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PRICING OF OCCUPATION TIME DERIVATIVES:
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Pricing of occupation time derivatives: continuous and discrete monitoring*

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Abstract

In the present work we use different numerical methods (multidimensional inverse Laplace transform, numerical solution of a PDE by finite diffference scheme, Montecarlo simulation) for pricing occupation time derivatives in order to examine the effect of continuous and discrete time monitoring of the underlying asset. In particular we treat the problem of the numerical inversion of a multidimensional Laplace transform and we show that it can be performed very fast and with great accuracy. We conduct also an analysis of the numerical method for the solution of the PDE with discrete monitoring and we show that the proposed method avoids unwanted oscillations in the solution arising near the monitoring dates due to the updating of the occupation time.

1 Introduction

In the present work we examine the pricing problem for occupation time derivatives comparing the case of discrete and continuous time monitoring. The payoff of these contracts depends on the time spent by an index below a given level (hurdle or switch derivatives) or inside a band (corridor and Parisian derivatives and range notes). In particular, we examine the case of the corridor bond, bond where the coupon is proportional to the time spent inside a given band, and the corridor option that guarantees a minimum coupon. The structure of their payoff is common to FX range floaters boost and step structures as described in Hull [15], Linetsky [18], Pechtl [22], Tucker et al.[28], Turnbull[29] and Bregagnolio[5]. Chacko and Das [8] (1997) have examined the case of the Asian corridor bond.

The focus of the article is then to examine the differences between the price of the contract assuming continuous or, the more realistic, discrete time monitoring. Indeed the difference between the two prices can be relevant mainly when the residual life of the option is short or when the monitoring frequency is low (e.g. monthly). Moreover, as observed in Broadie and Glasserman [6] for barrier options, in Heynen and Kat [14] for lookback options and in several articles on *Derivative Week,* a sizable portion of real contracts specify fixed times

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for monitoring the asset and this can introduce substantial differences between discrete and continuous monitoring.

Assuming continuous time monitoring and describing the evolution of the index by a Geometric Brownian Motion, we can obtain a closed form solution for the density of the occupation time below a given level and the double Laplace transform for the density function of the time spent inside a band. We propose two numerical methods for this inversion. We remark that this is the first work in finance where the numerical inversion of a bidimensional Laplace transform is discussed. We can show that this inversion, at least in the case studied in this paper, can be performed very quickly and with a great accuracy. Moreover the numerical methods adopted are remarkably easy to understand and perform.

In the case of discrete time monitoring we could give the option price as an iterate integral (one integral for every monitoring dates), but this method, adopted for example in [14] for studying lookback options, can be very expensive. So we follow a Partial Differential Equation approach à la Black and Scholes along the lines described in Wilmott et al. [11] and [30]. In order to numerically solve the PDE, taking into account the actual performance of high-speed computers, we draw our attention toward a proper finite difference scheme that has to satisfy the following requirements: the numerical solution a) exists; b) is positive; c) converges to the exact one as the discretization steps tend to zero; d) is free of unwanted oscillations arising at the monitoring dates, due to the discontinuity introduced by the influx of new information. We believe that these are minimal properties because they respect the physical nature (i.e. financial) of the problem. For this reason, in the present work, we resort to an upwind finite-difference scheme for the first spatial derivative and we show through a numerical analysis that it guarantees all of these requirements independently from the discretization step and the parameter values, i.e. all properties are satisfied unconditionally.

In the second section we describe the main contracts. In the third section we show how to compute analytically the price for the corridor bond. In the fourth section we examine the pricing problem for the corridor option and we show the convenience of computing the density function of the occupation time assuming continuous time monitoring, whilst with discrete monitoring we prefer adopt a PDE approach. In the final section we compare the different methods. As Appendices we give the main results concerning the double Laplace transform of the density function and its inversion and the numerical analysis of the adopted scheme for the PDE. We discuss also, Appendix D, the so called digital corridor option, where at the expiry the holder of the option receives a fixed amount if the occupation time of the interval has been greater than a prefixed level.

2 The contracts

There are several forms of occupation time derivatives. The more common are the so called hurdle (or switch or range) and corridor derivatives, Hull [15], Pechtl [22], Tucker and Wei [28], Turnbull[29], Bregagnolio and Iori [5], Linetski [18]. Miura [19] and Akahori [2] introduce the quantile option. Davydov and Linetsky [10] in a recent and independent work extends the single barrier case to the double barrier step options.

Let us define x be the index level at the current time t, and $Y(x,T,t;u,l)$ the time spent by the index inside the band $[l;u]$, in the time interval $[t;T]$. With continuous time monitoring, we can write:

$$Y(x,T,t;u,l) = \int_t^T 1_{(l<X_s<u)} ds$$

where $1_{(l<X_s<u)}$ stands for the indicator function of the set $[l;u]$. If $l=0$, we are considering the time spent below the level u. Assuming discrete time monitoring, with n monitoring dates t_1,..., t_n, where $t = t_0 < t_1 < ... < t_n = T$, we have:

$$Y(x,T,t;u,l) = \sum_{i=1}^{n} 1_{(l<X_{t_i}<u)} (t_i - t_{i-1})$$

A corridor bond pays at time T the amount:

$$N * \frac{Y(x,T,t;u,l)}{T-t}$$

where N is the nominal value of the bond. The corridor option guarantees a minimum amount $N * mc$ at the expiry, so the payoff is given by:

$$N * \max\left[\frac{Y(x,T,t;u,l)}{T-t}; mc\right]$$

In the case of the hurdle derivative, we set $l=0$.

In Appendix D we discuss the evaluation of a digital corridor option that pays at time T the amount:

$$N * 1_{Y(x,T,t;u,l)>K}$$

i.e. a fixed amount N if the occupation time is greater than K.

In all previous cases, if the lower barrier goes to zero we have hurdle derivatives.

3 The corridor bond

In order to price the corridor bond, we can use the well known fact that in absence of arbitrage opportunities the price is given by the expected value under the risk neutral measure of the discounted payoff, [13]. We can easily obtain the price of the contract either with continuous time monitoring either with discrete time monitoring.

3.1 Continuous and discrete time monitoring

The price is given by the expected value of the discounted payoff. Assuming a constant risk free interest rate r,we obtain:

$$\begin{aligned} E_{t,x}\left[e^{-r(T-t)} * N * \tfrac{Y(x,T,t;u,l)}{T-t}\right] &= e^{-r(T-t)} * \tfrac{N}{T-t} E_{t,x}\left[\int_t^T 1_{(l<X_s<u)} ds\right] \\ &= e^{-r(T-t)} * \tfrac{N}{T-t} \int_t^T E_{t,x}\left(1_{(l<X_s<u)}\right) ds \\ &= e^{-r(T-t)} * \tfrac{N}{T-t} \int_t^T \Pr_{t,x}\left(l < X_s < u\right) ds \end{aligned} \quad (1)$$

where in the second line we have used the Fubini's theorem for changing expectation and integral. The same result can be found in [22].

Assuming that the dynamic of the underlying price is described by a GBM, $dX = rXdt + \sigma X dW_t$, we obtain, taking the log of the prices:

$$\Pr_{t,x}(l < X_s < u) = \Phi(h(x,u,s-t)) - \Phi(h(x,l,s-t))$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$ and $h(x,l,\tau) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{l}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau\right)$.

In the case of discrete monitoring, similarly to the previous case, setting $\Delta = t_{i+1} - t_i$, i.e. we have a constant distance between monitoring dates, we obtain:

$$E_{t,x}\left[e^{-r(T-t)} * N * \frac{Y_d(x,T,t;u,l)}{T-t}\right] =$$
$$= e^{-r(T-t)} * \frac{N}{T-t} \sum_{i=1}^{n} \left(\Phi(h(x,u,t_i-t)) - \Phi(h(x,l,t_i-t))\right)\Delta$$

Remark: Recalling the fact that the local time at a level x for a Standard Brownian Motion is the amount of time spent by the Brownian path in the vicinity of the point x, compare Karatzas and Shreve (KS henceforth) [16] pag. 203, we can obtain a representation of the occupation time in terms of integral of the local time respect to the space variable instead as integral of the indicator function respect to time. Using the representation of the semimartingale local time for a continuous semimartingale given in KS, page 218 formula (7.3), we have for every Borel measurable function $k : \Re \to [0,\infty)$:

$$\int_t^T k(X_s)\, d < X >_s = 2\int_0^{+\infty} k(\xi) L_t(\xi) d\xi$$

where $< X >_s$ denotes the quadratic variation of the price process. In the case of GBM d$< X >_s = \sigma^2 X^2 ds$ and then choosing $k(X) = 1_{(l<X<u)}/\sigma^2 X^2$, we obtain:

$$\int_t^T 1_{(l<X_s<u)} ds = 2\int_0^{+\infty} \frac{1_{(l<\xi<u)}}{\sigma^2\xi^2} L_t(\xi) d\xi = 2\int_l^u \frac{L_t(\xi)}{\sigma^2\xi^2} d\xi$$

Unfortunately this representation does not seem to be useful for obtaining the price of the corridor option, because we should know the joint density law of the local times $L_t(\xi)$, for $l < \xi < u$. Obviously, for pricing the corridor bond, we need just the expected value of $L_t(\xi)$. Some more results, but with reference to the Brownian motion with no drift, can be found in Takacs [26]. Important results have been found in Carr and Jarrow [7] in the discussion of the stop loss strategy.

4 The corridor option

In order to compute the price for the options we can follow different ways:

1. find the density function of the r.v. $Y(x,T,t;u,l)$ and compute the expected value of the discounted payoff;

2. write and numerically solve the PDE describing the price of the contingent claim;

3. run a MonteCarlo simulation for the dynamics of the underlying asset and compute for every possible path the final value of the occupation time and the payoff of the contingent claim and then average the results of different paths.

The first approach results useful assuming continuous time monitoring, whilst the second one becomes preferable in the case of discrete time monitoring. The MonteCarlo simulation could be preferred in the case of discrete monitoring, because in the simulation the underlying can be checked only at discrete, albeit small, intervals. Unfortunately, there is no closed form expression for the density function, but we are able to obtain its double Laplace transform. In next section, we will see that from this double transform we can compute the double Laplace transform of the corridor option price.

4.1 Continuous time monitoring

Assuming continuous time monitoring can be realistic when the residual life of the contract is quite long or when the monitoring frequency is high (e.g. daily). In this case, it is easier to find the density function of the occupation time, rather than to solve a PDE with two state variables (asset price and occupation time) or to run a MonteCarlo simulation. The problem with the PDE is that we need to augment the state space introducing a new variable, locally risk-free, that takes into account the time spent inside the band. This fact makes the PDE a degenerate parabolic equation (the covariance matrix is singular) and the numerical method can suffer. The MonteCarlo method has a limit in the intrinsic discreteness of the simulation so we do not know if the process has crossed or not the barriers and we cannot compute the time spent inside the barriers during each step of the simulation[1]. For this reason it should be preferable to use the MonteCarlo method in presence of discrete monitoring.

We observe that to find the time spent inside the band $[l;u]$ by the GBM X_t, $X_0 \equiv x$, is equivalent to find the time spent inside the band $[L \equiv \ln(l)/\sigma; U \equiv \ln(u)/\sigma]$ by the Arithmetic Brownian Motion (ABM), starting at z=$\ln(x)/\sigma$, with drift $m = \left(r - \sigma^2/2\right)/\sigma$ and unitary diffusion coefficient. In the following, we set $\tau = T - t$.

Using the Feynman-Kac formula, [16] chapter 4.4, Fusai [12] has obtained an expression for the Laplace transform with respect τ of a function appearing in the expression of the moments generating function $v(z,\tau)$ of the r.v. Y:

$$\begin{aligned} v(z,\tau) &= \mathbf{E}_{t,z}\left[e^{-\mu Y(z,t+\tau,t;U,L)}\right] \\ &= \int_t^{t+\tau} e^{-\mu y}\,\mathrm{Pr}_{t,z}\left(Y(z,t+\tau,t;U,L) \in dy\right) + 1 \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = 0\right] + \\ &+ e^{-\mu(T-t)} \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = T-t\right] \end{aligned} \tag{2}$$

where we have token into account that the time spent inside the band can be equal to $T-t$ or equal to 0. In Theorem 1 in Appendix A we give the analytical expression for the second

[1]For a related problem of pricing double barrier options, Baldi et al. [4] illustrate how to improve the Montecarlo method providing approximations for the exit probability from an interval. Other important suggestions can be found in Andersen and Brotherton-Ratcliffe [3]. We do not have investigated, if the suggested methods can be used in the present problem.

and third terms in the expression above[2] and the Laplace transform of the first term:

$$\omega(\gamma,\mu,z) \equiv \omega(\gamma,\mu,z;L,U,m) = \mathcal{L}[\Omega(z,\tau);\tau\to\gamma] = \int_0^{+\infty} e^{-\gamma\tau}\Omega(z,\tau)\,d\tau$$

where:

$$\Omega(z,\tau) \equiv \Omega(z,\tau;L,U,m) = \int_t^{t+\tau} e^{-\mu y}\,\Pr_{t,z}(Y(z,t+\tau,t;U,L)\in dy)$$

We can use then the expression for $\omega(\gamma,\mu,z)$ for obtaining the double Laplace transform of the corridor option price. Indeed, we observe that we can write the undiscounted price of the corridor option with strike K and residual live τ as:

$$\begin{aligned}&C(\tau,K) \equiv C(\tau,K;x,t,l,u) =\\ &\textstyle\int_K^\tau (y-K)\Pr_{t,x}[Y(t+\tau,x;u,l)\in dy] + (\tau-K)^+ \times \Pr_{t,x\in(l,u)}[Y(t+\tau,x;u,l)=\tau]\end{aligned}$$

and with some algebra, we obtain:

$$\begin{aligned}&C(\tau,K) =\\ &= E_{t,x}[Y(t+\tau,x;u,l)] - K(1-\Pr_{t,x}[Y(t+\tau,x;u,l)=0])\\ &+\textstyle\int_0^K (K-y)\Pr_{t,z}(Y(t+\tau,z;U,L)\in dy)\end{aligned}$$

where $E_{t,x}[Y(t+\tau,x;u,l)]$, the expected value of the r.v. $Y(t+\tau,x;u,l)$, can be obtained by (1).

If we consider now the Laplace transform with respect to K of the third term and we exploit the convolution property of the Laplace transform, we obtain:

$$\begin{aligned}&\mathcal{L}\left[\textstyle\int_0^K (K-y)\Pr_{t,z}(Y(t+\tau,z;U,L)\in dy);K\to\mu\right]\\ &= \mathcal{L}[K;K\to\mu]\,\mathcal{L}[\Pr_{t,z}(Y(t+\tau,z;U,L)\in dK);K\to\mu]\\ &= \tfrac{1}{\mu^2}\Omega(t,\mu,x;l,u,m)\end{aligned}$$

and then $C(\tau,K)$ is given by:

$$\begin{aligned}&C(\tau,K) =\\ &= E_{t,x}[Y(t+\tau,x;u,l)] - K(1-\Pr_{t,x}[Y(t+\tau,x;u,l)=0]) + \mathcal{L}^{-1}\left[\tfrac{\Omega(t,\mu,x;l,u,m)}{\mu^2};\mu\to K\right]\\ &= E_{t,x}[Y(t+\tau,x;u,l)] - K(1-\Pr_{t,x}[Y(t+\tau,x;u,l)=0]) + \mathcal{L}^{-1}\left[\tfrac{\omega(\gamma,\mu,x;l,u,m)}{\mu^2};\mu\to K,\gamma\to t\right]\end{aligned}$$

So the pricing of the corridor option requires the inversion of the double Laplace transform[3] of the quantity $\omega(\gamma,\mu,x;l,u,m)/\mu^2$. We can remark that in order to calculate the Greeks of the contract we can simply calculate the derivatives of the Laplace transform of the double Laplace transform and invert them.

[2] In the case of only one barrier, $L=-\infty$ $(l=0)$, the density function admits a closed form expression very easy to compute, compare Takacs [27], and Fusai [12] that simplify the results in Akahori [2].

[3] For numerical purposes, it is convenient to divide in (8) and (9) the numerator and the denominator by $\sinh(a\pi)$ and to use the fact that $\tanh(a\pi/2) = (\cosh(a\pi)-1)/\sinh(a\pi) = \sinh(a\pi)/(\cosh(a\pi)+1)$

As suggested in Abate and Whitt [1], for numerically computing the inverse of the above quantity, we have considered two different methods[4] in order to have a cross check on the results. The two methods are: a) the Fourier-series method firstly introduced for multidimensional transform inversion by Choudhury et al. [9], and b) the Padé approximation as suggested in Singhal et al. [23].

We describe the two inversion methods in Appendix, whilst Tables 1-5 compare the inversion for different strikes, index, volatility and time to maturity. We can see that the two methods give results quite similar, sometimes up to the seventh digit. Using the Padé inversion technique, the inversion is performed very quickly, requiring less than one second. As described in Appendix, the Fourier-series method, although a little slower, has the advantage of permitting a control of the different type of errors (aliasing, truncation and roundoff) that can arise in the numerical inversion. In every case, both inversion techniques appear very accurate.

[INSERT TABLES 1-5]

We remark that we believe that this is the first paper in finance to use the numerical inversion of a multidimensional Laplace transform[5]. Moreover with success. The very good performance and the easy implementation of the proposed methods[6] contradicts the common idea that to perform the numerical inversion of a Laplace transform is difficult to implement and represents "a well known ill-conditioned problem"[7]. As well documented in Abate and Whitt [1] this common misconcept arises from extrapolating the problems of some numerical inversion to all available inversion techniques.

Finally, the following table compares the prices of the corridor option, assuming continuous monitoring, obtained by the Laplace inversion and MC simulation.

index level	90	100	105	110	120
Padé	0.04630	0.12472	0.14692	0.11613	0.04573
MC	0.04635	0.12474	0.14751	0.11642	0.04607
std. error*1000	0.427	0.670	0.711	0.660	0.435
MC+AV	0.04661	0.12495	0.14728	0.11583	0.04607
std. error*1000	0.265	0.518	0.665	0.358	0.271

Table 6: *Price for the corridor option with strike price K=0.2, and parameters as in table 1. The inverse Laplace transform is computed using the Padé inversion technique with* $n = 4$ and $m = 18$. *The MC estimate is obtained by 50000 simulations. MC+AV stands for the MonteCarlo simulation performed with antithetic variate. In brackets we have 1000xstandard error.*

[4] We have also tried a third method presented in Moorthy [20] that makes use of the Fourier series representation, but the inversion was quite sensible to the choice of the parameters and so we do not report the results here.

[5] At the moment this paper was completed and submitted, we learned that the same multidimensional inversion algorithm proposed by CLW has been used by Davydov and Linetski [10].

[6] All calculations have been done on a Compaq Presario Notebook P233MMX. The code for the numerical inversion has been written in C using Microsoft Visual C++ 5.0. In the Padé inversion the poles and the residues have been previously computed using Mathematica 3.0.

[7] This statement is reported in [1], pag. 7.

The Table shows that the MonteCarlo simulation, without AV, is usually accurate to the third or fourth digit. Moreover the use of the antithetic variate does not seem to improve substantially the estimate[8]. Obviously the average computing time has been greatest for the MC simulation[9].

4.2 Discrete time monitoring

In the case of discrete time monitoring, we could try to use the pricing formula for the continuous time case in order to approximate the solution in the discrete time case, but this approximation could work well only if the monitoring frequency is high or if the option has a long time to expiry. So we prefer to discuss alternative approaches: a PDE method and a MonteCarlo simulation. Another possibility is to express the option price as a multiple iterate integral as done in Heynen and Kat [14] for lookback options: one integral for each monitoring date, so that with monthly monitoring, we should iterate over twelve integrals and the computational effort and the time required can be quite high.

4.2.1 The PDE approach

Along the lines in Wilmott et al. [30] and [11], we can show that the price $V(x,\tau,y)$ of the corridor option satisfies the following system of PDE's:

$$-\frac{\partial V(x,\tau,y)}{\partial \tau}+rx\frac{\partial V(x,\tau,y)}{\partial x}+\frac{\sigma^2}{2}x^2\frac{\partial^2 V(x,\tau,y)}{\partial x^2}=rV(x,\tau,y) \tag{3}$$

where $\tau \geq 0, 0<x<+\infty$ $0 \leq y \leq n$, where we have redefined the time t in time to expiry of the option, $\tau = T-t$ [10] and n gives the number of monitoring dates, $n=[\tau/\Delta]$, with Δ the time distance between two consecutive monitoring dates. The initial condition to be satisfied is[11]:

$$V(x,0,y)=\max(y-K;0)*\Delta \tag{4}$$

In the PDE's above y is an integer number representing the number of times the index has spent inside the band at the monitoring dates. We treat (3) as a system of PDE's indexed on y: between monitoring dates y is fixed, and, for each fixed y, $V(x,\tau,y)$ evolves according to (3). So we could solve different PDE simultaneously, one for each possible value of y,

[8] We encountered a similar problem when we have used as control variate the price of the corridor bond. The standard error in this case increased.

[9] It took around 42' to run the 50000 simulations with five different starting points for the index, and considering simultaneously different monitoring frequencies (step by step, daily, weekly and monthly). We used 1200 steps for year.

[10] As a consequence, if the times t_1, ..., t_n are the monitoring dates, now they correspond to $\tau_i = T-t_i$. So $\tau_0 = 0$, and when the time to expiry of the option is $\tau_{i-1} \leq \tau < \tau_i$ it means that we have again i monitoring dates and the residual life of the contract is $i*\Delta$.

[11] The initial condition depends on the fact that we can write:

$$N*\max\left[\frac{Y}{(T-t)};mc\right]*\Delta=\frac{N}{T-t}*(\max[Y-K;0]+K)*\Delta$$

where $K=(T-t)\,mc$. So the payoff can be viewed as $N/(T-t)$ call options on Y plus the same quantity of bonds of nominal value K. In the following we will concentrate the attention on the quantity $\max[Y-K;0]*\Delta$.

until the next monitoring date. An arbitrage argument, Wilmott et al. [30], requires that the option price has to be continuous across the monitoring dates i.e.:

$$V\left(x, \tau_j^+, y^+\right) = V\left(x, \tau_j^-, y^-\right)$$

or if $y^+ = y$:

$$V\left(x, \tau_j^+, y\right) = V\left(x, \tau_j^-, y + 1_{(l<x<u)}\right) \tag{5}$$

The "jump condition" (5) links at the monitoring dates PDE's for different values of y: depending on the position of the index respect to the band, there will be an exchange of information between the different PDE's for adjacent values of y, as illustrated in Figure 1.

[INSERT FIGURES 1 and 2]

Figure 2 illustrates the updating process that occurs after one third of year and after four monitoring dates, assuming $K = 2.4$. The PDE indexed with $y = 0$ at the monitoring date receives new information from the PDE indexed with $y = 1$ if the index is inside the barriers, otherwise updates itself. Similarly, the PDE indexed with $y = 1$ ($y = 2$) receives new information from the PDE indexed with $y = 1$ ($y = 3$) if the index is inside the barriers, otherwise updates itself. The PDE with $y = 3$ does not need an updating because at the expiry the option will be surely exercised ($y > K$), so we have an analytical expression for it (compare eq. 8 below). The updated values for each PDE can be then used to start to solve again separately the different PDE's between monitoring dates.

Regarding the boundary conditions, we observe that for a GBM process, if $x = 0$ the price will be forever equal to 0, and then the stock cannot spend any more time inside the interval and the final occupation time will be equal to y. So we know with certainty the final payoff and we obtain the condition:

$$V(0, \tau, y) = e^{-r\tau} \max(y - K; 0) * \Delta \tag{6}$$

Similarly, for the case $x \to +\infty$, we will get:

$$V(+\infty, \tau, y) = e^{-r\tau} \max(y - K; 0) * \Delta \tag{7}$$

In conclusion, we have to solve the PDE (3), with initial condition (4) and boundary conditions (6) and (7) and continuity condition (5) at times τ_i. The computational domain is $\tau > 0, 0 < x < +\infty$.

Let us observe that when $y \geq [K] + 1 \geq K$, where $[K]$ is the greatest integer strictly smaller than K, and when the time to maturity is τ the option will be surely exercised, it is like a corridor bond, and we have an analytical solution given by:

$$V(x, \tau, y) = e^{-r\tau}\left((y - K)\Delta + \sum_{i=1}^{n}\left(\Phi(h(x, u, i * \Delta)) - \Phi(h(x, l, i * \Delta))\right)\Delta\right) \tag{8}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$ and $h(x, l, \tau) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln(l/x) - \left(r - \frac{\sigma^2}{2}\right)\tau\right)$. This remark allows us to solve only a restricted number of PDE's instead of having to solve a PDE for each possible value of y. For example with monthly monitoring and with a strike equal to 2.4, we need to solve just three equations (corresponding to $y = 0, 1$and 2) instead of thirteen and the computational time is not prohibitive[12].

At the monitoring dates τ_i, taking into account the known solution for $y > K$, the updating condition can be written:

$$\begin{aligned} V\left(x, \tau_j^+, y\right) &= \begin{cases} V\left(x, \tau_j^-, y\right) & x \notin [l, u] \\ V\left(x, \tau_j^-, y+1\right) & x \in [l, u] \end{cases} \\ \text{when } y &= 0, 1, ..., [K]-1,\ j = 1, ..., n \end{aligned}$$

and:

$$V\left(x, \tau_j^+, [K]\right) = \begin{cases} V\left(x, \tau_j^-, [K]\right) & x \notin [l, u] \\ V\left(x, \tau_j^-, [K]+1\right) & x \in [l, u] \end{cases}$$

with $j = 1, ..., n$.

When the time to maturity is τ, i.e. there are again n monitoring times, the expression $V\left(x, \tau_j^-, [K]+1\right)$ is given by (8). When we have again j monitoring dates, $j = n-1, n-2$, ..., 2, 1, the analytical solution is always given by (8), once we have set $\tau = \tau_j, y = [K]+1$, $n = j$. When $\tau_0 = 0$ the option is expired and we apply the initial condition $\max(y-k; 0) * \Delta$.

We remark that the change of information between PDE's implies that, for the nature of the updating of the occupation time, we are introducing a "discontinuity" in the new initial condition to be used at every monitoring date as well illustrated in figure (2). This fact could generate unwanted spurious oscillations in the numerical solution if a scheme is adopted without paying attention to the nature of the problem. For example a Crank-Nicholson scheme is usually the preferred one, just because it is reputed unconditionally stable and its local precision is maximal for PDE's of the parabolic type. But if adopted in the present context, this natural choice forgets the nature of the problem. Indeed as shown in Smith [24], pag. 122-124, "numerical studies indicate that very slowly decaying finite oscillations can occur with the Crank-Nicholson method in the neighborhood of discontinuities in the initial values...". So for our problem it is necessary to devote some attention in choosing the numerical scheme because to adopt a Crank-Nicholson scheme without a preliminary analysis could be very dangerous[13]. In effect doing a numerical analysis similar to that one in Appendix C, it is easily proved that the absence of spurious obscillations in the Crank-

[12] To solve numerically the PDE with monthly monitoring took around 1'05" for 1000 different index levels and 300 discretization steps respect to τ and K=12*0.2=2.4. If we increase the strike price, we need solve more PDE: e.g. if K=12*0.4=4.8, the time required becomes 1'50". If we increase the monitoring frequency, also the computational time increases: e.g. with daily monitoring, K=0.2 and 150 discretization steps respect to τ, it took more than 2h30'. But in this case it is more convenient to approximate the solution using the continuous time formula. The code for the numerical solution of the PDE has been written in Fortran.

[13] We know of only a paper by Zvan and al. [31] documenting the limits of a Crank-Nicholson scheme when applied to the pricing of barrier options.

Nicholson scheme requires the following relationship between spatial and time step:

$$\frac{\Delta\tau}{(\Delta x)^2} \leq \frac{1}{\sigma^2}$$

and depending on the values of the volatility, this restriction can become very demanding. For example if we choose Δx =0.01 and $\sigma = 0.2$, we have that $\Delta\tau \leq$0.0025 (i.e. 400 time steps between two monitoring dates), whilst if Δx =0.001 then we need to require $\Delta\tau \leq$0.000025 (i.e. 40000 time steps) and higher accuracy can be obtained only with a very small time step.

For this reason, we would like to use a proper numerical method that guarantees the respect of some minimal conditions very natural from the financial point of view: a) the solution exists, b) it is positive, c) it converges to the exact one as the discretization steps tend to zero, d) it is free of unwanted oscillations arising at the prefixed updating dates, due to exchange of information between PDE's. In particular in Appendix C we describe a proper numerical scheme that can be used in the present case. Through a detailed numerical analysis, we can show that all above requirements are satisfied unconditionally and in particular the oscillations due to discontinuities in the updating process are eliminated very quickly and do not perturb the numerical solution. The cost to be paid is a lower accuracy than in the Crank-Nicholson: for the proposed scheme the order of convergence is $o(\Delta\tau, \Delta x)$.

4.2.2 The MonteCarlo simulation

The advantage of using MonteCarlo simulation is its practicality, in the sense that is easy to understand and applicable without effort to different problems and assuming more general processes for the underlying. Moreover in the case of discrete monitoring the bias due to the underestimation of the maximum and overestimation of the minimum does not appear, because the simulation can be devised to match the actual observation dates. But as remarked in Andersen and Brotherton-Ratcliffe [3] simulation could become prohibitively lengthy if there are too many monitoring dates, although this problem is common to the PDE approach as well. Moreover, when the dimensionality of the problem (number of state variables) is low as in the present case, a finite-difference method can be preferable because more accurate. But the real advantage of using a PDE approach is that we can solve simultaneously for different initial index levels (1000 points in the numerical examples) and we can calculate quite easily and with great accuracy the Greeks of the contract. In particular the numerical solution can be of support in calculating an hedging strategy that in the case of exotic options appear very important, whilst in the case of the MonteCarlo simulation this is not possible. For example in the present case, particular care has to be devoted to the calculation of the gamma near the barriers, because it presents a jump.

Table 7 compares the results from the MonteCarlo simulation (with and without control variate) and the numerical solution of the PDE, assuming monthly monitoring. The MC estimate has a higher standard error in this case than it did for continuous monitoring (Table 6), even though the latter could not match the observation dates. Moreover also in this case, the use of the Antithetic Variate reduces the standard error, but it appears to give a more biased estimate respect to the numerical solution of the PDE. Also the use of the corridor bond as Control Variate was not useful in reducing the standard error and we do not report the relative results.

index level	PDE (monthly)	MC	MC+AV
90	0.05235	0.05328;(0.482)	0.05355;(0.298)
100	0.12120	0.12161;(0.697)	0.12207;(0.541)
105	0.13281	0.13404;(0.726)	0.13351;(0.659)
110	0.11328	0.11387;(0.687)	0.11316;(0.387)
120	0.05160	0.05218;(0.487)	0.05234;(0.304)

Table 7: *Price for the corridor option with strike price K=0.2, and parameters as in table 6. In the PDE we have used 1000 discretization steps w.r.to x and 300 respect to τ. The MC estimate is obtained by 50000 simulations.*

5 A comparison between discrete and continuous monitoring

In this section we compare the results coming from the inversion of the Laplace transform, with the numerical solution of the PDE.

From table 8 and figure 3 we can appreciate the difference between the price with continuous (inverse Laplace) and discrete time monitoring (PDE). If we consider the index level varying in the interval 85-125, we can see that the percentage difference between continuous and monthly monitoring can go from +10% to -11%, depending on the position of the index respect to the barriers. This difference is reduced a lot if we compare daily and continuous monitoring.

In every case the price of the contract with continuous time monitoring is the highest (lowest) when the index is inside (outside) the band[14]. This fact is due to the nature of the contract: if the index is inside the band, and we assume continuous time monitoring, then the passage of the time will increase the value of the contract until the moment in which the index crosses the barriers. Instead if we assume discrete time monitoring, we cannot exploit completely the passage of time: we register the position of the index only at discrete dates and if they are quite distant (e.g. a month), it is possible that the index in the meantime has moved outside the band and so the occupation time cannot increase. In this case the time between two monitoring dates can be entirely lost if the index moves outside the band. Viceversa if we are outside the band, and between two monitoring dates the index moves inside the band the occupation time increases by the distance Δ : the contract earns the entire time distance between monitoring dates. Instead with continuous time monitoring we lose every instant until the process crosses the barriers. So the continuous time formula will overvalue (undervalue) the discrete time formula when the index is inside (outside) the band. Then it will be important to distinguish between discrete and continuous monitoring time.

In figure 4 and table 9 we illustrate the effect of the monitoring frequency on the delta of the contract: in this case we can see that higher the monitoring frequency, higher the absolute value of the delta. So the monitoring frequency assumes importance for replicating the contract, mainly when the index level is near the barriers.

[14]This result depends also on the value of the drift of the process, so that the overvaluation (undervaluation) does not occur exactly at the extremes of the band.

[INSERT FIGURES 3-4]

We also investigated if it is useful to shift the barriers in the continuous time formula in order to get an accurate price for the case of discrete monitoring. This is for example the idea in Broadie and al. [6]: they examine the pricing problem for several kinds of barrier and lookback options and they show that shifting in an opportune way the barrier in the continuous formula they can get an accurate price for the case of discrete time monitoring. Basically the idea we pursued was to find the lower barrier that equates the price of the corridor bond with discrete and continuous time monitoring and then use it as input in the double Laplace transform for obtaining an approximate price for the discrete corridor option. The percentage difference between the numerical solution of the PDE and the price obtained by this procedure was reduced for example from 11% to around the 3% when the index level was around 90, but in general this procedure was not completely satisfactory. So more work has to be done in this case.

6 Conclusion

In the present paper we have examined the effect of the monitoring frequency on the price and the delta of corridor derivatives, using either a numerical solution of a PDE either the double inverse Laplace transform. We have shown that the two methods can be fruitfully coupled. Indeed as the monitoring frequency increases, it can be too expensive to solve the PDE and so it becomes more convenient to approximate the solution using the inverse Laplace transform: the difference between the prices and deltas with daily and continuous monitoring is very small. Viceversa for low monitoring frequency we can use the numerical solution of the PDE. In both cases, as check test we can always use MonteCarlo simulation, but it is much more expensive relatively to the time required and the standard variance reduction techniques do not seem work very well. It remains to be investigated if we can approximate the discrete time formula using the much more efficient continuous time formula using a shift argument similar to that used in Broadie et al.[6].

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A The double Laplace transform of the density law

If we define the moment generating function (mgf) $v(z,\tau)$ of the r.v. Y

$$\begin{aligned} v(z,\tau) &\equiv v(z,\tau;\mu,m,U,L) = \mathbf{E}_{t,z}\left[e^{-\mu Y(z,t+\tau,t;U,L)}\right] \\ &= \int_t^{t+\tau} e^{-\mu y}\Pr_{t,z}\left(Y(z,t+\tau,t;U,L)\in dy\right) + 1\times\Pr_{t,z}\left[Y(z,T,t;U,L)=0\right] + \\ &+ e^{-\mu(T-t)}\times\Pr_{t,z}\left[Y(z,T,t;U,L)=T-t\right] \end{aligned} \tag{9}$$

it can be shown that, using the Feynman-Kac formula (Karatzas and Shreve [16] chapter 4.4), the function $v(z,\tau)$ satisfies the following PDE:

$$-\frac{\partial v(t,z)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t,z)}{\partial^2 z} + m\frac{\partial v(t,z)}{\partial z} - \mu\mathbf{1}_{(L<z<U)}v(t,z) = 0 \tag{10}$$

with initial condition:

$$v(z,0) = 1, \forall z\in(-\infty;+\infty) \tag{11}$$

and boundary conditions:

$$v(\pm\infty,\tau) = 1,\ \forall\tau > 0 \tag{12}$$

The above PDE can be solved taking the Laplace transform with respect to time and solving the second differential equation requiring continuity and differentiability of the solution at the points L and U and boundedness of the solution at $\pm\infty$. This requires to solve a linear system with four equations and four unknowns. A more efficient way for obtaininig the solution is presented in Fusai [12].

In Theorem 1 below, given a function $G(\tau)$, we denote with $g(\gamma)$ its Laplace transform with respect to the variable $\tau = T - t$:

$$g(\gamma) \equiv \mathcal{L}[G(\tau);\tau \to \gamma] = \int_0^{\tau} e^{-\gamma\tau} G(\tau)\, d\tau$$

and with $\mathcal{L}^{-1}[g(\gamma);\gamma \to \tau]$ its inverse Laplace transform. We have:

Theorem 1 *:The Laplace transform (moment generating function) of the density law of the occupation time of the interval $[L;U]$ by the ABM has the following representation:*

$$v(z,\tau) = \Omega(z,\tau;L,U,m) + \\ + \begin{cases} 1 \times \Pr_{0,z\in[U,+\infty)}\left(\inf\limits_{0\le s\le\tau} m*s + W(s) > U\right) \\ \\ e^{-\mu\tau} \times \Pr_{0,z\in(L,U)}\left(\sup\limits_{0\le s\le\tau} m*s + W(s) < U;\ \inf\limits_{0\le s\le t} m*s + W(s) > L\right) \\ \\ 1 \times \Pr_{0,z\in(-\infty,L]}\left(\sup\limits_{0\le s\le\tau} m*s + W(s) < L\right) \end{cases} \tag{13}$$

where:

$$\begin{aligned} &\Pr_{0,z\in[U,+\infty)}\left(\inf_{0\le s\le\tau} m*s + W(s) > U\right) = \\ &= 1 - \left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{z-U+m\tau}{\sqrt{2\tau}}\right) + \tfrac{e^{-2m(z-U)}}{2}\operatorname{Erfc}\left(\tfrac{z-U-m\tau}{\sqrt{2\tau}}\right)\right]; \end{aligned}$$

$$\begin{aligned} &\Pr_{0,z\in(L,U)}\left(\sup_{0\le s\le\tau} m*s + W(s) < U;\ \inf_{0\le s\le t} m*s + W(s) > L\right) = \\ &= 2e^{-mz-\frac{m^2}{2}\tau}\int_0^1\left[\sum_{n=1}^{\infty} e^{-(cn\pi)^2\tau}\sin(n\pi z)\sin(n\pi\xi)\right] e^{m(\xi(U-L)+L)}d\xi; \end{aligned}$$

$$\begin{aligned} &\Pr_{0,z\in(-\infty,L]}\left(\sup_{0\le s\le\tau} m*s + W(s) < L\right) = \\ &= 1 - \left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{L-z-m\tau}{\sqrt{2\tau}}\right) + \tfrac{e^{2m(L-z)}}{2}\operatorname{Erfc}\left(\tfrac{L-z+m\tau}{\sqrt{2\tau}}\right)\right] \end{aligned}$$

and:

$$\begin{aligned} \Omega(z,\tau;L,U,m) &\equiv \textstyle\int_0^{\tau} e^{-\mu y}\Pr_{t,z}\left(y(z,\tau,0;U,L) \in dy\right) \\ &= e^{-mz-\frac{m^2}{2}\tau}\mathcal{L}^{-1}\left[\varpi(\gamma,z;L,U,m);\tau\right] \end{aligned}$$

$$\varpi\left(\gamma,z;L,U,m\right)=$$

$$=\begin{cases}\mathbf{1}_{(z\geq U)}e^{-\sqrt{2}(z-U)\sqrt{\gamma}}\mathcal{L}\left[A\left(\tau\right);\gamma\right]\\ \\ \mathbf{1}_{(L<z<U)}\frac{\mathcal{L}[A(\tau);\gamma]\sinh\left(a\pi\left(\frac{U-z}{U-L}\right)\right)+\mathcal{L}[B(\tau);\gamma]\sinh\left(a\pi\left(\frac{z-L}{U-L}\right)\right)}{\sinh(a\pi)}\\ \\ \mathbf{1}_{(z\leq L)}e^{-\sqrt{2}(L-z)\sqrt{\gamma}}\mathcal{L}\left[B\left(\tau\right);\gamma\right]\end{cases}$$

$$\mathcal{L}\left[A\left(\tau\right);\tau\rightarrow\gamma\right]=\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{c}{2\sqrt{\gamma}}\left(s\left(\gamma\right)+d\left(\gamma\right)\right)$$

$$\mathcal{L}\left[B\left(\tau\right);\tau\rightarrow\gamma\right]=\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}+\frac{c}{2\sqrt{\gamma}}\left(s\left(\gamma\right)-d\left(\gamma\right)\right)$$

with:

$$\begin{aligned}\frac{c}{\sqrt{\gamma}}d\left(\gamma\right)=&\frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\left(\sqrt{\gamma+\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}+\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}+\right.\\&\left.+\frac{\left(\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha L}-e^{-\alpha U}\right)(\cosh(a\pi)+1)-\sqrt{\gamma+\mu}\left(e^{-\alpha U}+e^{-\alpha L}\right)\sinh(a\pi)\right)}{\sqrt{\gamma+\mu}\sinh(a\pi)\left(\gamma+\mu-\frac{\alpha^2}{2}\right)}\right)\\ \frac{c}{\sqrt{\gamma}}s\left(\gamma\right)=&\frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\sqrt{\gamma+\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)}\left(\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}+\right.\\&\left.-\frac{\left(\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha U}+e^{-\alpha L}\right)(\cosh(a\pi)-1)+\sqrt{\gamma+\mu}\left(e^{-\alpha U}-e^{-\alpha L}\right)\sinh(a\pi)\right)}{\sqrt{\gamma+\mu}\sinh(a\pi)\left(\gamma+\mu-\frac{\alpha^2}{2}\right)}\right)\end{aligned}\tag{14}$$

$$a\pi=\sqrt{\frac{\gamma+\mu}{c^2}};\quad\alpha=-m;\quad\beta=-\frac{m^2}{2};\quad c^2=\frac{1}{2(U-L)^2}\ \blacklozenge\tag{15}$$

The above result can be obtained

B The numerical inversion

In this section, we describe the idea underlying the two algorithms adopted for the numerical inversion of the double Laplace transform. As remarked in Choudhury et al. [9] little attention has been given to inversion of multidimensional transforms and moreover nothing at all regard their application in finance.

B.1 The Fourier series method

The inversion formula proposed in the Choudhury et al. (henceforth CLW) [9] is a multidimensional version of the algorithm in Abate and Whitt [1], with an enhancement in order to control simultaneously the aliasing and the round-off errors. The authors damp the given function multiplying by a two dimensional decaying exponential function and then approximate the damped function by a periodic function constructed by aliasing. The two exponential parameters allow to control the aliasing error in the approximation by the periodic function. The inversion formula is then the two dimensional Fourier series of the periodic function. Moreover expressing the two dimensional series as an alternating series nested within a second alternating series CLW can apply the Euler transformation to compute the infinite series from finitely many terms. If $\tilde{f}(s_1, s_2)$ is the double Laplace transform of the function $f(t_1, t_2)$, their inversion formula is given by:

$$\bar{f}(t_1,t_2) = \frac{\exp\left(\frac{A_1}{2l_1}+\frac{A_2}{2l_2}\right)}{4t_1t_2l_1l_2} \times$$
$$\times \left\{ \tilde{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2}\right) + 2\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k \operatorname{Re}\left[e^{-\frac{ik_1\pi}{l_2}}\tilde{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{t_2l_2} - \frac{ik\pi}{t_2}\right)\right] + \right.$$
$$+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\left[\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}+\frac{ik_1\pi}{l_2}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij_1\pi}{t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{t_2l_2} - \frac{ik\pi}{t_2}\right)\right] +$$
$$+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\left[e^{-\left(\frac{ij_1\pi}{l_1}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij_1\pi}{t_1}, \frac{A_2}{2l_2t_2}\right) + \right.$$
$$\left.\left. + \sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}-\frac{ik_1\pi}{l_2}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij_1\pi}{t_1}, \frac{A_2}{2l_2t_2} + \frac{ik_1\pi}{t_2l_2} + \frac{ik\pi}{t_2}\right)\right]\right\}$$

and

$$f(t_1,t_2) = \bar{f}(t_1,t_2) + \bar{e}$$

CLW are able to show that, if the $|f(t_1,t_2)| \leq C$ for some constant C and for all $t_1$and t_2, then the aliasing error $\bar{e}$ can be bounded as

$$|\bar{e}| \leq C\left(e^{-A_1} + e^{-A_2}\right)$$

and then with A_1 and A_2 we can control the aliasing error. This is en effect our case, because for large times the discounting factor will have prevalence on the linear increase of the occupation time and then theoretically we could find the constant C. In the numerical examples we have set $A_1 = A_2 = 20$.

In the inversion formula above we can remark the presence of infinite sums of the form $\sum_{k=0}^{\infty}(-1)^k a_k$, where a_k is real or complex. In this case we can apply the Euler transformation and approximate the infinite sum by:

$$\begin{aligned} E(m,n) &= S_n + \sum_{k=0}^{m}\binom{m}{k}2^{-m}S_{n+k} \\ S_n &= \sum_{k=0}^{n}(-1)^k a_n \end{aligned}$$

So the Euler transformation requires the computation of just $m+n+1$ terms in the sum and then the final sum is extrapolated by the $m+1$ extraterms $n,...,n+m$. CLW state that, in their experience, using the Euler summation technique with $n = 38$ and $m = 11$ can reduce to 10^{-13} or lower the truncation error coming from using a finite number of terms. In the inversion, we have verified that we can obtain an accurate answer with $n = 20$ and $m = 20$ terms, i.e. a total of just 41 terms. Although we have implemented the inversion in C using Microsoft Visual C++, we remark that the Euler algorithm can be found in Mathematica version 3.0 in the package NumericalMath'NLimit'.

The parameters l_1 and l_2 can be used in order to control the roundoff error, due to multiplying large numbers by small ones. In particular the quantity $\exp\left(A_1/2l_1 + A_2/2l_2\right) / \left(4t_1t_2l_1l_2\right)$ can be large and we can decrease it increasing $l_1$and l_2, although this fact increases the computational time that is proportional to the product l_1l_2. The authors suggest that the roundoff and the aliasing errors can be set about of the same order of magnitude and that for two-dimensional inversion $l_1 = l_2 = 2$ is adequate. In effect this choice worked well in our problem.

B.2 The Padé inversion method

This method, proposed by Singhal and al. (henceforth SVV) [23], seems to have been for some time the only known inversion technique for the numerical inversion of multidimensional Laplace transforms. Fortunately its implementation is very easy and moreover the computational time required for the inversion has resulted to be less than 1". Its success depends strongly on the smoothness property of the original function.

This technique starts from the inversion formula in the complex plane. If $\tilde{f}\left(s_1, s_2\right)$ is the double Laplace transform, the inverse Laplace transform $f\left(t_1, t_2\right)$ is obtained applying the inversion formula in two variables:

$$f\left(t_1, t_2\right) = \left(\frac{1}{2\pi j}\right)^2 \int_{c_1 - j\infty}^{c_1 + j\infty} \int_{c_2 - j\infty}^{c_2 + j\infty} e^{s_1 t_1} e^{s_2 t_2}\, \tilde{f}\left(s_1, s_2\right) ds_1 ds_2$$

where c_1 and c_2 are the right-most singularities. Substituting $s_k t_k = w_k$, $k = 1, 2$, we have:

$$f\left(t_1, t_2\right) = \frac{1}{t_1 t_2}\left(\frac{1}{2\pi j}\right)^2 \int_{c_1' - j\infty}^{c_1' + j\infty} \int_{c_2' - j\infty}^{c_2' + j\infty} e^{w_1} e^{w_2}\, \tilde{f}\left(\frac{w_1}{t_1}, \frac{w_2}{t_2}\right) dw_1 dw_2 \qquad (16)$$

Then SVV approximate the exponential function e^{z_k} by a Padé rational function:

$$e^{w_k} \simeq \psi_{n_k, m_k}\left(w_k\right) = \frac{\sum\limits_{i=0}^{n_k} \left(n_k + m_k - i\right)! \binom{n_k}{i} w_k^i}{\sum\limits_{i=0}^{m_k} \left(-1\right)^i \left(n_k + m_k - i\right)! \binom{m_k}{i} w_k^i}$$

where $n_k < m_k$ in order that the function:

$$\tilde{f}\left(\frac{w_1}{t_1}, \frac{w_2}{t_2}\right) \psi_{n_1, m_1}\left(w_1\right) \psi_{n_2, m_2}\left(w_2\right)$$

has two more poles than zeros in each variable. Then

$$\psi_{n_k,m_k}(w_k) = \sum_{i=1}^{m_k} \frac{r_{ki}}{w_k - w_{ki}}$$

where w_{ki} are the poles of the approximation and r_{ki} are the corresponding residues. Substituting $\psi_{n_k,m_k}(w_k)$ for e^{w_k} in (16), we get an approximation $\bar{f}(t_1, t_2)$ to $f(t_1, t_2)$:

$$\bar{f}(t_1,t_2) = \frac{1}{t_1 t_2}\left(\frac{1}{2\pi j}\right)^2 \int_{c_1'-j\infty}^{c_1'+j\infty}\int_{c_2'-j\infty}^{c_2'+j\infty} \tilde{f}\left(\frac{w_1}{t_1},\frac{w_2}{t_2}\right)\sum_{i=1}^{m_1}\frac{r_{1i}}{w_1 - w_{1i}}\sum_{i=1}^{m_2}\frac{r_{2i}}{w_2 - w_{2i}} dw_1 dw_2$$

Interchanging the sums with the integrals, using the residue calculus and closing the path in the right half plane we get the final formula:

$$\bar{f}(t_1,t_2) = \frac{1}{t_1 t_2}\sum_{i=1}^{m_1}\sum_{i=1}^{m_2} r_{1i} r_{2i}\, \tilde{f}\left(\frac{w_{1i}}{t_1},\frac{w_{2i}}{t_2}\right)$$

The inversion reduces to a double summation and requires just $m_1 \times m_2$ function evaluations. This inversion formula can be easily programmed once we have calculated the poles and the residues of the Padé approximant. This can be done very easily in Mathematica version 3.0 using the function Pade[] for computing the approximant, the function NSolve[] to find the poles and the function NResidue[] to find the corresponding residues, as illustrated here below, where we give the Mathematica 3.0 code for computing the poles and the residues of the Padé approximant. It requires the use of the packages NResidue and Pade.

```
<< NumericalMath`NResidue`;
<< Calculus`Pade`;
poles[nk_, mk_] := x/.NSolve[Denominator[Pade[Exp[x], {x, 0, nk, mk}]] == 0, x]
res[nk_, mk_, i_] := NResidue[Pade[Exp[x], {x, 0, nk, mk}], {x, poles[nk, mk][[i]]}]
```

Once we have calculated poles and residues[15] we can store them and perform the numerical inversion very quickly[16]. Regarding the choice of the degree of the numerator and denominator, we have seen that if $m_k > 20$ we can incur in roundoff errors because the residues and the poles can assume very large values. A check test is to verify that the sum of the residues is zero. For our problem a good choice was to set n_k equal to 4 and $m_k = 18$.

C The finite difference scheme

Our main system of PDE's (3)

$$-V_\tau + rxV_x + \frac{\sigma^2}{2}x^2 V_{xx} = rV \tag{17}$$

[15] This is quite time consuming if the degree of the denominator is high.

[16] However, the computation of poles and residues can be done easily in C or Fortran as well, using double precision arithmetic.

with continuity condition across the sampling dates τ_j,

$$V\left(x,\tau_j^+,y\right)=V\left(x,\tau_j^-,y+1_{(l<x<u)}\right) \tag{18}$$

and initial condition$V(x,0,y)=\max(y-K;0)*\Delta$ is solved by a proper finite difference scheme in order to satisfy the requirements specified in the main text, i.e.: existence, positivity, convergence to the exact one and absence of spurious oscillations.
From a computational point of view the change of variable

$$X=\frac{1}{1+x}$$

which maps the x-interval $[0,+\infty)$ onto the X-interval $[0,1]$ avoids assuming a finite arbitrarily large computational domain. We get the more convenient boundary value and initial conditions problem:

$$\begin{array}{l} -\frac{\partial V}{\partial \tau}+\left[r\left(X^2-X\right)+\sigma^2X\left(1-X\right)^2\right]\frac{\partial V}{\partial X}+\frac{\sigma^2}{2}X^2(1-X)^2\frac{\partial^2 V}{\partial X^2}-rV=0 \\ V(\tau,0)=V(\tau,1)=e^{-r\tau}\max(y-K,0), \qquad V(0,X)=\max(y-K,0) \end{array} \tag{19}$$

Equally spaced points $\tau_n=n\Delta\tau$, $n=1,...,M_\tau$, $X_j=j\Delta X$, $j=1,...,M_X$ have been chosen, with steps size $\Delta\tau$, ΔX respectively. $M_\tau\cdot M_X$ grid points are so obtained.

- The first derivative $\frac{\partial V}{\partial \tau}$ is discretized by a first order backward-difference scheme;
- the first derivative $\frac{\partial V}{\partial X}$ has been treated by a *first-order upwind scheme*, as the convection term coefficient can take both negative and positive values;
- for the second derivative $\frac{\partial^2 V}{\partial X^2}$a centered-difference formula has been chosen.

a) The difference equation is as follows.
If $r(X_j^2-X_j)+\sigma^2X_j(1-X_j)^2>0$, then:

$$\begin{array}{l} -\frac{V_j^{n+1}-V_j^n}{\Delta\tau}+\left[r(X_j^2-X_j)+\sigma^2X_j(1-X_j)^2\right]\frac{V_{j+1}^{n+1}-V_j^{n+1}}{\Delta X}+ \\ +\frac{1}{2}\sigma^2X_j^2(1-X_j^2)^2\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{(\Delta X)^2}-rV_j^{n+1}=0 \end{array} \tag{20}$$

otherwise if $r(X_j^2-X_j)+\sigma^2X_j(1-X_j)^2<0$:

$$\begin{array}{l} -\frac{V_j^{n+1}-V_j^n}{\Delta\tau}+[r(X_j^2-X_j)+\sigma^2X_j(1-X_j)^2]\frac{V_j^{n+1}-V_{j-1}^{n+1}}{\Delta X}+ \\ +\frac{1}{2}\sigma^2X_j^2(1-X_j)^2\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{(\Delta X)^2}-rV_j^{n+1}=0 \end{array} \tag{21}$$

Here $j=1,...,M_X$, $X_j=X(j\Delta X)$, $V_j^n=V(n\Delta\tau,j\Delta X)$, represents the approximate solution at the point $(n\Delta\tau,j\Delta X)$, $V_0^n=V_{M_X+1}^n=e^{-rn\Delta\tau}\max(y-K,0)$.

A set of M_τ uncoupled M_X-order linear systems are so obtained. Each system $AV_{n+1}=b_n$, $n=0,...,M_\tau$ contains the grid points having a $\tau=\tau_{n+1}$ constant value. Some interesting features of each A matrix are the following:

- A is tridiagonal (with diagonal positive components and off diagonal non positive components), nonsymmetric, irreducible (its associated graph is strongly connected), strictly diagonally dominant;
- b_n is non negative and contains both the approximate values of the solution previously calculated at the time τ_n and the boundary conditions at the time τ_{n+1}. Therefore:
- A is nonsingular having eigenvalues with positive real parts (from Gerschgorin's Theorem);
- A is an M-matrix (compare Ortega [21], p. 110);
- $A^{-1} > 0$ strictly.

b) Then $V_{n+1} = A^{-1}b_n$ is positive. Thus the numerical solution in all grid points is positive.

c) For what concerns the convergence we resort to the following *Lax's equivalence theorem* (compare Smith [24]): Given a properly posed linear initial-value problem and a linear finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

1. Consistency

Now we rewrite (19) into operational form $L(V) = 0$, where V indicates the exact solution. Let V_j^n the approximate solution at the point $(n\Delta\tau, j\Delta X)$ and T_j^n the local truncation error. Expand each term V_{j+1}^{n+1}, V_j^{n+1}, V_{j-1}^{n+1} by Taylor's series about $(n\Delta\tau, j\Delta X)$. The principal part of the local truncation error

$$\begin{aligned} T_j^n &= L\left(V\left(n\Delta\tau, j\Delta X\right)\right) + \\ &+\Delta\tau\left[-r\tfrac{\partial V}{\partial\tau} - \tfrac{1}{2}\tfrac{\partial^2 V}{\partial\tau^2} + \left(r(X_j^2 - X_j) + \sigma^2 X_j(1 - X_j)^2\right)\tfrac{\partial^2 V}{\partial\tau\partial X} + \right. \\ &\left. +\tfrac{1}{2}\sigma^2 X_j^2(1 - X_j)^2\tfrac{\partial^3 V}{\partial\tau\partial X^2}\right] + \Delta X\tfrac{\left(r(X_j^2 - X_j) + \sigma^2 X_j(1 - X_j)^2\right)}{2}\tfrac{\partial^2 V}{\partial X^2} \end{aligned} \tag{22}$$

is obtained. Being V solution of the differential equation, then $L(V_j^n) = 0$ holds. In the hypothesis that $\frac{\partial^2 V}{\partial\tau^2}, \frac{\partial^2 V}{\partial\tau\partial X}, \frac{\partial^2 V}{\partial X^2}, \frac{\partial^3 V}{\partial\tau\partial X^2}$ are bounded then as $(\Delta\tau, \Delta X) \to 0$ we have $T_j^n \to 0$, so that the numerical scheme given by (20) and (21) is consistent with (19).

2. Stability

Writing (20) and (21) into matrix form at the value $\tau = \tau_{n+1} = (n+1)\Delta\tau$ we obtain:

$$AV_{n+1} = V_n + C \tag{23}$$

where $A = [a_{ij}]$ is a tridiagonal diagonally dominant matrix, with $a_{ii} - \sum_{j\neq i}^{M_x} \mid a_{ij} \mid = 1 + r\Delta\tau$, $i = 1, ..., M_x$, whilst the vector C includes the boundary conditions. From Gerschgorin theorem the eigenvalues $\lambda_i(A)$ satisfy $\lambda_i(A) \geq 1 + r\Delta\tau$ and then

$$0 < \lambda_i(A^{-1}) \leq \frac{1}{1 + r\Delta\tau} < 1 \tag{24}$$

Thus the scheme is unconditionally stable and then the convergence, via Lax's equivalence theorem, is proved.

c) The matrix A is tridiagonal, diagonally dominant, with all its off-diagonal entries strictly negative. Then A is similar to a real symmetric tridiagonal matrix DAD^{-1} with non-zero off-diagonal entries (Ortega, p. 113), so that all the eigenvalues $\lambda_i(A)$ are real. Here D is a diagonal matrix.

The characteristic polynomial $P_{M_x}(\lambda)$ associated to DAD^{-1} admits M_x real and distinct zeros (Stoer, p.43) so that A admits M_x real distinct eigenvalues. Hence the M_x eigenvectors v_s of A are linearly independent and can be used as a basis for the M_x dimensional space of the vector V_0 of initial values. In other words, V_0 can be expressed as $V_0 = \sum_{j=1}^{M_x} c_j v_j$ where the c_j are constant. From (23) we have

$$\begin{aligned} V_n &= \left(A^{-1}\right) V_{n-1} + A^{-1}C = ... = \left(A^{-1}\right)^n V_0 + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \\ &= \left(A^{-1}\right)^n \sum_{j=1}^{M_x} c_j v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j = \sum_{j=1}^{M_x} c_j \left(A^{-1}\right)^n v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \\ &= \sum_{j=1}^{M_x} c_j \lambda_j^n v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \end{aligned} \tag{25}$$

Taking into account (24) then the numerical scheme (20),(21) is L_0-stable (Smith, p.121) and unwanted finite oscillations in the numerical solution are rapidly dampened.

In summarizing, the scheme satisfies all the requirements $\forall(\Delta\tau, \Delta x)$.

D The digital corridor option

In this section we discuss the pricing of the digital corridor option that pays a cash amount if the occupation time at the expiry of the option is greater than a fixed level K :

$$payoff_{t+\tau} = 1_{(Y(t+\tau,x;u,l)>K)}$$

We remark on the importance of this contract as basic element in the construction of the corridor option, in exactly the same manner as the digital option for the plain vanilla options. Indeed the corridor option with residual life τ, can be expressed as sum of corridor digital options with ascending strikes:

$$corropt_t\left(K\right) = \sum_{n=0}^{\left[\left(\tau - \bar{K}\right)/\Delta K\right]} digcorropt_t\left(\bar{K} + n\Delta K\right) \Delta K$$

where [] stands for the integer part. In the limit, we have:

$$corropt_t\left(\bar{K}\right) = \int_{\bar{K}}^{\tau} digcorropt_t\left(K\right) dK$$

In order to price this contract we need to discount the following quantity[17]:

$$\begin{aligned}
&E_{t,x}\left[1_{(Y(t+\tau,x;u,l)>K)}\right] = \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) > K\right] \\
&= \int_K^{t+\tau} \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) \in dy\right] + \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) = \tau\right] \\
&= 1 - \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) = 0\right] - \int_0^K \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) \in dy\right]
\end{aligned}$$

In this expression, the only problem can come from the integral term, because all the other quantities have a known closed form expression. If we consider the Laplace transform with respect of K of this quantity, this transform is given by the Laplace transform of $\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) \in dy\right]$ divided by μ and then the undiscounted price is given by:

$$\begin{aligned}
&= 1 - \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) = 0\right] - \mathcal{L}^{-1}\left[\frac{\Omega(t,\mu,x;l,u,m)}{\mu};\mu \to K\right] \\
&= 1 - \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right) = 0\right] - \mathcal{L}^{-1}\left[\frac{\omega(\gamma,\mu,x;l,u,m)}{\mu};\mu \to K,\gamma \to t\right]
\end{aligned}$$

so in order to price this option we need to compute the double Laplace inverse of the quantity $\omega\left(\gamma,\mu,x;l,u,m\right)/\mu$. The numerical routines adopted for the inversion in the case of the corridor option can then be modified very quickly to cope with this pricing problem. In the following table we compare the prices obtained using the two different numerical procedures and we can verify that the accuracy is very high and the computational time very low. The parameters are $U = 110, L = 100, r = 0.05, \sigma = 0.2$, $\tau = 1yr$.

Index Level	Fourier	Padé
90	0.286586	0.286588
95	0.425404	0.425388
100	0.582029	0.582005
105	0.646513	0.646487
110	0.5421136	0.542088
115	0.387798	0.387776
120	0.269909	0.269911
Average CPU	3.85"	0.25"

Regarding the discrete time monitoring case we remark that we can apply the same procedure adopted for the corridor option, i.e. solving the PDE (3), but now the initial condition is given by:

$$V\left(x,0,y\right) = 1_{(y>K)} \tag{26}$$

and the boundary conditions:

$$V\left(0,\tau,y\right) = e^{-r\tau}1_{(y>K)} \tag{27}$$
$$V\left(+\infty,\tau,y\right) = e^{-r\tau}1_{(y>K)} \tag{28}$$

In this case y is an integer number representing the number of times the index has spent inside the band at the monitoring dates. We observe that when $y \geq [K]+1 \geq K$, where $[K]$

[17]In the following we use the fact that 1= $\Pr_{t,x}\left[Y_{t+\tau} = 0\right]$ + $\Pr_{t,x}\left[Y_{t+\tau} = \tau\right]$ + $\int_K^{t+\tau} \Pr_{t,x}\left[Y_{t+\tau} \in dy\right]$ + $\int_0^K \Pr_{t,x}\left[Y_{t+\tau} \in dy\right]$.

is the greatest integer strictly smaller than K, and when the time to maturity is τ the option will be surely exercised, and we have an analytical solution given by $V(x, \tau, y) = e^{-r\tau}$.

We remark that for this kind of contract the discontinuities are introduced not only at the monitoring dates, but at the initial date as well and the numerical scheme described above is then a correct one for the problem at hand.

In the following table we report the price of this contract in presence of continuous and discrete time monitoring. In the discrete time case, the value of K_d has been set equal to $K_d = [K_c * n]$ where [] is the integer part, n the number of monitoring dates and K_c the fixed level (as ratio to the residual life) used in the continuous time formula. In this way if $K_c = 0.2$ and with monthly monitoring, we have $K_d = [0.2 * 12] = [2.4] = 2$. The remaining parameters are the same as in the previous table.

Index Level	Monthly	Continuous	%Difference
90	0.267553	0.286588	-6.64%
95	0.400571	0.425388	-5.83%
100	0.519848	0.582005	-10.68%
105	0.553547	0.646487	-14.38%
110	0.481742	0.542088	-11.13%
115	0.361531	0.387776	-6.77%
120	0.250799	0.269911	-7.08%

In this case the price of the option in the discrete time monitoring appears to be systematically lower than in the corresponding continuous case. The likely reason is in the discrete distribution of the occupation time in the case of discrete monitoring, which concentrates the masses of probability in 0, Δ,2Δ.

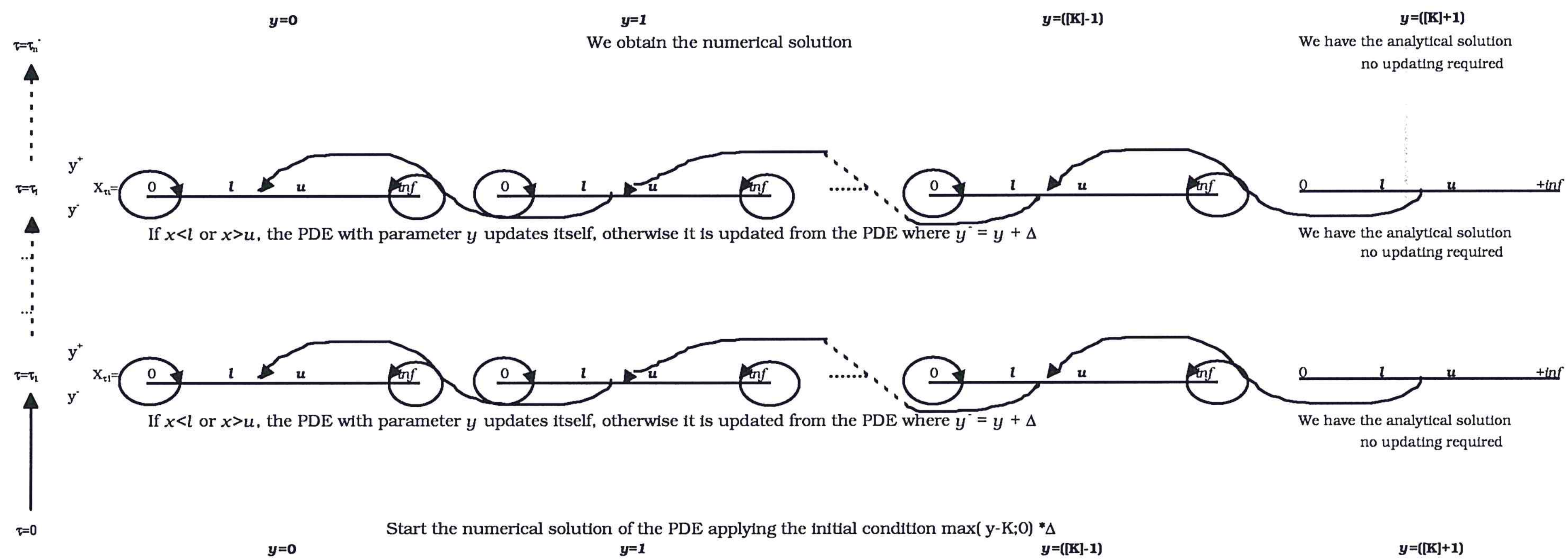

Figure 1: Updating process for the Corridor Option with Discrete Time Monitoring

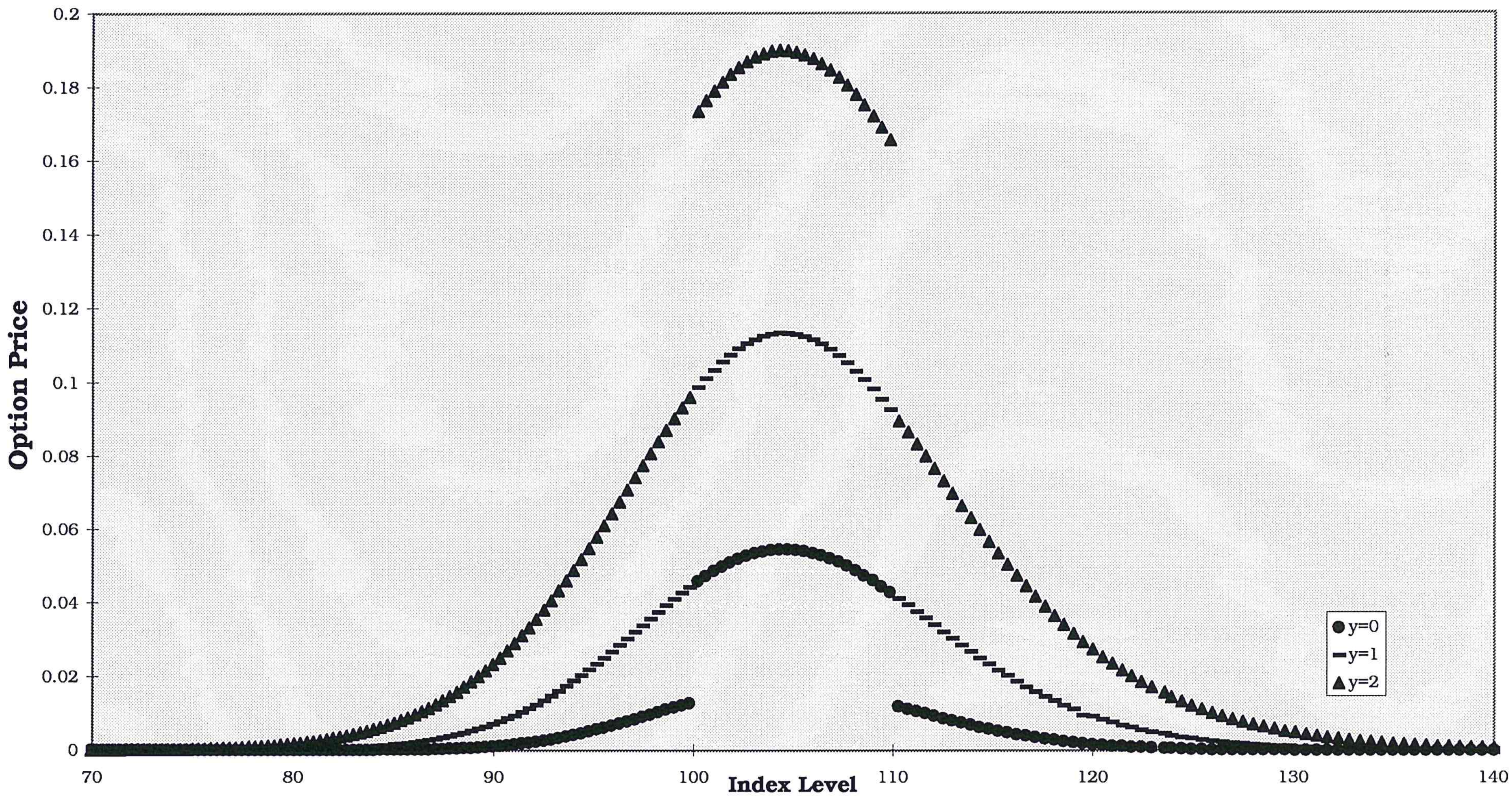

Figure 2: The jump condition for the updating process after four monitoring dates and for different values of the parameter y (r=0.05, σ=0.2, K=2.4, l=100, u=110, τ = 1/3 years)

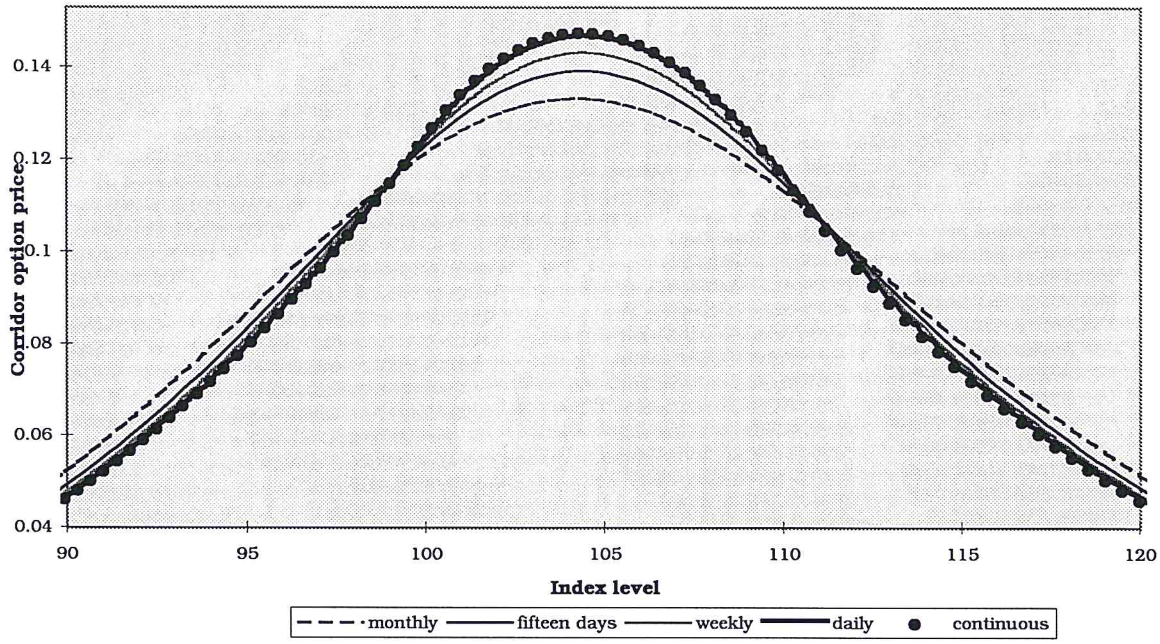

Figure 3: Price of the corridor option and monitoring frequency
(l=100, u=110, K*mon. freq=0.2, τ=1, r=5%, σ=0.2)

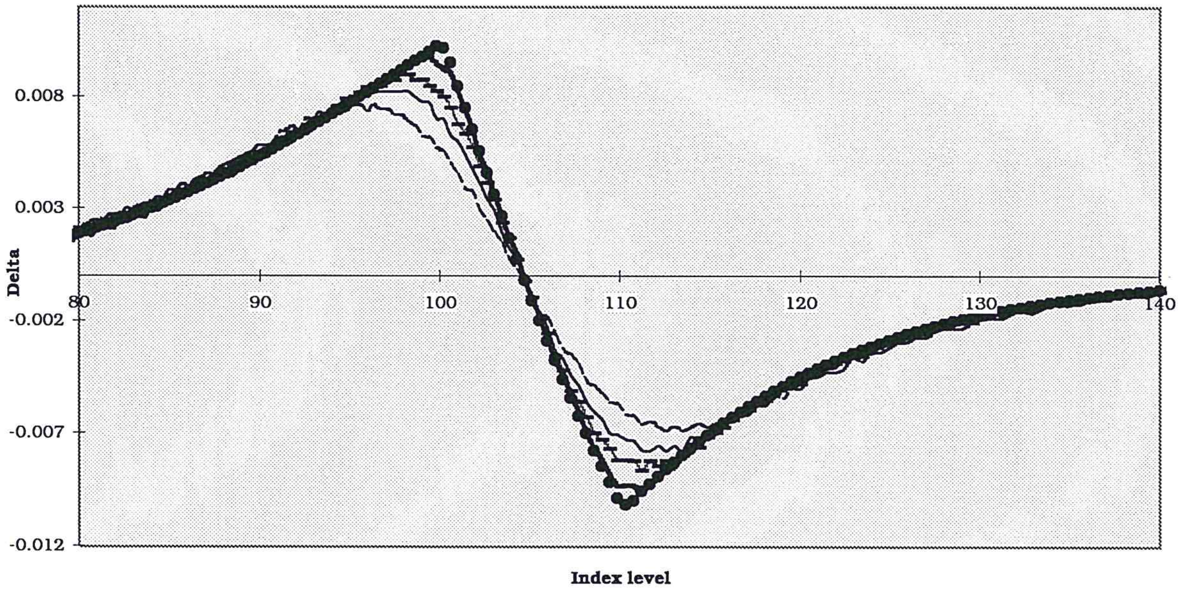

Figure 4 : Delta of the corridor option and monitoring frequency
(l=100, u=110, K*mon. freq=0.2, τ=1, r=5%, σ=0.2)

Price of the corridor option with the CLW-Fourier inversion method

l =100, u =110, r =0.05, σ =0.2, time to expiry = 1yr

n=	20	20	38	50	50
n+m=	30	40	49	60	70
K=0.2	Percentage Difference				Price
Index Level					
90	0.00%	0.00%	0.00%	0.00%	0.0463
95	-0.02%	0.00%	0.00%	0.00%	0.0792
100	-0.02%	0.00%	-0.05%	-0.01%	0.1247
105	-0.02%	0.00%	0.00%	0.00%	0.1469
110	-0.02%	0.00%	-0.05%	-0.01%	0.1161
115	-0.02%	0.00%	0.00%	0.00%	0.0736
120	0.00%	0.00%	0.00%	0.00%	0.0457
Avg. CPU	2"29/100	4"43/100	7"14/100	8"43/100	12"15/100
K=0.6	Percentage Difference				Price
Index Level					
90	0.78%	0.00%	-0.03%	-0.02%	0.0009
95	-1.29%	0.00%	0.28%	0.06%	0.0027
100	-1.62%	0.00%	-3.02%	-0.41%	0.0068
105	-0.88%	0.00%	0.08%	0.00%	0.0095
110	-1.74%	0.00%	-3.24%	-0.44%	0.0063
115	-1.87%	0.00%	0.10%	-0.01%	0.0027
120	0.70%	0.00%	0.09%	0.03%	0.0011
Avg. CPU	2"29/100	4"	6"	8"57/100	11"57/100

n=	20	20	38	50	50
n+m=	30	40	49	60	70
K=0.4	Percentage Difference				Price
Index Level					
90	0.05%	0.00%	0.00%	0.00%	0.0101
95	-0.11%	0.00%	0.02%	0.00%	0.0213
100	-0.17%	0.00%	-0.33%	-0.04%	0.0400
105	-0.11%	0.00%	0.01%	0.00%	0.0503
110	-0.18%	0.00%	-0.36%	-0.05%	0.0373
115	-0.17%	0.00%	0.01%	0.00%	0.0203
120	0.05%	0.00%	0.01%	0.00%	0.0108
Avg. CPU	2"43/100	4"14/100	5"57/100	8"43/100	11"43/100
K=0.8	Percentage Difference				Price
Index Level					
90	118.13%	0.14%	-4.10%	-2.33%	0.0000
95	-87.61%	-0.11%	19.29%	4.33%	0.0001
100	-61.23%	-0.07%	-110.29%	-14.93%	0.0002
105	-26.18%	-0.03%	2.63%	0.24%	0.0004
110	-65.90%	-0.07%	-118.52%	-16.05%	0.0002
115	-116.75%	-0.15%	6.70%	-0.72%	0.0001
120	82.59%	0.12%	9.92%	3.00%	0.0000
Avg. CPU	2"71/100	4"	5"86/100	8"43/100	13"29/100

Table 1: In the first two lines n and m are the numbers of terms used in the Euler summation technique.
The inversion has been performed with A1=A2=20 and l1=l2=2.
In the table under the column "Price" there are the prices when the inversion is done setting n=50 and m+n=70 whilst in the other places appear the percentage difference

Price of the corridor option with the Padé inversion method

l =100, u =110, r =0.05, σ =0.2, time to expiry = 1yr

Num. Degr	1	2	3	3	4
Den. Degre	20	19	18	19	18
K=0.2	Percentage Difference				Price
Index Level					
90	0.00%	0.00%	0.00%	0.00%	0.0463
95	0.00%	0.00%	0.00%	0.00%	0.0792
100	0.00%	0.00%	0.00%	0.00%	0.1247
105	0.00%	0.00%	0.00%	0.00%	0.1469
110	0.00%	0.00%	0.00%	0.00%	0.1161
115	0.00%	0.00%	0.00%	0.00%	0.0736
120	0.00%	0.00%	0.00%	0.00%	0.0457
Avg. CPU	14/100	14/100	14/100	29/100	14/100

Num. Degr	1	2	3	3	4
Den. Degre	20	19	18	19	18
K=0.4	Percentage Difference				Price
Index Level					
90	0.00%	0.00%	0.00%	0.00%	0.0101
95	0.00%	0.00%	0.00%	0.00%	0.0213
100	0.00%	0.00%	0.00%	0.00%	0.0400
105	0.00%	0.00%	0.00%	0.00%	0.0503
110	0.00%	0.00%	0.00%	0.00%	0.0373
115	0.00%	0.00%	0.00%	0.00%	0.0203
120	0.00%	0.00%	0.00%	0.00%	0.0108
Avg. CPU	29/100	14/100	14/100	29/100	29/100

K=0.6	Percentage Difference				Price
Index Level					
90	0.02%	0.01%	-0.05%	-0.05%	0.0009
95	0.08%	0.00%	-0.05%	-0.03%	0.0027
100	0.02%	-0.02%	-0.03%	0.02%	0.0068
105	-0.01%	-0.03%	-0.02%	0.01%	0.0095
110	0.02%	-0.02%	-0.03%	0.02%	0.0063
115	0.07%	0.00%	-0.05%	-0.03%	0.0027
120	0.05%	0.01%	-0.05%	-0.05%	0.0011
Avg. CPU	14/100	14/100	14/100	29/100	14/100

K=0.8	Percentage Difference				Price
Index Level					
90	-100.00%	-15.48%	-13.19%	-4.23%	0.0000
95	-3.13%	2.03%	-1.48%	3.00%	0.0001
100	4.41%	3.42%	0.39%	1.92%	0.0003
105	4.36%	2.59%	0.39%	0.75%	0.0004
110	4.40%	3.42%	0.33%	1.94%	0.0002
115	-1.25%	2.55%	-1.17%	2.92%	0.0001
120	-23.10%	-6.42%	-7.63%	0.21%	0.0000
Avg. CPU	29/100	29/100	14/100	29/100	14/100

Table 2: In the first two lines num and den degree are the numerator and denominator degrees for the Padé approximants
In the table are given the prices when the inversion is done setting num=4 and den=18 whilst
in the other places appear 100 x the percentage difference

Comparison between the Padé and the CLW Inversion

l =100, u =110, r =0.05, σ =0.5, time to expiry = 1yr

	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion
	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18
Index Level	K = 0.2, T-t =1yr			K = 0.4, T-t =1yr			K = 0.6, T-t =1yr		
90	0.00%	0.0104	0.00%	-0.02%	0.0002	0.16%	-46.02%	0.00000	361.16%
95	0.00%	0.0136	0.00%	-0.03%	0.0003	0.09%	185.87%	0.00000	-676.72%
100	0.00%	0.0174	0.00%	-0.02%	0.0005	0.01%	-25.67%	0.00000	18.91%
105	0.00%	0.0196	0.00%	-0.02%	0.0006	0.03%	-22.46%	0.00000	37.17%
110	0.00%	0.0180	0.00%	-0.01%	0.0005	0.00%	-20.37%	0.00000	13.80%
115	0.00%	0.0149	0.00%	-0.02%	0.0004	0.07%	-23.12%	0.00000	76.76%
120	0.00%	0.0123	0.00%	-0.03%	0.0003	0.13%	-43.71%	0.00000	246.26%
Avg. CPU	3"86/100	11"43/100	29/100	4"	11"29/100	14/100	3"86/100	11"29/100	14/100

Index Level	K = 0.2, T-t =2 yr			K = 0.5, T-t =2 yr			K = 0.8, T-t =2 yr		
90	0.00%	0.0376	0.00%	0.00%	0.0017	0.02%	-1.57%	0.00001	7.59%
95	0.00%	0.0443	0.00%	0.00%	0.0021	0.01%	-1.02%	0.00002	3.25%
100	0.00%	0.0516	0.00%	0.00%	0.0027	0.00%	-0.79%	0.00002	0.73%
105	0.00%	0.0556	0.00%	0.00%	0.0030	0.00%	-1.02%	0.00001	1.98%
110	0.00%	0.0531	0.00%	0.00%	0.0028	0.00%	-0.76%	0.00002	0.66%
115	0.00%	0.0478	0.00%	0.00%	0.0023	0.01%	-0.97%	0.00002	2.84%
120	0.00%	0.0431	0.00%	0.00%	0.0020	0.02%	-1.57%	0.00001	6.48%
Avg. CPU	3"86/100	11"43/100	14/100	3"86/100	11"29/100	14/100	3"86/100	11"43/100	14/100

Table 3: In the central column of each block there are the prices when the inversion is done setting n=50 and m+n=70 in the CLW algorithm. In the other places appear the percentage difference.

Comparison between the Padé and the CLW Inversion

l =100, u =110, r =0.05, σ =0.3152, time to expiry = 1yr

	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion
	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18
Index Level	**K = 0.2, T-t =1yr**			**K = 0.4, T-t =1 yr**			**K = 0.6, T-t =1 yr**		
90	0.00%	0.0283	0.00%	0.00%	0.0030	0.01%	-0.01%	0.0001	0.75%
95	0.00%	0.0410	0.00%	0.00%	0.0052	0.01%	-0.07%	0.0002	0.35%
100	0.00%	0.0569	0.00%	0.00%	0.0084	0.00%	-0.04%	0.0003	0.03%
105	0.00%	0.0654	0.00%	0.00%	0.0103	0.00%	-0.05%	0.0004	0.11%
110	0.00%	0.0568	0.00%	0.00%	0.0084	0.00%	-0.04%	0.0003	0.03%
115	0.00%	0.0428	0.00%	0.00%	0.0055	0.00%	-0.07%	0.0002	0.31%
120	0.00%	0.0322	0.00%	0.00%	0.0036	0.01%	-0.03%	0.0001	0.60%
Avg. CPU	3"86/100	11"29/100	14/100	3"86/100	11"29/100	14/100	3"86/100	11"29/100	14/100

Index Level	**K = 0.2, T-t =2 yr**			**K = 0.5, T-t =2 yr**			**K = 0.8, T-t =2 yr**		
90	0.00%	0.0841	0.00%	0.00%	0.0143	0.00%	-0.01%	0.0010	0.07%
95	0.00%	0.1047	0.00%	0.00%	0.0196	0.00%	-0.01%	0.0015	0.04%
100	0.00%	0.1276	0.00%	0.00%	0.0260	0.00%	-0.01%	0.0023	0.00%
105	0.00%	0.1391	0.00%	0.00%	0.0294	0.00%	-0.01%	0.0027	0.01%
110	0.00%	0.1275	0.00%	0.00%	0.0260	0.00%	-0.01%	0.0023	0.00%
115	0.00%	0.1075	0.00%	0.00%	0.0203	0.00%	-0.01%	0.0016	0.03%
120	0.00%	0.0906	0.00%	0.00%	0.0159	0.00%	-0.01%	0.0011	0.06%
Avg. CPU	3"86/100	11"29/100	14/100	3"71/100	11"14/100	14/100	3"86/100	11"57/100	14/100

Table 4: In the central column of each block there are the prices when the inversion is done setting n=50 and m+n=70 in the CLW algorithm. In the other places appear the percentage difference.

Comparison between the Padé and the CLW Inversion

l =100, u =110, r =0.05, σ =0.2, time to expiry = 1yr

	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion	Fourier -Series Inversion		Padé inversion
	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18	n=20 n+m=40	n=50 n+m=70	n=4 m=18
Index Level	K = 0.2, T-t =1yr			K = 0.4, T-t = 1 yr			K = 0.6, T-t =1 yr		
90	0.00%	0.0463	0.00%	0.00%	0.0101	0.00%	0.00%	0.0009	0.05%
95	0.00%	0.0792	0.00%	0.00%	0.0213	0.00%	0.00%	0.0027	0.02%
100	0.00%	0.1247	0.00%	0.00%	0.0400	0.00%	0.00%	0.0068	0.00%
105	0.00%	0.1469	0.00%	0.00%	0.0503	0.00%	0.00%	0.0095	0.00%
110	0.00%	0.1161	0.00%	0.00%	0.0373	0.00%	0.00%	0.0063	0.00%
115	0.00%	0.0736	0.00%	0.00%	0.0203	0.00%	0.00%	0.0027	0.02%
120	0.00%	0.0457	0.00%	0.00%	0.0108	0.00%	0.00%	0.0011	0.04%
Avg. CPU	3"86/100	11"43/100	14/100	3"86/100	13"29/100	29/100	4"	11"29/100	14/100

Index Level	K = 0.2, T-t =2 yr			K = 0.5, T-t =2 yr			K = 0.8, T-t =2 yr		
90	0.00%	0.1402	0.00%	0.00%	0.0448	0.00%	0.00%	0.0091	0.01%
95	0.00%	0.1900	0.00%	0.00%	0.0672	0.00%	0.00%	0.0157	0.00%
100	0.00%	0.2473	0.00%	0.00%	0.0958	0.00%	0.00%	0.0255	0.00%
105	0.00%	0.2711	0.00%	0.00%	0.1090	0.00%	0.00%	0.0305	0.00%
110	0.00%	0.2303	0.00%	0.00%	0.0892	0.00%	0.00%	0.0237	0.00%
115	0.00%	0.1716	0.00%	0.00%	0.0615	0.00%	0.00%	0.0146	0.00%
120	0.00%	0.1274	0.00%	0.00%	0.0421	0.22%	0.00%	0.0090	0.01%
Avg. CPU	3"86/100	11"29/100	14/100	4"	11"43/100	14/100	3"86/100	12"	14/100

Table 5: In the central column of each block there are the prices when the inversion is done setting n=50 and m+n=70 in the CLW algorithm. In the other places appear the percentage difference.

Table 8: Discrete and Continuous Monitoring: effect on the price

	Monitoring Frequency				
	Percentage difference				*Price*
Index Level	monthly	beweekly	weekly	daily	continuos
80	19.52%	10.81%	5.58%	1.33%	0.0115
85	15.96%	8.23%	4.15%	1.29%	0.0245
90	13.05%	6.53%	3.29%	0.92%	0.0463
95	9.28%	4.95%	2.47%	0.70%	0.0793
100	-2.81%	-1.45%	-0.64%	0.04%	0.1247
102	-7.74%	-4.66%	-2.57%	-0.37%	0.1406
105	-9.58%	-5.54%	-2.77%	-0.25%	0.1469
108	-7.31%	-4.25%	-2.17%	-0.02%	0.1338
110	-2.45%	-1.04%	-0.21%	0.59%	0.1161
115	9.09%	5.45%	3.14%	1.37%	0.0736
120	12.85%	6.72%	3.66%	1.69%	0.0457
125	14.95%	8.15%	4.57%	1.75%	0.0279

Parameter values: l=100, u=110, σ=0.2, r=0.05, τ=1yr, K*mon.freq=0.2.

Table 9: Discrete and Continuous Monitoring: effect on the delta

	Monitoring Frequency				
	Percentage difference				*Delta*
Index Level	monthly	beweekly	weekly	daily	continuous
80	10.34%	10.34%	10.34%	-3.71%	0.0019
85	11.62%	2.98%	2.98%	2.98%	0.0034
90	7.58%	2.46%	-1.81%	-1.81%	0.0054
95	-3.65%	-0.29%	-1.35%	-2.50%	0.0078
100	-45.94%	-32.73%	-21.92%	-9.90%	0.0104
102	-39.72%	-20.00%	-4.33%	6.91%	0.0056
105	-42.81%	-29.84%	-42.81%	-42.81%	-0.0013
108	-38.10%	-25.22%	-15.57%	-6.74%	-0.0072
110	-43.82%	-31.45%	-21.25%	-10.32%	-0.0104
115	-3.20%	1.28%	3.13%	3.13%	-0.0069
120	12.77%	8.07%	8.07%	3.37%	-0.0044
125	13.20%	6.90%	6.90%	6.07%	-0.0028

Parameter values: l=100, u=110, σ=0.2, r=0.05, τ=1yr, K*mon.freq=0.2.