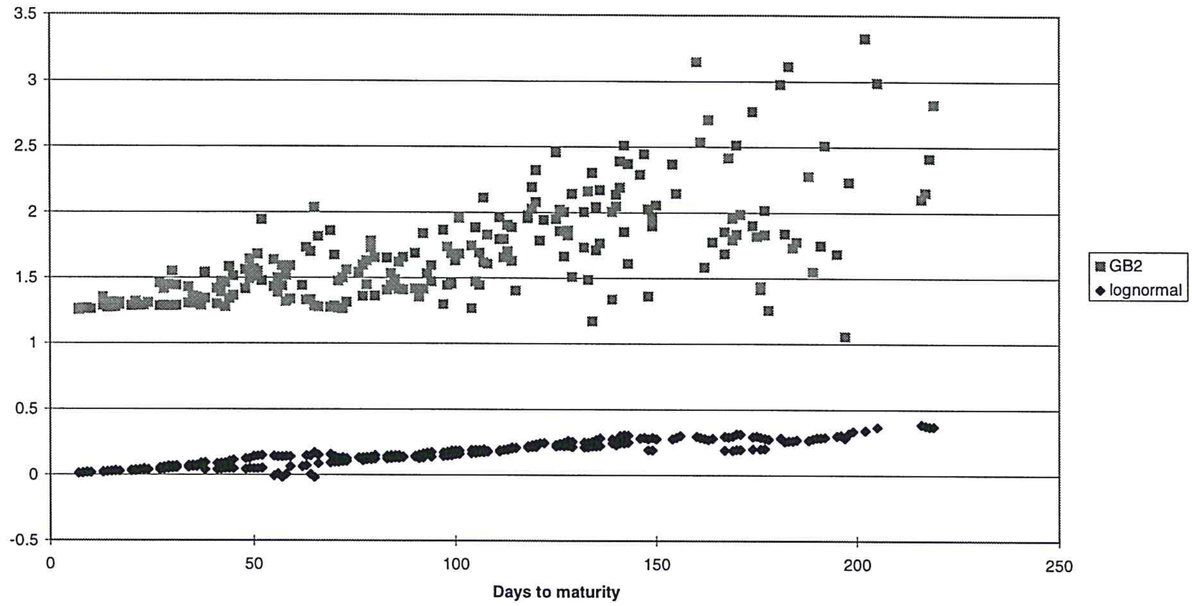


Figure 6: Skewness of the implied probabilities distributions March, June, September and December 1992 contracts

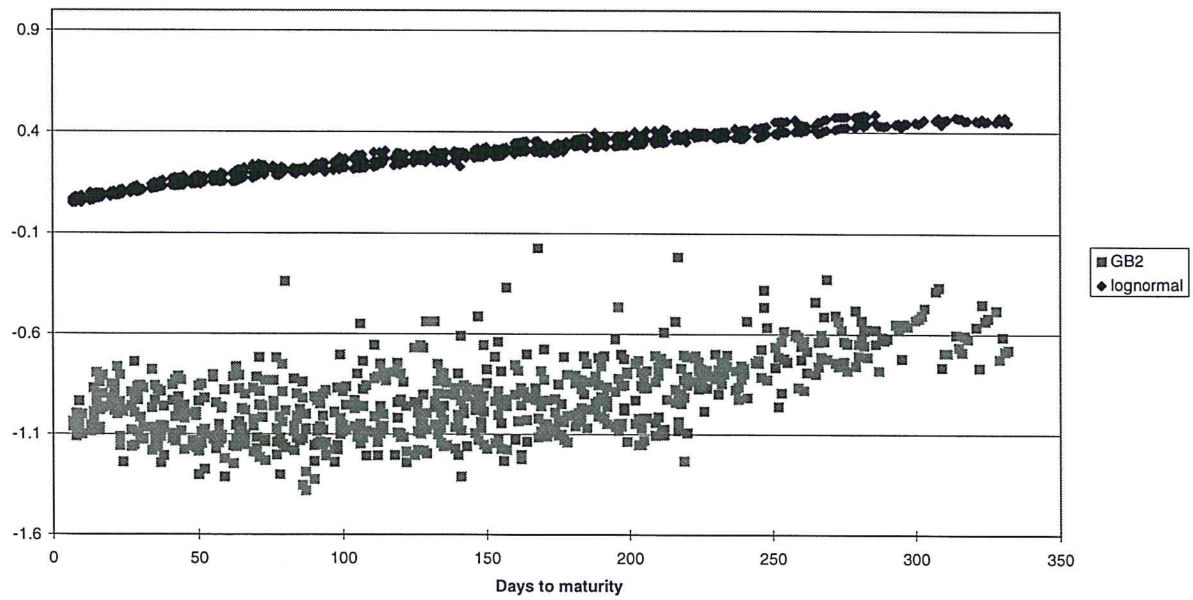


Figure 7: Excess kurtosis of the implied probabilities distributions March, June, September and December 1992 contracts

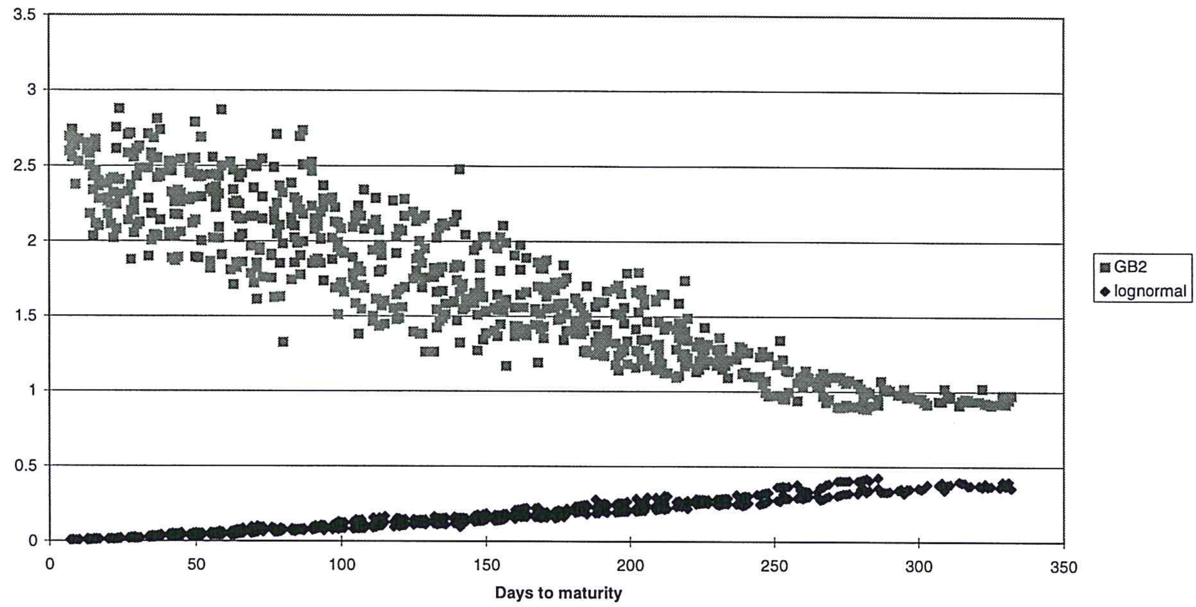


Figure 8: Implied Volatility Approximation CME June 1992 contract 11th of June 1992

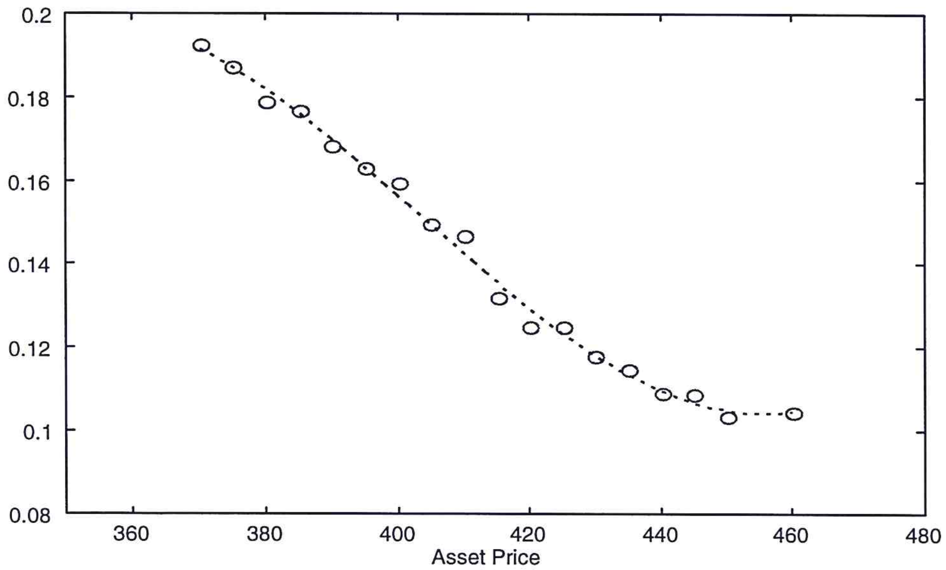


**Figure 9: S&P 500 Index Future truncated risk-neutral density function
CME June 1992 contract
11th of June 1992**

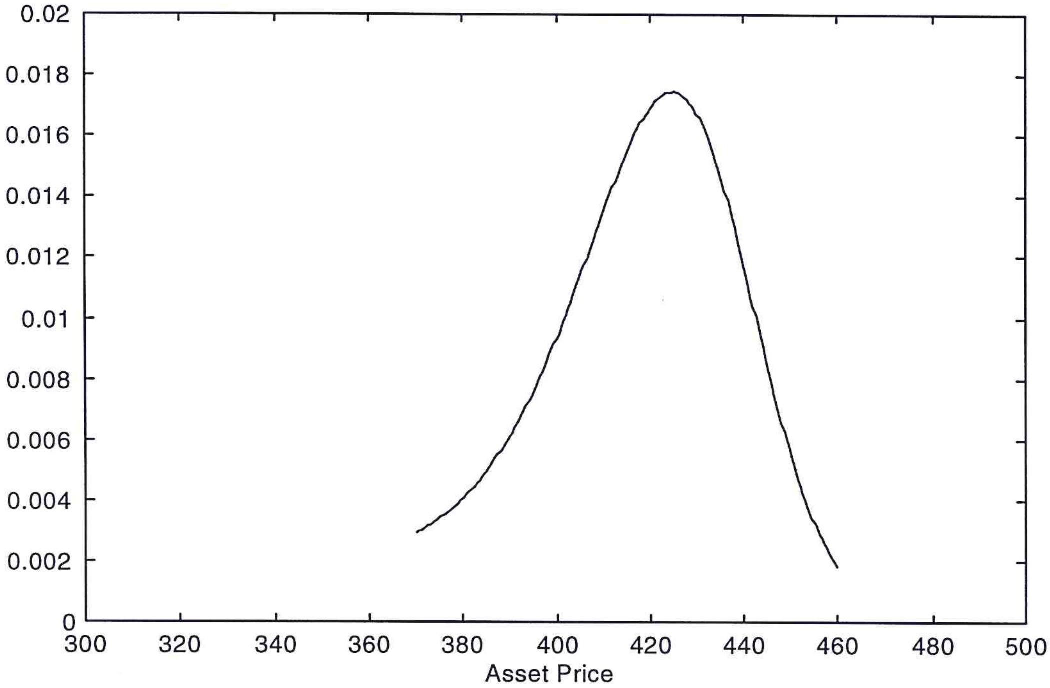
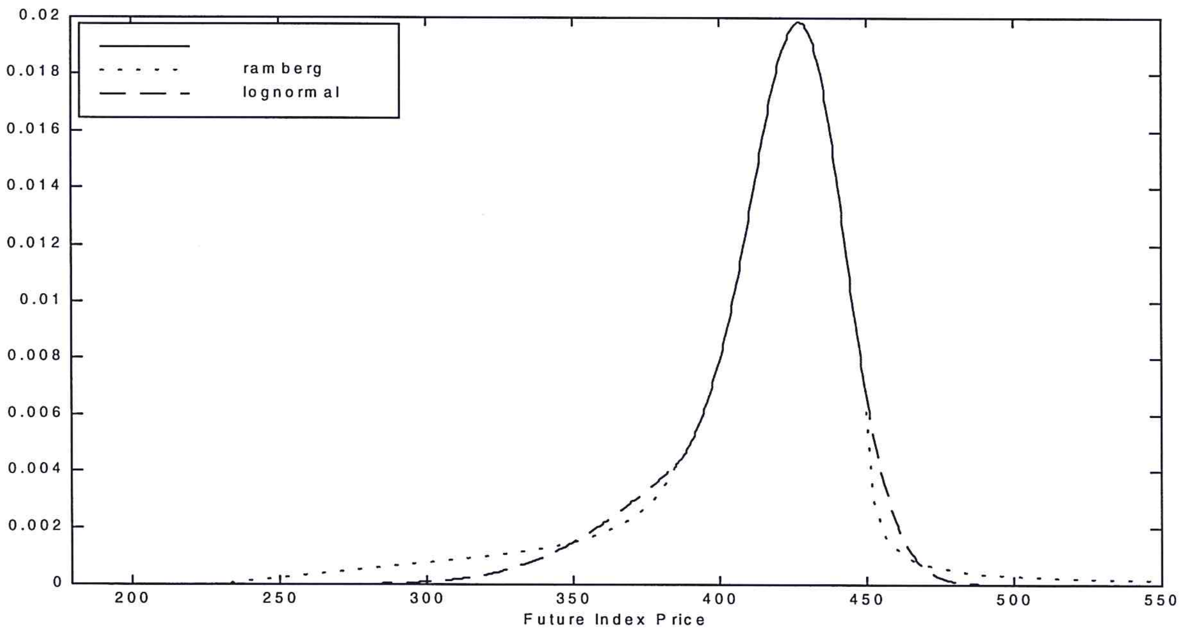


Figure 10: S&P 500 Index Future risk-neutral density function:



Tails approximated with Ramberg (maximum kurtosis) and Lognormal distributions
CME June 1992 contract (11th of June 1992)

Figure 12: Pre-crash implied volatilities

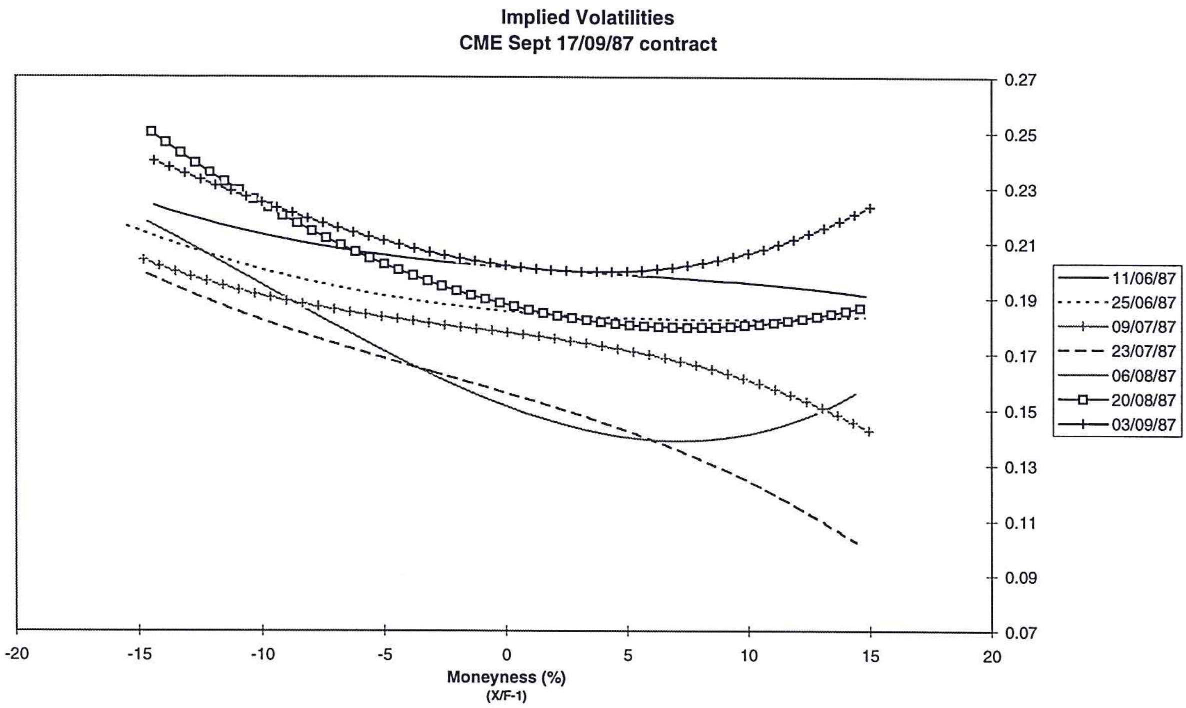


Figure 13: Post-crash implied volatilities

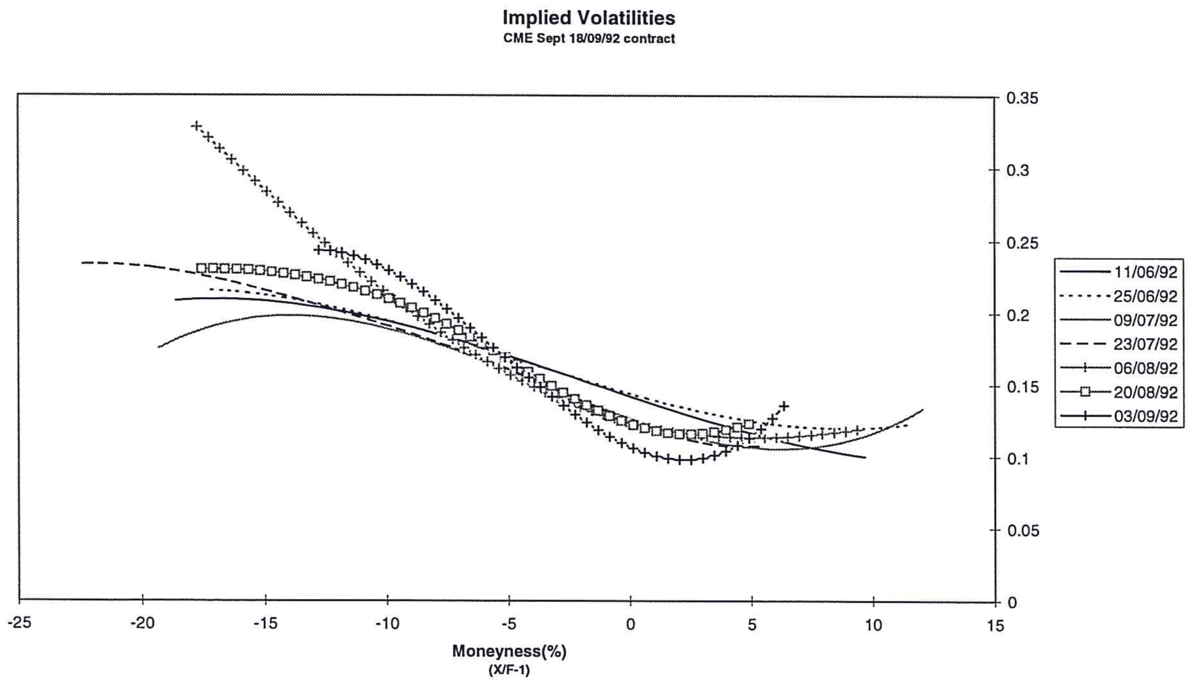


Figure 14: Pre-crash implied density functions B-spline approximation approach

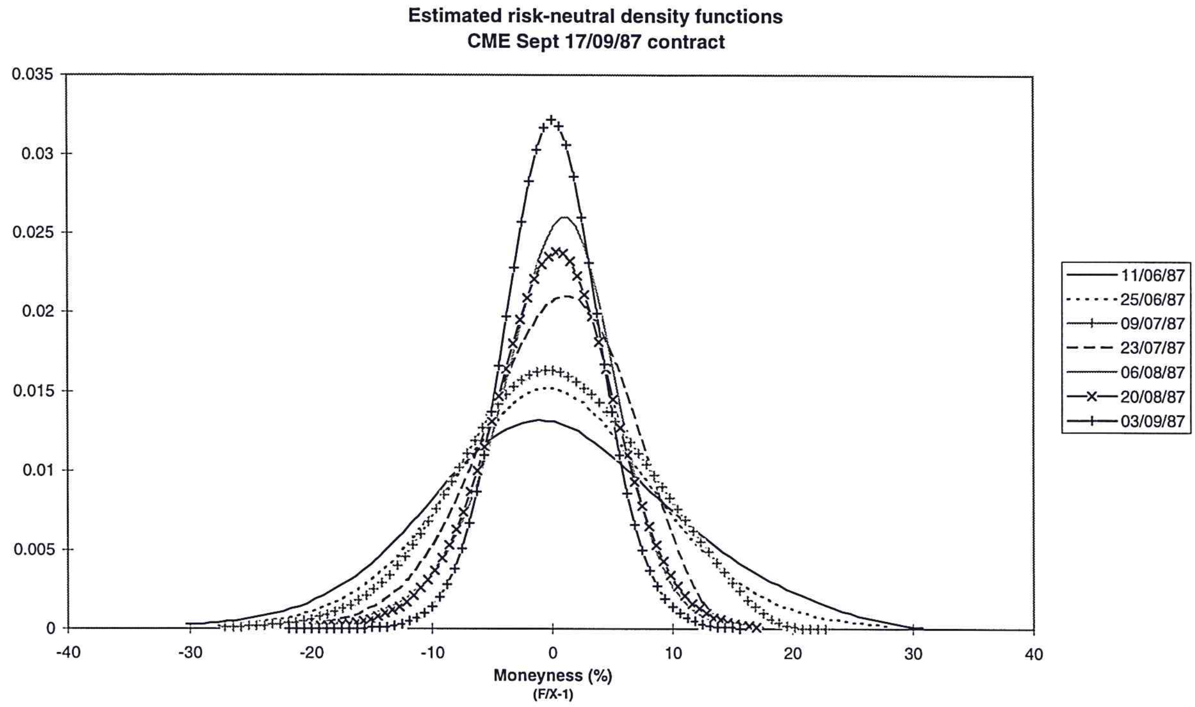


Figure 15: Post-crash implied density functions B-spline approximation approach

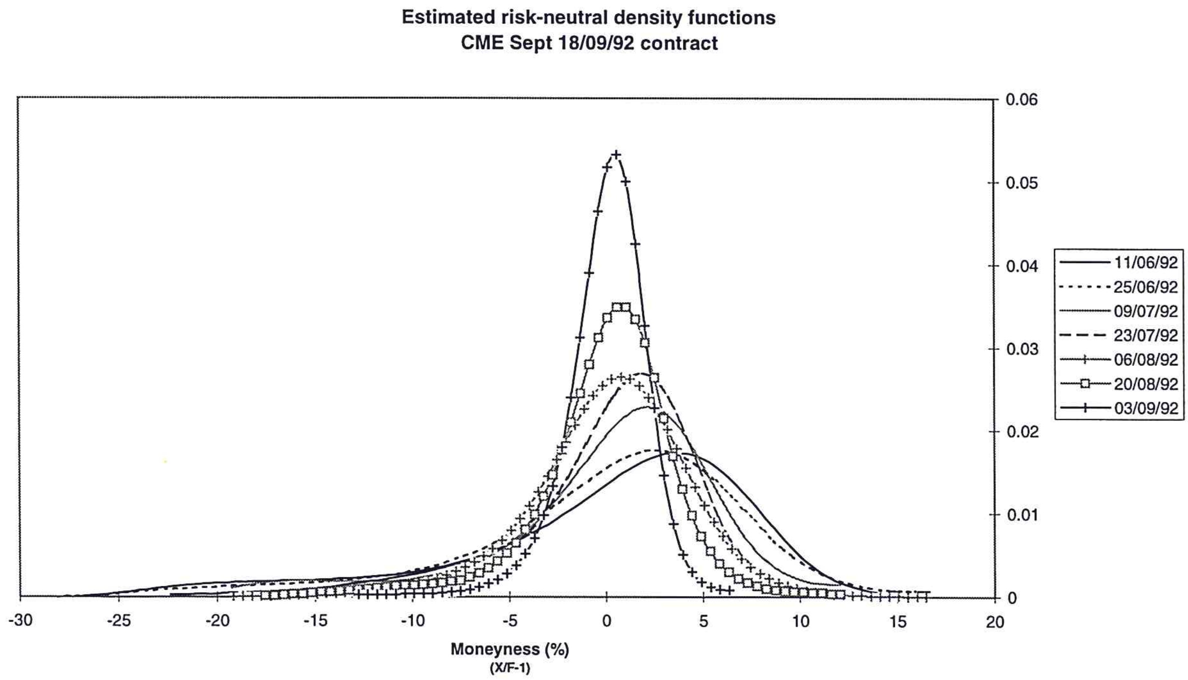


Figure 16: Future Index time series

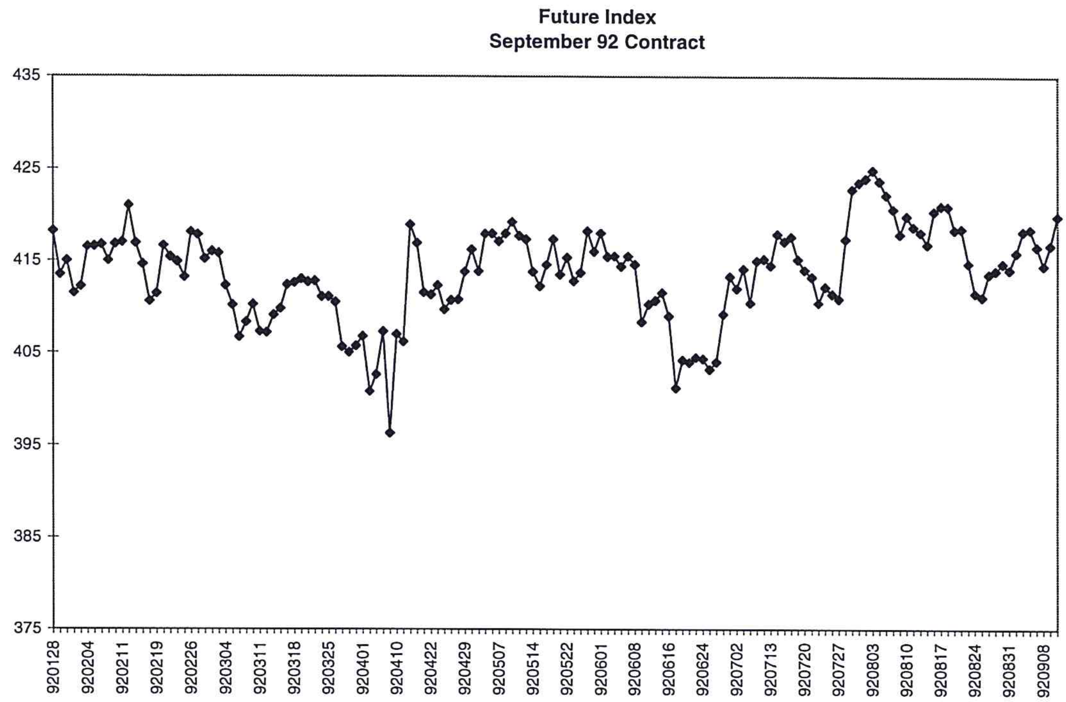


Figure 17: Probability density time series for the strike interval 410-420 B-splines estimation method

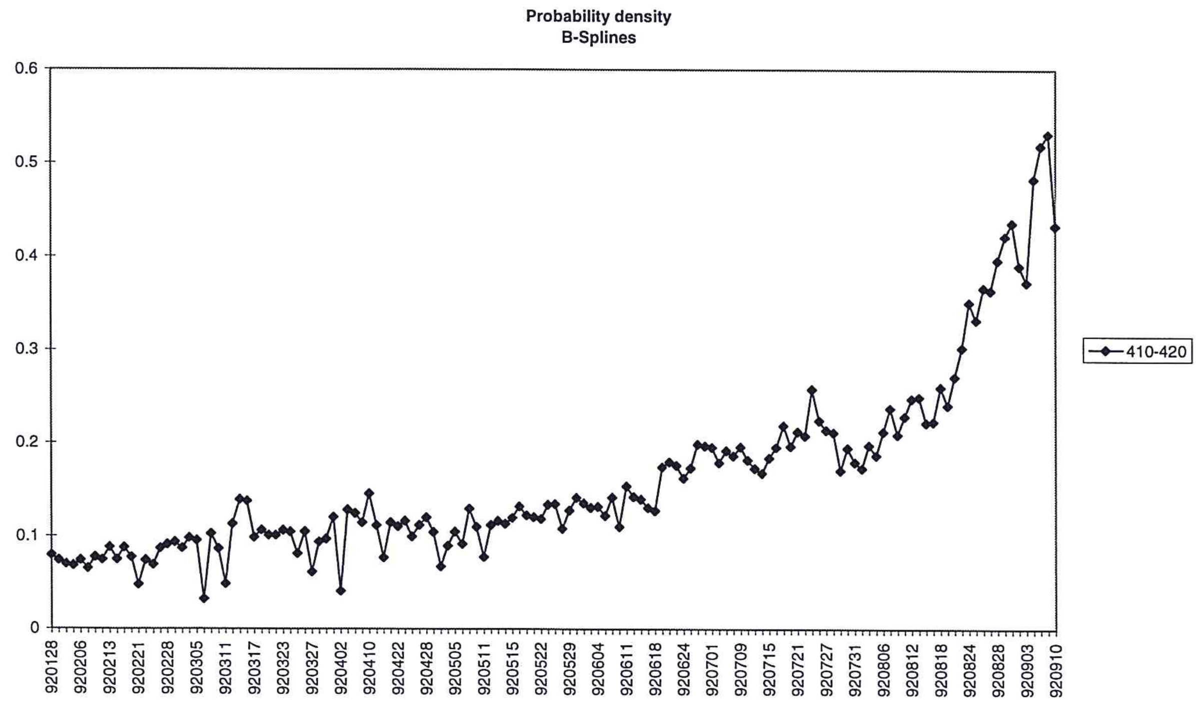


Figure 18: Probability density time series for the strike interval 410-420
GB2 estimation method

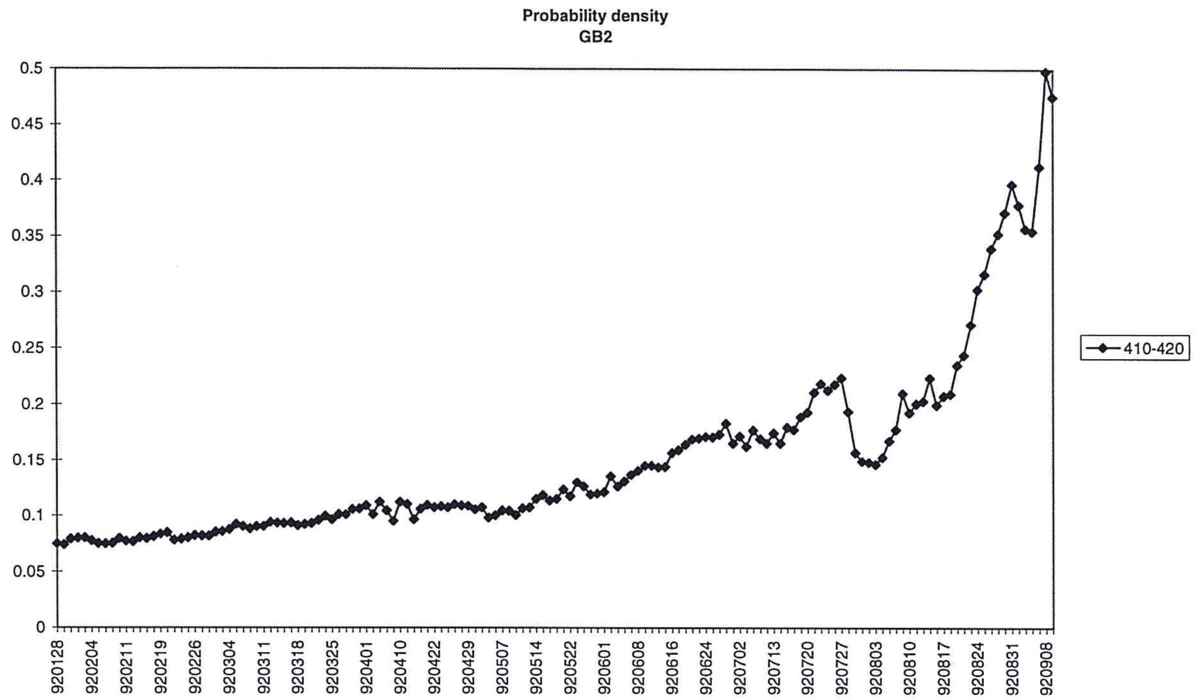


Figure 19: Probability density time series for the strike interval 410-420
Lognormal estimation method

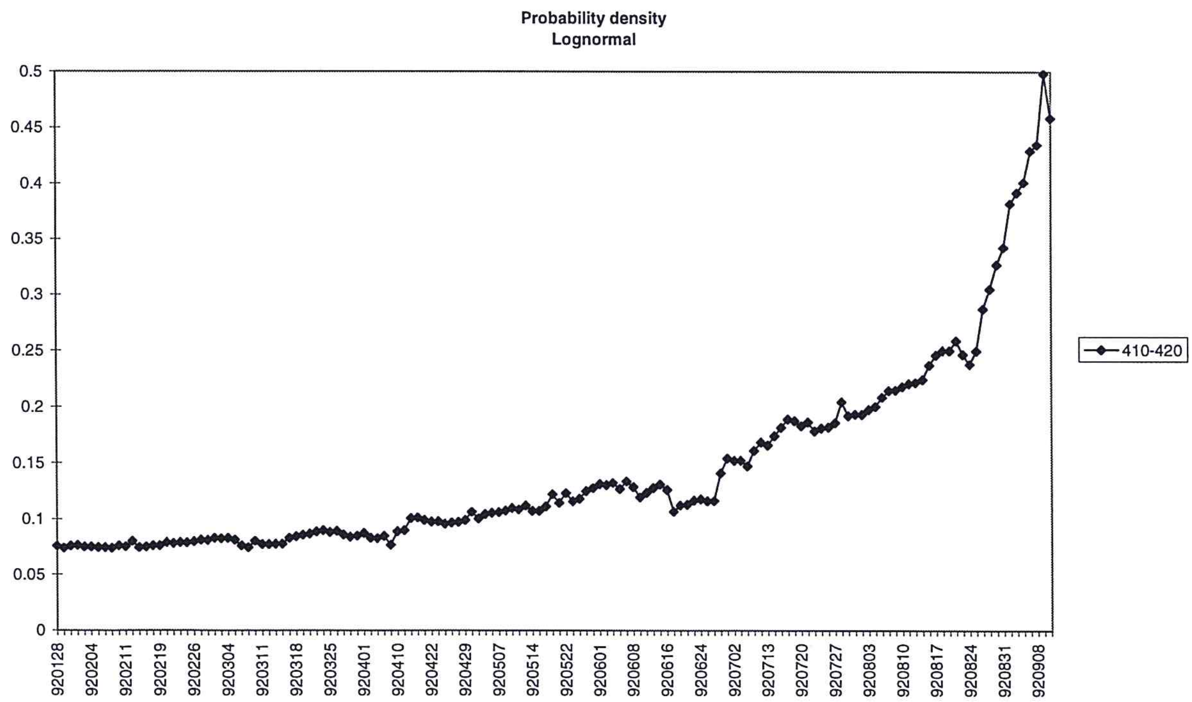


Figure 20: Probability density time series innovations for the strike interval 410-420 B-splines estimation method

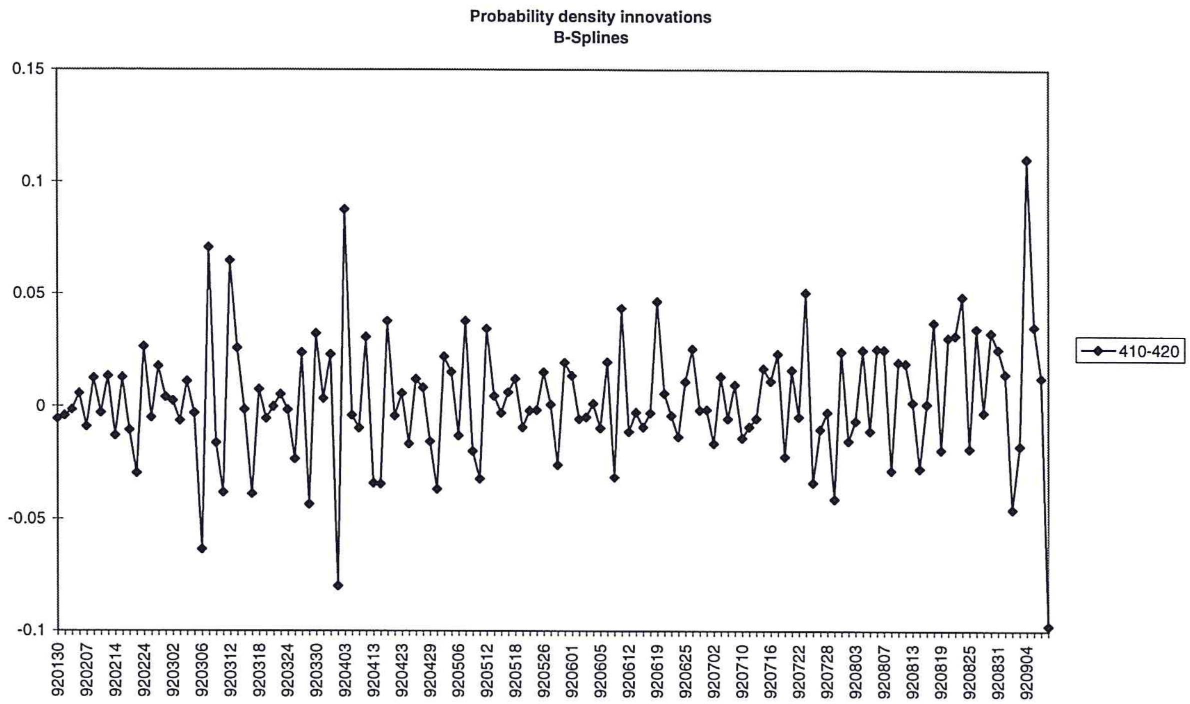


Figure 21: Probability density time series innovations for the strike interval 410-420 GB2 estimation method

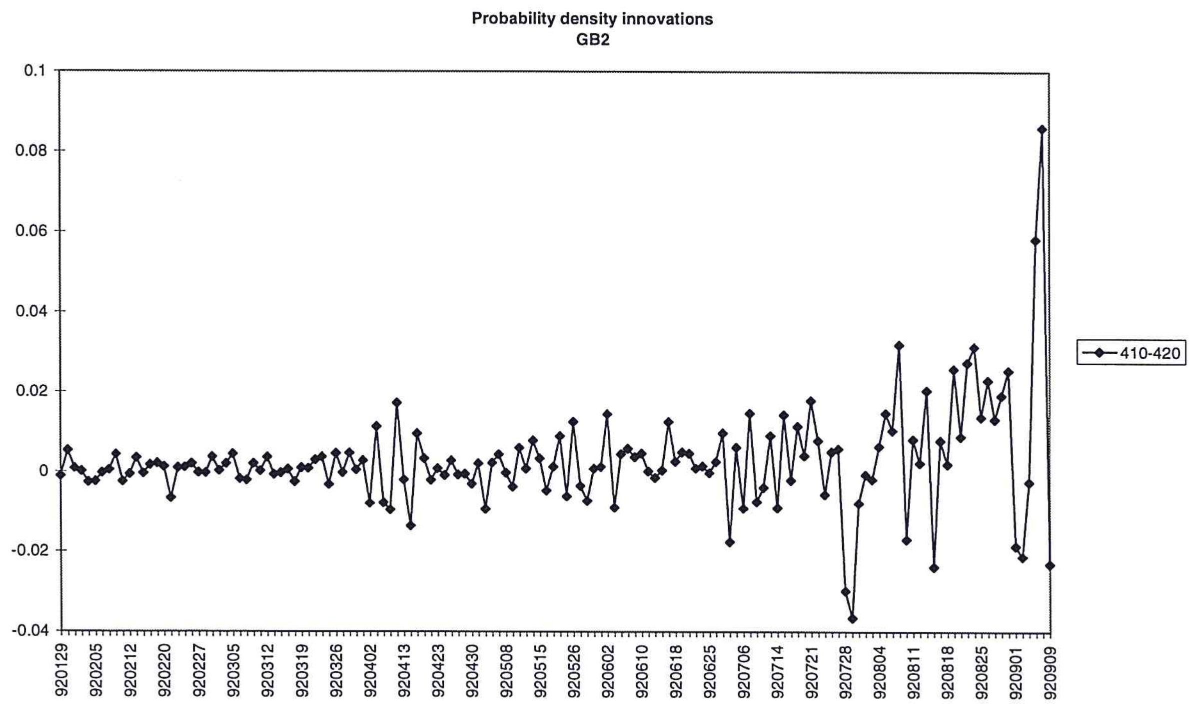
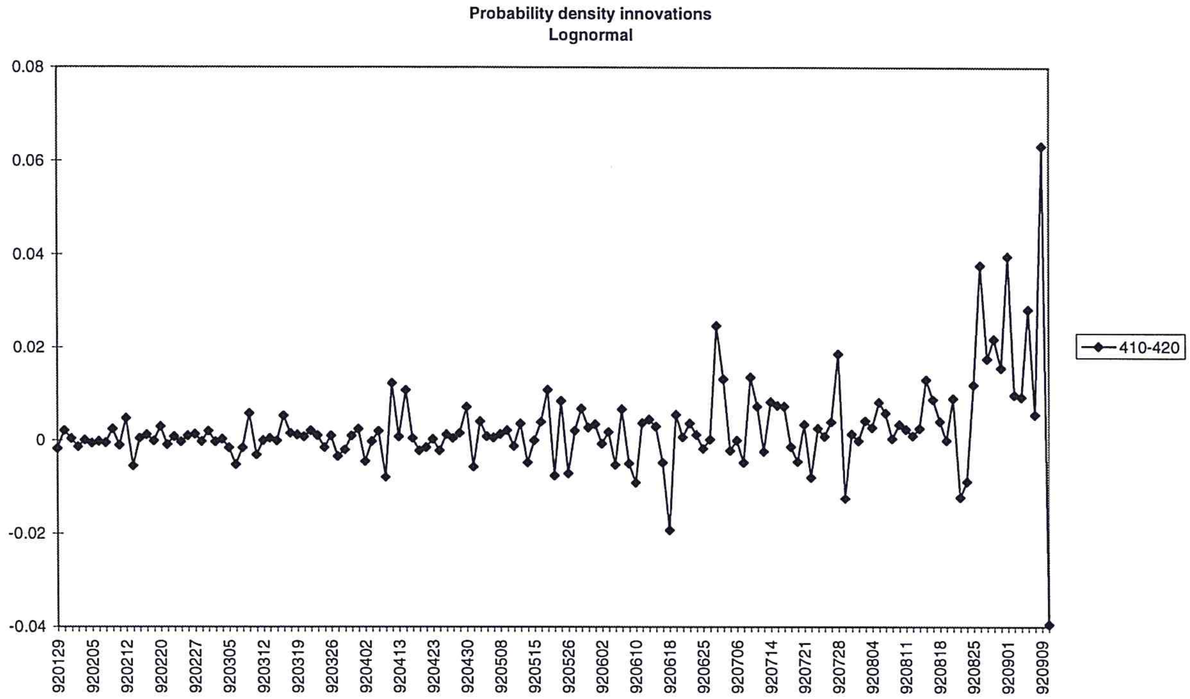


Figure 22: Probability density time series innovations for the strike interval 410-420 Lognormal estimation method



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