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Taxonomy of Algorithms

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A Taxonomy of Algorithms

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Abstract

Valuation and risk management systems use a variety of algorithms to determine the price and the hedge parameters (Greeks) for different types of contingent claims. The motivation for this study is to investigate the "optimal" (in some sense) algorithm for pricing a given contingent claim with given modeling assumptions.

This work catalogues the properties of various contingent claim pricing and hedging algorithms. The study includes the nature of the required input parameters (for the instrument, the valuation model, the numerical method and the parameters implicit in model calibration) and the number of stochastic processes each algorithm accommodates. Issues of calibration, model complexity and isomorphism are also discussed.

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1 Introduction: Optimal Algorithms

A comprehensive knowledge of existing algorithms and their properties enables one to select the optimal (in some sense) algorithm for a given contingent claim and modeling assumptions. As a corollary to this one can also identify situations and contingent claims where there are no current satisfactory algorithms. These holes highlight areas that require future algorithm research and contingent claims which should not be traded or where greater care is essential.

The problem of derivative valuation can often be considered as having a number of possible solutions, for example the equivalent martingale measure approach or the partial differential equation approach with fixed initial boundary conditions. Different modeling view points and different types of contingent claim problems consequently produce different optimal algorithms. For example, a free boundary problem necessitates a PDE or lattice based solution whereas a fixed boundary problem with multiple stochastic variables is best approached using Monte Carlo simulation.

Traditionally, due to legacy issues and the diversity of needs both between banks and within an individual bank's front, middle and back offices, there have developed a multitude of proprietary financial software tools. However, the desire of regulators to see a move away from spreadsheet based solutions (towards coherent firm wide risk management) and the attraction of Internet based electronic trading, confirmation and portfolio analysis, have motivated banks to move in the direction of contract language standardization. Practitioners are becoming increasingly interested in developing languages to describe financial contracts for example, Ibikunle *et al* (1999) [59] with the Financial-Product-Mark-Up-Language (FpML) being developed by J.P. Morgan and PricewaterhouseCoopers. Moreover, Eber (1999) at Société Générale [37] is developing a functional programming language to both describe, price, hedge and risk manage contingent claims.

If one wishes to extend the exercise of contract language standardization to the automatic valuation of portfolios with algorithm selection "on the fly" then the algorithms themselves and their parameters must also be cataloged, standardized and described. For an introduction to the run-time optimization of financial algorithms, where the performance model of the program is combined with knowledge of run-time conditions (such as input data and system state) see Perry, Grimwood, Kerbyson, Papaefstathiou and Nudd (2000) [75]. The aim of this paper then is to catalogue and analyze algorithm properties with the objective of facilitating algorithm selection given a particular type of contingent claim and modeling assumptions.

Section 2 describes the main pricing algorithms applied to the example of a European call option in a Black-Scholes [7]-Merton [68] world. Section 3 gives a catalogue of different algorithms. Section 4 includes algorithm properties such as their input parameters, the number of stochastic processes they can accommodate and their suitability for different contingent claims. There is also a discussion of model calibration as an implicit algorithm parameter and a discussion of model complexity and isomorphism. Finally Section 5 sets out our conclusions.

2 Standard Algorithms

The following section examines the main algorithms for contingent claim pricing using the example of a European call option[1] in a valuation model consistent with the Black-Scholes assumptions. It is manifest from this section that simple cases such as determining the price of a European call have a diverse set of possible solution algorithms. Indeed all the main families of algorithms are discussed here: analytical solutions, lattices, finite difference methods, quadrature and Monte Carlo simulation. The majority of contingent claims can be valued using at least one of these algorithms. The terminology developed in this section will be used throughout the following sections when algorithm input parameters are discussed.

2.1 Analytical Solutions

The price of a European call option C on a non-dividend paying stock is given by the Black-Scholes[2] equation,

$$C = S_0 N(d_1) - X e^{-rT} N(d_2), \tag{1}$$

where,

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{2}$$

$$d_2 = d_1 - \sigma\sqrt{T}, \tag{3}$$

and $N(.)$ is the cumulative normal distribution function which is normally calculated using a polynomial approximation, see Hull (2000) [55]. The contingent claim contract parameters are the strike price X, the maturity T, the fact that it is European style and the fact that it is of call class. The valuation model parameters are the underling asset price S_0, the interest rate r and the volatility σ. Analytical solutions are by far the most elegant method of pricing contingent claims being both fast and compact. However, one often has to make strict modeling assumptions in order to derive a closed form solution and this can be at the expense of realism.

[1] The optimal algorithm (in terms of processing time and memory requirements) in a Black-Scholes world is clearly the Black-Scholes formula itself. However, the *raison d'être* of this section is to give an overview of the main algorithms when applied to the same problem.

[2] The analytical formulae are all considered members of one class of algorithm except where more complicated functions than the cumulative normal distribution are used, for example, the cumulative bivariate normal distribution. An approximation for the cumulative bivariate normal distribution can be found in Drezner (1978) [35].

2.2 Lattices

The binomial method is perhaps the best known lattice based method primarily because it provides a simple and intuitive numerical solution. It was introduced by Sharpe (1978) [90] and developed independently by Cox, Ross & Rubinstein (1979) [28] and Rendleman & Bartter (1979) [80]. In the binomial method, the asset price jumps up or down (i.e., follows a binomial process), by a fixed proportion, at each of a number of discrete time steps during the option's life. The length of each time step Δt, is determined when the user specifies the number of time steps, N, i.e., $\Delta t = T/N$ where T is the option's maturity. The greater the number of time steps, the more precise the method. The cost of the increased precision, however, is computational speed and memory.

2.2.1 Incorrect $E[S_T]\forall N$

It seems natural to construct the binomial tree so as to be consistent with the Black-Scholes-Merton model for European options, and therefore to choose u, d and p to match the risk-neutral mean and variance of the GBM process,

$$dS = rSdt + \sigma Sdz. \tag{4}$$

Since we are working in a risk-neutral world, the expected return from a stock is the risk-free interest rate, r. Hence the expected return from a stock price at the end of a time interval Δt is $Se^{r\Delta t}$, where S is the stock price at the beginning of the time interval. It follows that,

$$Se^{r\Delta t} = pSu + (1-p)Sd. \tag{5}$$

The variance of the change in stock price in a small time interval Δt is $S^2\sigma^2\Delta t$. Since the variance of a variable Q is defined as $E(Q^2) - (E(Q))^2$, where E denotes expected value, it follows that,

$$S^2\sigma^2\Delta t = pS^2u^2 + (1-p)S^2d^2 - S^2(pu + (1-p)d)^2. \tag{6}$$

This is the standard approach for choosing the binomial parameters. However, since there are three parameters u, d and p and we are only trying to match two values (the mean and variance) we have a free choice for one of the parameters. This results in different specifications for the binomial model.

2.2.2 Cox, Ross and Rubinstein (CRR)

$$u = exp\left(\sigma\sqrt{\Delta t}\right), \tag{7}$$

$$d = exp\left(-\sigma\sqrt{\Delta t}\right), \tag{8}$$

and

$$p = \frac{1}{2} + \frac{r - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}. \tag{9}$$

2.2.3 Jarrow and Rudd (JR) [61]

$$u = exp\left(\left(r - 1/2\sigma^2\right)\Delta t + \sigma\sqrt{(\Delta t)}\right), \tag{10}$$

$$d = exp\left(\left(r - 1/2\sigma^2\right)\Delta t - \sigma\sqrt{(\Delta t)}\right), \tag{11}$$

and

$$p = 1/2. \tag{12}$$

2.2.4 Hull and White (HW) [56]

$$u = exp\left(\sigma\sqrt{\Delta t}\right), \tag{13}$$

$$d = exp\left(-\sigma\sqrt{\Delta t}\right), \tag{14}$$

and

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}. \tag{15}$$

2.2.5 Correct $E[S_T]\forall N$

A problem with the previous formulations for the binomial method is that the approximation is only good over a small time interval, we cannot freely choose arbitrarily large time steps. That is to say, the approximation is only consistent in the limit as $\Delta t \to 0$. To solve this problem we can reformulate the model in terms of the natural logarithm of the asset price ($x = ln(S)$). The natural logarithm of the asset price under GBM is normally distributed with a constant mean and variance. This approach is consistent for any step size, not just in the limit. The continuous time risk-neutral process for x can be shown to be,

$$dx = (r - \frac{1}{2}\sigma^2)dt + \sigma dz. \tag{16}$$

With the discrete time binomial model for x the variable x can either go up to a level of $x + \Delta x_u$ with a probability of p or down to a level of $x + \Delta x_d$ with a probability of $1 - p$. Equating the mean and the variance of the binomial process for x with the mean and the variance of the continuous time process over the time interval Δt leads to the following equations,

$$(r - \frac{1}{2}\sigma^2)\Delta t = p\Delta x_u + (1 - p)\Delta x_d, \tag{17}$$

$$\sigma^2\Delta t = p\Delta x_u^2 + (1 - p)\Delta x_d^2 - (r - \frac{1}{2}\sigma^2)\Delta t. \tag{18}$$

We have two equations in three unknowns. So, again we have a free choice for one of the parameters. We can make the analogous choices to CRR and JR i.e. set the probabilities or the jump sizes to be equal.

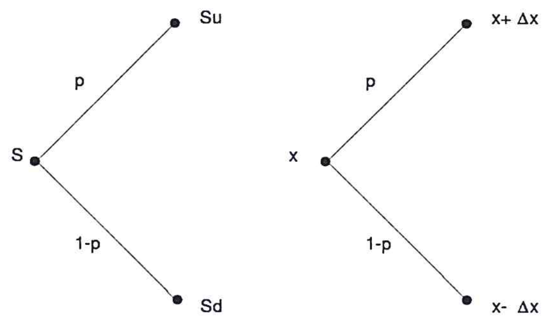

Figure 1: Multiplicative and additive binomial tree elements.

2.2.6 Trigeorgis (1992) (TRG) [93]

Equal jump sizes lead to,

$$\Delta x_u = \Delta x_d = \sqrt{\sigma^2 \Delta t + (r - \frac{1}{2}\sigma^2)^2 \Delta t^2}, \tag{19}$$

and

$$p = \frac{1}{2} + \frac{1}{2}\frac{(r - \frac{1}{2}\sigma^2)\Delta t}{\Delta x}. \tag{20}$$

2.2.7 Equal Probabilities (EQP)

Equal probabilities leads to the following,

$$\Delta x_u = \frac{1}{2}(r - \frac{1}{2}\sigma^2)\Delta t + \frac{1}{2}\sqrt{4\sigma^2 \Delta t - 3(r - \frac{1}{2}\sigma^2)^2 \Delta t^2}, \tag{21}$$

$$\Delta x_d = \frac{3}{2}(r - \frac{1}{2}\sigma^2)\Delta t - \frac{1}{2}\sqrt{4\sigma^2 \Delta t - 3(r - \frac{1}{2}\sigma^2)^2 \Delta t^2}, \tag{22}$$

and

$$p = 1/2. \tag{23}$$

2.2.8 Binomial Tree Option Valuation

Using either the parameters S, u and d or x, Δx_u and Δx_d an asset price tree can be constructed starting from the initial value of either S_0 or x_0. At any node (i, j) in the tree the asset price is either $S_0 u^j d^{i-j}$ where $j = 0, 1, \ldots, i$ or $x_0 + j\Delta x_u + (i-j)\Delta x_d$. See Figure 1 for the multiplicative and additive binomial tree elements. The tree recombines in that an up movement followed by a down movement leads to the same asset price as a down movement followed by an up movement. This is important from a computational point of view because if the tree did not recombine (which would be the case for a non-Markovian process) then the amount of nodes would grow as 2^N rather than N^2.

Starting at the end of the tree at time T one knows the value of the option. For a European call option the value is $\max(S_{T,j} - X, 0)$. As one is working in a risk-neutral world, the value at each node at time $T - \Delta t$ can be calculated as the expected value at time T discounted at rate r for a period Δt. This leads to the following relationship,

$$C_{i,j} = exp(-r\Delta t)(pC_{i+1,j+1} + (1-p)C_{i+1,j}). \tag{24}$$

Eventually by working back through all the nodes in the tree (a process called dynamic programming) the value of the option at time zero is obtained, $C_{0,0}$. If the option were of American style then at each node one checks whether early exercise is preferable to holding the option for a further time period Δt i.e., $C_{i,j} = \max(C_{i,j}, S_{i,j} - X)$.

The lattice methods require as inputs the contract parameters (the maturity T and the strike price X, the type: call and the class: European), the model parameters (the volatility σ, the interest rate r and the underlying asset price S_0) and the numerical algorithm parameters which are the number of time steps N and any stability and convergence criteria e.g. for trinomial tree $\Delta x = \sigma\sqrt{3\Delta t}$.

2.2.9 Trinomial Trees

For a given number of time steps N, trinomial trees have faster convergence than binomial trees although they require more memory $N(2N+1)$ as opposed to $N(N+1)$. This faster convergence is due to the extra branch which provides a better approximation to the continuous time process. The extra degree of freedom also makes this method more flexible, allowing relatively easy extension to time-varying drift and volatility parameters. The trinomial tree method is equivalent to the fully explicit finite difference method which is described in the next section therefore it is not outlined in any further detail here.

2.3 Finite Difference Methods

2.3.1 The θ-Method

Starting with the Black-Scholes PDE,

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} = rC(S,t), \tag{25}$$

we use the natural logarithm of the underlying asset price i.e., $S = ln(x)$ and $W(x,t) = C(S,t)$ to transform the PDE so that we have constant coefficients for the partial derivatives,

$$\frac{1}{2}\sigma^2 \frac{\partial^2 \mathrm{W}(x,\, t)}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial \mathrm{W}(x,\, t)}{\partial x} + \frac{\partial \mathrm{W}(x,\, t)}{\partial t} = r\,\mathrm{W}(x,\, t). \tag{26}$$

In finite difference methods we replace the partial derivatives with difference equations. For the partial derivative with respect to time we use a forward

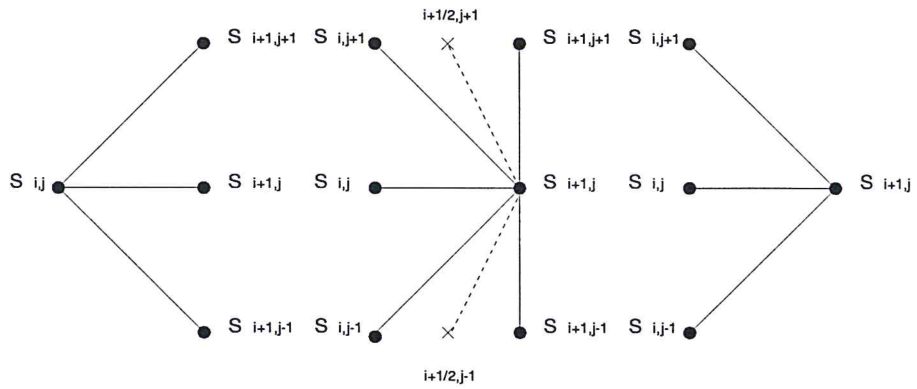

Figure 2: Fully explicit element centred around $i+1, j$, Crank-Nicolson element centred around $i+\frac{1}{2}, j$ and a fully implicit element centred around i, j.

difference equation,

$$\frac{\partial W(x,t)}{\partial t} \approx \frac{W_{i+1} - W_i}{\Delta t} + O(\Delta t), \tag{27}$$

for the first order partial derivative with respect to asset price we use a central difference equation,

$$\frac{\partial W(x,t)}{\partial x} \approx \frac{W_{j+1} - W_{j-1}}{2\Delta x} + O((\Delta x)^2), \tag{28}$$

and for the second order partial derivative with respect to asset price we use a symmetric central difference equation,

$$\frac{\partial W(x,t)}{\partial x^2} \approx \frac{W_{j+1} - 2W_j + W_{j-1}}{(\Delta x)^2} + O((\Delta x)^2). \tag{29}$$

The $O(\cdot)$ terms in the above difference equations represent the rest of the Taylor series expansion. Henceforth we will truncate the expansions by not including these terms. The smaller the Δt and Δx terms become (i.e., the finer the discretization) the more accurate our approximations become, that is to say the local truncation errors $O(\cdot)$ become smaller.

Depending on whether the difference equations are centred around time step $i+1$, i or $i+\frac{1}{2}$ determines whether the finite difference method is fully explicit, fully implicit[3] or Crank-Nicolson[4] respectively, see Figure 2. These three finite difference methods are nested within the θ-method. The Black-Scholes difference

[3]The explicit and implicit finite difference frameworks were first used in option pricing by Brennan and Schwartz (1978) [15]

[4]The Crank Nicolson method was first introduced into the contingent claim pricing literature by Courtadon (1982) [24]

equation for the θ-method is,

$$
\begin{aligned}
&1/2\,\frac{\sigma^2\,(W_{i+1,j+1}-2\,W_{i+1,j}+W_{i+1,j-1})}{(\Delta x)^2}\,(1-\theta)\\
&\quad+1/2\,\frac{\left(r-1/2\,\sigma^2\right)(W_{i+1,j+1}-W_{i+1,j-1})}{\Delta x}\,(1-\theta)\\
&\quad+\frac{W_{i+1,j}-W_{i,j}}{\Delta t}-rW_{i+1,j}\,(1-\theta)\\
&\quad=1/2\,\frac{\sigma^2\,(W_{i,j+1}-2\,W_{i,j}+W_{i,j-1})}{(\Delta x)^2}\theta\\
&\quad+1/2\,\frac{\left(r-1/2\,\sigma^2\right)(W_{i,j+1}-W_{i,j-1})}{\Delta x}\theta+rW_{i,j}\theta,
\end{aligned}
\tag{30}
$$

where $0\leq\theta\leq 1$. Wilmott, Dewynne and Howison (1993) [96] point out that this can be thought of as a θ weighted average of the explicit and fully implicit finite difference methods.

2.3.2 $\theta = 0$ Fully Explicit Finite Difference Method

When $\theta = 0$, the θ-method gives the fully explicit method. The fully explicit finite difference method has the disadvantage that it is only stable and convergent with the imposition of the restriction $\frac{\sigma^2\Delta t}{\Delta x^2}\geq\frac{1}{2}$. The restriction implies that in order to have the large number of asset price steps necessary for accurate prices ridiculously small time steps are required. The accuracy of the fully explicit method is $O(\Delta x^2+\Delta t)$.

The difference equation for the explicit finite difference method is,

$$
\begin{aligned}
&\frac{1}{2}\sigma^2\,\frac{(\mathrm{W}_{i+1,\,j+1}+\mathrm{W}_{i+1,\,j-1}-2\,\mathrm{W}_{i+1,\,j})}{(\Delta x)^2}\\
&\quad+(r-\frac{1}{2}\sigma^2)\,\frac{(\mathrm{W}_{i+1,\,j+1}-\mathrm{W}_{i+1,\,j-1})}{2\Delta x}\\
&\quad+\frac{\mathrm{W}_{i+1,\,j}-\mathrm{W}_{i,\,j}}{\Delta t}=r\,\mathrm{W}_{i,j}.
\end{aligned}
\tag{31}
$$

Collecting terms we arrive at the following equation,

$$
\begin{aligned}
\mathrm{W}_{i,\,j}=\frac{1}{(1+r\,\Delta t)}&\left(\frac{1}{4}\frac{(2\,\sigma^2\,\Delta t-\sigma^2\,\Delta x\,\Delta t+2\,r\,\Delta x\,\Delta t)}{(\Delta x)^2}\mathrm{W}_{i+1,\,j+1}\right.\\
&+\frac{1}{4}\frac{(-4\,\sigma^2\,\Delta t+4\,(\Delta x)^2)}{(\Delta x)^2}\mathrm{W}_{i+1,\,j}\\
&\left.+\frac{1}{4}\frac{(-2\,r\,\Delta x\,\Delta t+2\,\sigma^2\,\Delta t+\sigma^2\,\Delta x\,\Delta t)}{(\Delta x)^2}\mathrm{W}_{i+1,\,j-1}\right),
\end{aligned}
\tag{32}
$$

where $\frac{1}{(1+r\Delta t)}$ is an approximate discount term[5]. For fully explicit finite difference we can interpret the coefficients of the terms $W_{i+1,j+1}$, $W_{i+1,j}$ and $W_{i+1,j-1}$

[5] We now substitute this for the continuous discount term $exp^{(-r\Delta t)}$.

as the probabilities $p_{u,i,j}$, $p_{m,i,j}$ and $p_{d,i,j}$. Finally we have the relationship that,

$$W_{i,j} = exp^{-r\Delta t}\left(p_{u,i,j}W_{i+1,j+1} + p_{m,i,j}W_{i+1,j} + p_{d,i,j}W_{i+1,j-1}\right), \tag{33}$$

where,

$$p_{u,i,j} = \frac{1}{4}\frac{2\,\sigma^2\,\Delta t - \sigma^2\,\Delta x\,\Delta t + 2\,r\,\Delta x\,\Delta t}{(\Delta x)^2}, \tag{34}$$

$$p_{m,i,j} = \frac{1}{4}\frac{-4\,\sigma^2\,\Delta t + 4\,(\Delta x)^2}{(\Delta x)^2}, \tag{35}$$

and

$$p_{d,i,j} = \frac{1}{4}\frac{-2\,r\,\Delta x\,\Delta t + 2\,\sigma^2\,\Delta t + \sigma^2\,\Delta x\,\Delta t}{(\Delta x)^2}. \tag{36}$$

2.3.3 $\theta = 1$ Fully Implicit Finite Difference Method

When $\theta = 1$, the θ-method gives the fully implicit method. The accuracy of the fully implicit method is $O(\Delta x^2 + \Delta t)$. Unlike the fully explicit method the implicit method is unconditionally stable and convergent. The difference equation for the implicit finite difference method is,

$$\begin{aligned}&\frac{1}{2}\sigma^2\,\frac{(\mathrm{W}_{i,\,j+1} + \mathrm{W}_{i,\,j-1} - 2\,\mathrm{W}_{i,\,j})}{(\Delta x)^2}\\ &\quad + (r - \frac{1}{2}\,\sigma^2)\,\frac{(\mathrm{W}_{i,\,j+1} - \mathrm{W}_{i,\,j-1})}{2\Delta x} + \frac{\mathrm{W}_{i+1,\,j} - \mathrm{W}_{i,\,j}}{\Delta t} = r\,\mathrm{W}_{i,j}.\end{aligned} \tag{37}$$

Collecting terms we arrive at the following equation,

$$\begin{aligned}\mathrm{W}_{i+1,\,j} = &\left(\frac{1}{4}\frac{(\sigma^2\,\Delta x\,\Delta t - 2\,\sigma^2\,\Delta t - 2\,r\,\Delta x\,\Delta t)}{(\Delta x)^2}\mathrm{W}_{i,\,j+1}\right.\\ &+ \frac{1}{4}\frac{(4\,\sigma^2\,\Delta t + 4\,(\Delta x)^2 + 4r\,(\Delta x)^2\Delta t)}{(\Delta x)^2}\mathrm{W}_{i,\,j}\\ &\left.+ \frac{1}{4}\frac{(2\,r\,\Delta x\,\Delta t - 2\,\sigma^2\,\Delta t - \sigma^2\,\Delta x\,\Delta t)}{(\Delta x)^2}\mathrm{W}_{i,\,j-1}\right).\end{aligned} \tag{38}$$

The coefficients of $W_{i,j+1}$, $W_{i,j}$ and $W_{i,j-1}$ sum to $(1 + r\Delta t)$, an approximation of the inflation term[6]. The coefficients for implicit finite difference methods may not be interpreted as probabilities. Nevertheless we will factor out the inflation term from the coefficients so that they sum to unity. We now have the relationship,

$$W_{i+1,j} = exp^{r\Delta t}\left(P_{u,i+1,j}W_{i,j+1} + P_{m,i+1,j}W_{i,j} + P_{d,i+1,j}W_{i,j-1}\right), \tag{39}$$

where $P_{u,i+1,j}$, $P_{m,i+1,j}$ and $P_{d,i+1,j}$ represent the fully implicit coefficients.

[6] From now on we will replace this approximation with the continuous inflation term $exp^{r\Delta t}$.

2.3.4 $\theta = \frac{1}{2}$ Crank-Nicolson Finite Difference Method

When $\theta = \frac{1}{2}$, the θ-method gives the Crank-Nicolson method. The accuracy of the Crank-Nicolson method is $O(\Delta x^2 + \Delta t^2)$ and again it is an unconditionally stable and convergent method.

Setting $\theta = \frac{1}{2}$ in equation 30 gives,

$$
\begin{aligned}
1/4\,&\frac{\sigma^2\left(W_{i+1,j+1} - 2\,W_{i+1,j} + W_{i+1,j-1} + W_{i,j+1} - 2\,W_{i,j} + W_{i,j-1}\right)}{(\Delta x)^2} \\
&+ 1/4\,\frac{\left(r - 1/2\,\sigma^2\right)\left(W_{i+1,j+1} - W_{i+1,j-1} + W_{i,j+1} - W_{i,j-1}\right)}{\Delta x} \\
&+ \frac{W_{i+1,j} - W_{i,j}}{\Delta t} - r\left(1/2\,W_{i+1,j} + 1/2\,W_{i,j}\right) = 0, \qquad (40)
\end{aligned}
$$

which can be rewritten as,

$$
\begin{aligned}
&\left(1/4\,\frac{\Delta t r}{\Delta x} - 1/8\,\frac{\Delta t \sigma^2}{\Delta x} + 1/4\,\frac{\Delta t \sigma^2}{(\Delta x)^2}\right) W_{i+1,j+1} \\
&\quad + \left(1 - 1/2\,\frac{\Delta t \sigma^2}{(\Delta x)^2} - 1/2\,\Delta t r\right) W_{i+1,j} \\
&\quad + \left(1/4\,\frac{\Delta t \sigma^2}{(\Delta x)^2} + 1/8\,\frac{\Delta t \sigma^2}{\Delta x} - 1/4\,\frac{\Delta t r}{\Delta x}\right) W_{i+1,j-1} \\
&\quad + \left(1/4\,\frac{\Delta t r}{\Delta x} - 1/8\,\frac{\Delta t \sigma^2}{\Delta x} + 1/4\,\frac{\Delta t \sigma^2}{(\Delta x)^2}\right) W_{i,j+1} \\
&\quad + \left(-1/2\,\Delta t r - 1/2\,\frac{\Delta t \sigma^2}{(\Delta x)^2} - 1\right) W_{i,j} \\
&\quad + \left(1/4\,\frac{\Delta t \sigma^2}{(\Delta x)^2} + 1/8\,\frac{\Delta t \sigma^2}{\Delta x} - 1/4\,\frac{\Delta t r}{\Delta x}\right) W_{i,j-1} = 0. \qquad (41)
\end{aligned}
$$

If we separate the terms at time step i and $i+1$ onto either side of the equality and sum the coefficients of the W_i terms they come to $(1 + (r\Delta t)/2)$ which is an approximation to the inflation term[7]. Factoring out the inflation term from the coefficients and writing the latter as $P_{u,i+1,j}$, $P_{m,i+1,j}$ and $P_{d,i+1,j}$ we get,

$$
\begin{aligned}
&exp((r\Delta)t)/2)\left(P_{u,i+1,j}W_{i,j+1} + P_{m,i+1,j}W_{i,j} + P_{d,i+1,j}W_{i,j-1}\right) \\
&\quad = exp((r\Delta)t)/2) \times \\
&\quad \left(-P_{u,i+1,j}W_{i+1,j+1} - (P_{m,i+1,j} - 2)W_{i+1,j} - P_{d,i+1,j}W_{i+1,j-1}\right). \qquad (42)
\end{aligned}
$$

The RHS of the above Equation 42 is made up of known option prices (we know the option's payoff at maturity) and (once we have calculated them) known

[7]We now substitute the continuous time equivalent $exp^{(r\Delta t)/2}$.

constants $P_{u,i+1,j}$, $P_{m,i+1,j}$ and $P_{d,i+1,j}$; the RHS can therefore be considered as a known constant, $Z_{i+1,j}$,

$$exp((r\Delta)t)/2)\,(P_{u,i+1,j}W_{i,j+1} + P_{m,i+1,j}W_{i,j} + P_{d,i+1,j}W_{i,j-1}) = Z_{i+1,j}. \quad (43)$$

2.3.5 $\theta = \frac{1}{2} - \frac{\Delta x^2}{12\Delta t}$ Douglas Finite Difference Method

When $\theta = \frac{1}{2} - \frac{\Delta x^2}{12\Delta t}$, the θ-method gives the Douglas method. The accuracy of the Douglas method is $O(\Delta x^4 + \Delta t^2)$. For a deeper discussion of the Douglas and other θ weighted schemes see Morton and Mayers (1998) [71].

2.3.6 Multi-Dimensional Finite Difference Methods

Finite difference methods can also be used for problems with multiple stochastic processes. However, two stochastic process problems are tricky to implement and three stochastic process problems are very difficult. Anymore than three processes and one is forced to either reduce the dimensionality of the problem[8] or use Monte Carlo simulation.

There are numerous methods for implementing finite difference in multiple dimensions: locally one dimensional (LOD) methods, hop skotch and alternating direction implicit (ADI).

2.3.7 Finite Difference Option Valuation

Option valuation in the finite difference framework proceeds in a similar fashion to lattices. The initial boundary condition is known at maturity T where the European call option is worth $C_{T,j} = \max(S_{T,j} - X, 0)$. The system of linear equations is then solved to determine the vector of option prices at time step $(i-1, j)$ and so on by backward induction to time zero. However, with finite difference methods we have a grid rather than a tree so that there exist upper and lower boundary conditions which feature as the first and last of the equations in the linear system. For the European call option as $S \to \infty$ then $\frac{\partial C}{\partial S} \to 1$ and as $S \to 0$ then $\frac{\partial C}{\partial S} \to 0$.

Finite difference methods are typically used for solving the same kinds of problems as lattices. They can be used for American and European style contracts but it is difficult to extend there use to pricing path dependent contingent claims where the payoff depends on the past history of an underlying variable. As mentioned they can be used to solve problems with multiple state variables however, the grid then becomes multi-dimensional and there is a considerable increase in the required memory and computer processing time.

[8]Practitioners do not like estimating parameters and calibrating models with more that two stochastic processes.

2.4 Quadrature

For a call option the price C_t at $t = 0$ is given by,

$$C_t = E_t^{\tilde{P}} \left(e^{(-rT)} C_T \right), \tag{44}$$

where E_t is the expectation operator at time t and P is the risk-neutral probability measure. C_T is the value of the European option at maturity,

$$C_T = \max \left(S_T - X, 0 \right). \tag{45}$$

This can be rewritten as,

$$C_0 = \int_{-\infty}^{\infty} e^{-rT} g(S) \max \left(S_T - X, 0 \right) dS, \tag{46}$$

where $g(S)$ is the probability density function. Figures 3, 4 and 5 represent $g(S)$, $\max(S_T - X, 0)$ and $g(S)\max(S_T - X, 0)$, respectively. In the Black-Scholes model $g(S)$ is a lognormal distribution and therefore $ln(S)$ is normally distributed. Using this fact the integral can be solved to produce the Black-Scholes equation. In general however, we may have a distribution implied by market data which can only be integrated numerically.

Let us define a sequence of abscissas, denoted $x_0, x_1, \ldots, x_N, x_{N+1}$ which are spaced apart by a constant h such that $x_i = x_0 + ih$ where $i = 0, 1, \ldots, N+1$. A function $f(x_i)$ has known values at the x_i's, $f(x_i) \equiv f_i$. One wants to integrate the function $f(x)$ between a lower limit a and an upper limit b, where a and b are each equal to one or the other of the x_i's.

Classical quadrature routines such as the trapezoidal and Simpson's rule have the form,

$$\int_{x_1}^{x_N} f(x)dx = h\left[\frac{1}{2}f_1 + f_2 + f_3 + \ldots + f_{N-1} + \frac{1}{2}f_N\right] + O\left(\frac{1}{N^2}\right), \tag{47}$$

and

$$\int_{x_1}^{x_N} f(x)dx = h\left[\frac{1}{3}f_1 + \frac{4}{3}f_2 + \frac{2}{3}f_3 + \frac{4}{3}\ldots + \right. \tag{48}$$

$$\left. \frac{2}{3}f_{N-2} + \frac{4}{3}f_{N-1} + \frac{1}{3}f_N\right] + O\left(\frac{1}{N^4}\right), \tag{49}$$

respectively. In these classical quadrature routines the integral of the function is approximated by the sum of its functional values at a set of equally spaced points, multiplied by certain aptly chosen weighting coefficients. Gaussian quadrature routines give one the freedom to choose not only the weighting coefficients, but also the location of the abscissas at which the function can be evaluated. Thus one has twice the number of degrees of freedom. Another feature of Gaussian quadrature is that one can arrange the choice of weights and abscissas to make the integral exact for a class of integrands i.e., a polynomial

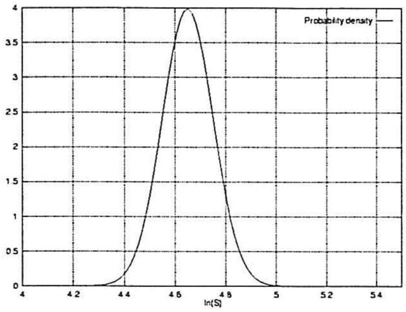

Figure 3: Probability Density Function $g(S)$.

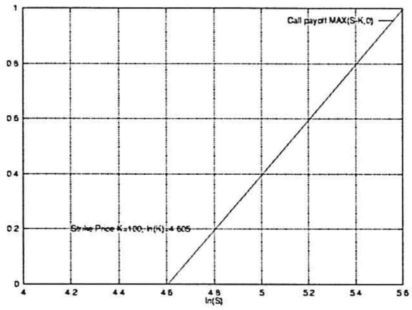

Figure 4: Call Payoff at Maturity $\max(S_T - X, 0)$

multiplied by some known function $W(x)$. The function $W(x)$ can be chosen to remove integrable singularities from the desired integral. Given $W(x)$ and an integer N, we can find a set of weights w_j and abscissas x_j such that the approximation,

$$\int_a^b W(x)f(x)dx \approx \sum_{j=1}^{N} w_j f(x_j), \tag{50}$$

is exact if $f(x)$ is a polynomial. For a given $W(x)$ one locates tabulated weights and abscissas in a book, see for example Stroud and Secrest (1966) [91].

Quadrature routines can be used to price European style fixed boundary problems. They require the contract parameters (the maturity T, the strike price K and the type, call), the model parameters (the volatility σ, the interest rate r and the underlying asset price S_0) and the numerical method parameters (the upper limit of integration a, the lower limit of integration b and either the number abscissas N or an error tolerance).

2.5 Monte Carlo Simulation

The value of an option is the risk-neutral expectation of its discounted pay-off, as stated in Equation 44. One can obtain an estimate of this expectation by

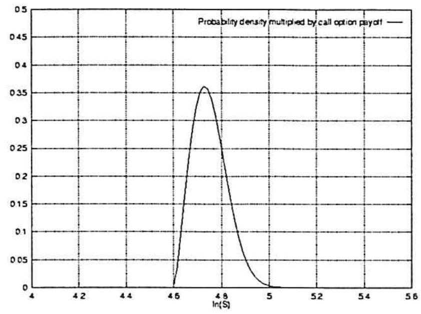

Figure 5: Integrand function $g(S) \times \max(S_T - X, 0)$

computing the average of a large number of discounted payoffs. For the example of a European call option one simulates the risk-neutral process for the state variable S from today at time zero until maturity at time T and then calculates its payoff $\max(S_T - X, 0)$. This calculation is then repeated M times to obtain many sample values for the expected payoff. One then calculates the mean of the sample to get an estimate of the expected payoff in a risk-neutral world. One then discounts this payoff at the risk-free rate r (which is constant in this example) to get an estimate of the value of the option.

In order to implement the Monte Carlo simulation one must simulate the geometric Brownian motion for the underlying asset S,

$$dS = rSdt + \sigma Sdz. \tag{51}$$

To simulate the path followed by S, the time to maturity is divided up into N short intervals of length Δt and the above equation is approximated by,

$$S(t + \Delta t) - S(t) = rS(t)\Delta t + \sigma S(t)\epsilon\sqrt{\Delta t}, \tag{52}$$

where ϵ is a random Gaussian deviate from a distribution with mean of 0 and standard deviation of 1.0[9]. This equation iterates the value of $S(t)$ along the path between time zero and T. One trajectory involves constructing a complete path for S using N random samples from a normal distribution.

[9]The most critical part of the Monte Carlo simulation algorithm is the generation of the Gaussian pseudo-random numbers, ϵ. The generation of pseudo-random numbers generally takes about 30% of the total execution time of the simulation. So relative execution time of the generator is very important. Generally speaking a standard uniform random number generator is employed which produces real numbers between zero and one. It is essential that the pseudo-random number generator produces serially uncorrelated numbers and that the repetition period of the sequence is greater than the number of deviates required. Care is needed as some generators are significantly better than others, see Press *et al* (1996) [76] for an excellent discussion of pseudo-random number generators and their testing. Repetition of the sequence produces serial correlation which will cause the prices generated to be biased. The standard uniform deviates are transformed into Gaussian deviates using the Box-Muller transformation or the polar rejection method. As the former transformation contains trigonometric functions it is slower than polar rejection which is therefore the preferred method.

The best way to simulate a variable following GBM is via the process for the natural logarithm of the variable which follows arithmetic Brownian motion and is normally distributed. Define $x = ln(S)$ then,

$$dx = (r - \frac{\sigma^2}{2})dt + \sigma dz, \tag{53}$$

which is approximated by,

$$x(t + \Delta t) = x(t) + (r - \frac{\sigma^2}{2})\Delta t + \sigma\epsilon\sqrt{\Delta t} \tag{54}$$

Monte Carlo simulation copes well with path dependent options and using the Cholesky decomposition (see Press *et al* [76])to generate correlated random variables it can be extended to pricing options dependent on multiple stochastic processes. Its major disadvantages are that it is computationally intensive[10] and that it cannot easily be extended to handle free boundary problems.

3 A Catalogue of Algorithms

The following catalogue of algorithms is not meant to be exhaustive but includes commonly used analytical formulae and generic numerical techniques. The analytical formulae are divided into two groups, those that extend the Black-Scholes model of an option on a stock to other instruments and underlying assets and those that extend the Black-Scholes model to other processes. The analytical formulae should really be thought of as a single algorithm class except where more complicated functions than the cumulative normal distribution function are employed. The division of contingent claim pricing algorithms into analytical and numerical forms is rather subjective. All algorithms require numerical procedures, for example when implementing the Black-Scholes model one needs a numerical evaluation for the cumulative normal distribution function and for options on the maximum or minimum of two risky assets one requires the bivariate distribution function. The names of models, commonly implemented with particular numerical approximation algorithms, are included.

- **Analytical formulae** are closed-form solutions.

  1. Extensions of the Black-Scholes (1973) [7] model of an option on a stock to other instruments and underlying assets:

     Arithmetic average-rate option approximation for example, Turnbull and Wakeman (1991) [94] and Curran (1992) [29].

     Barrier options for example, Rubinstein Reiner (1991) [78], Bhagavatula and Carr (1995) [2], Geman and Yor (1996) [41], Ikeda and Kunitomo (1992) [60], Heynen and Kat (1994c) [50], Heynen and Kat (1994a) [48] and Hart and Ross (1994) [45].

[10] Various variance reduction techniques have been developed to reduce the computing time for Monte Carlo simulation. These include the antithetic variable technique, control variates, importance sampling, stratified sampling and quasi-random sequences.

Binary options for example, Reiner and Rubinstein (1991) [79] and Heynen and Kat (1996) [51].

Chooser options (simple) for example, Rubinstein (1991c) [85].

Constant dividend yield stock options for example, Merton (1973) [68].

Currency options for example, Grabbe (1984) and Garman and Kohlhagen (1984) [40].

Currency translated options for example, Derman, Karasinski and Wecker (1990) [33], Reiner (1992) [77] and Dravid, Richardson and Sun (1993) [34].

Discrete barrier option approximation for example, Broadie, Glasserman and Kou (1995) [20].

Exchange options (an option to exchange one asset for another) for example Margrabe (1978) [67], Zhang (1994) [97] and Bjerksund and Stensland (1993) [3].

Executive stock options for example, Jennergren and Nasulund (1993) [62].

Forward start options for example, Rubinstein (1990) [83].

Futures options for example, Black (1976) [4][11] and Schaefer and Schwartz (1987) [89].

Geometric average options for example, Kemna and Vorst (1990) [64].

Lookback options for example, Goldman, Sosin and Gatto (1979) [44], Garman (1987) [39], Conze and Viswanthan (1991) [23] and Heynen and Kat (1994b) [49].

Options on the min. and the max. of two or more risky assets for example, Stulz (1982) [92], Johnson(1987) [63], Boyle, Evnine and Gibbs (1989) [11], Boyle and Tse (1990) [12], Rubinstein (1991d) [86] and Rich and Chance (1993) [81].

Time switch options for example, Pechtl (1995) [74].

2. Generalizations of the Black-Scholes (1973) model to other stochastic processes: Haug (1997) [46] states that the valuation of options on assets that are assumed to follow stochastic processes other than Brownian motion has received attention mainly only by academics and that few of these alternative processes are used by practitioners. One explanation is that most of these models require additional input parameters that are not directly observable in financial markets. They must therefore be estimated using various statistical techniques. Haug asserts that the additional accuracy offered by several of these

[11] Although Black (1976) was originally derived for pricing forwards and futures it is also widely used by practitioners to price caps, floors, swaptions and bond options. Its assumption of underlying lognormality is theoretically inconsistent when pricing both a cap and a swaption however its popularity suggests that this inconsistency is negligible in an economic sense. Black (1976) should only be used for short maturity bond options as it does not model the pull to par affect. Schaefer and Schwartz (1987) in their model adjust volatility with duration to capture the pull to par affect.

techniques is outweighed by the complexity of estimating the additional input parameters required.

Compound options by Geske (1979).

Constant volatility by Merton (1973).

Displaced diffusion by Rubinstein (1983) [82].

Jumps in the asset price level by Merton (1976) [69].

One-factor term structure models for example, Vasicek (1977) [95] and Cox, Ingersoll and Ross (1985) [26].

Constant elasticity of variance for example, Cox-Ross (1976) [27] and Cox (1996) [25].

- **Analytical approximations** Analytical approximation algorithms are quasi-analytical algorithms which have a closed-form solution but include a numerical approximation (typically a Newton-Raphson algorithm to find the location of a free boundary).

American options for example, Johnson (1983), Geske and Johnson (1984) [43], Macmillan (1986) and Barone-Adesi-Whaley (1987) [1].

Chooser options (complex) [12] for example, Rubinstein (1991c) [85].

Compound options [13] for example, Geske (1979) [42], Hodges and Selby (1987) [54] and Rubinstein (1991a) [84].

Exchange options (an option to exchange one option for another) for example, Carr (1988) [21].

Extendible maturity options for example, Longstaff (1990) [65].

- **Trees / Lattices**

  1. Binomial algorithms:

     Standard binomial for example, Cox, Ross and Rubinstein (1979) [28] or Rendleman and Bartter (1979) [80] based on the random walk approximation to the Brownian motion.

     Accelerated binomial by Breen (1991) [14] uses Richardson's extrapolation technique to reduce the number of steps.

     Black-Scholes formula 1 step before expiration by Broadie and Detemple (1996) [18].

     Binomial average The average price of Nth and $N+1th$ step tree.

     Binomial one-factor interest rate models [14] for example, Ho and Lee (1986) [52] and Black, Derman and Toy (1990) [5].

[12] The simple chooser option has the same strike for the call and put while for the complex chooser option the call and the put can have different strikes.

[13] A compound option can either be considered as a compound option on an asset following a Geometric Brownian Motion or a simple option on an asset with a complex process. As such it also appears in the section on generalizing the Black-Scholes model to other processes.

[14] The Ho and Lee routine has parameters which can be fitted to the yield curve (time-inhomogeneous) in which case the model is implemented in a tree. Alternatively the parameters can be kept constant (time-homogeneous) in which case an explicit closed form solution exists. The Black, Derman and Toy model does not have explicit closed form solutions.

Implied binomial for example, Derman and Kani (1994) [32] and Rubinstein (1994) [87].

Non-recombining or bushy binomial for example, a one-factor Heath, Jarrow and Morton (1992) [47] model.

2. Trinomial algorithms:

   Standard Trinomial for example, Boyle (1986) [9] and Parkinson (1977) [73].

   Trinomial one-factor interest rate models for example, Hull and White (1990) [57] and Black and Karasinski (1991) [6][15].

   Implied Trinomial for example, Dupire (1994) [36].

   Non-recombining or bushy trinomial for example a one-factor Heath, Jarrow and Morton (1992) [47] model.

3. Multi-Dimensional tree algorithms for example Boyle (1988) [10], Boyle, Evnine and Gibbs (1989) [11] and Rubinstein (1994) [88]:

   Two-factor interest rate models for example, Brennan and Schwartz (1982) [17], Fong and Vasicek (1991) [38] and Longstaff and Schwartz (1992) [66].

   Two-factor convertible bonds models for example, Ho and Pfeffer (1996) [53].

   Two-factor non-recombining or bushy binomial and trinomial models for example a two-factor Heath, Jarrow and Morton (1992) [47] model.

- **Partial differential equations**

1. Fully Explicit Finite Difference, first introduced into the finance literature by Brennan and Schwartz (1978) [15].
2. Fully Implicit Finite Difference, first introduced into the finance literature by Brennan and Schwartz (1978) [15].
3. Crank Nicolson Finite Difference, first introduced into the finance literature by Courtadon (1982) [24].
4. Alternating Direction Implicit, see Morton and Myers (1998) [71]:

   Two-factor interest rate models for example, Brennan and Schwartz (1982) [17], Fong and Vasicek (1991) [38] and Longstaff and Schwartz (1992) [66].

   Two-factor convertible bond models for example, Brennan and Schwartz (1980) [16].

5. Fourier Methods.

[15]The Hull and White routine has parameters which can be fitted to the yield curve (time-inhomogeneous) in which case it is modeled in a tree. Alternatively the parameters can be kept constant (time-homogeneous) in which case explicit closed form solutions exist. The Black and Karasinski model does not have an explicit closed form solution.

- **Quadrature** see for example, Press, Teukolsky, Vetterling and Flannery (1995) [76] and Parkinson (1977) [73].
    1. Trapezoidal Rule.
    2. Simpson's Rule.
    3. Romberg Integration.
    4. Gaussian Quadrature, for example in the integration of a non-central χ^2 distribution in CIR (1985) [26].
- **Monte Carlo simulation** was first introduced into the finance literature by Boyle (1977) [8].
    1. Antithetic Variates.
    2. Control Variates see for example, Clewlow and Caverhill (1994) [22].
    3. Importance Sampling.
    4. Stratified Sampling see for example Curran (1994) [30] and Moro (1985) [70].
    5. Quasi Random Numbers or Low Discrepancy Sequences (Faure, Sobol Halton) for example, Papageorgiou and Traub (1996) [72].
    6. Multi-Dimensional simulations using Cholesky decomposition.

       Multi-Factor Interest Rate Models for example, Heath, Jarrow and Morton (1992) [47] and Brace, Gatarek and Musiela (1997) [13].
- **Non-parametric methods**
    1. Neural Networks for example, Hutchinson and Poggio (1994) [58].

4 Algorithm Properties

4.1 Classification of Algorithms by their Suitability for Pricing Different Instruments, the Number of Stochastic Processes they can Accommodate and their Input Parameters.

The following section lists a number of common pricing algorithms. The maximum number of stochastic processes which the algorithm can accommodate is stated. For the analytical formulae algorithms the particular contingent claim which they price is detailed whereas for the numerical algorithms it is merely stated whether they can price American contingent claims or just European.

The motivation here is to classify analytical and analytical approximation algorithms by their suitability for pricing different instruments and to classify generic numerical algorithms by their ability to handle multiple stochastic processes. The later property is important if one needs to model multiple underlying assets, for example with a Rainbow option and / or more complicated assumptions about the nature of the world for example stochastic volatility or stochastic interest rates.

For the analytical formulae and the analytical approximations where it is stated that the maximum number of stochastic processes is 2 the algorithm makes uses of Magrabe's trick of using a change of numeraire. This reduces the problem from one of modeling two stochastic processes S_1 and S_2 to one of modeling one stochastic process $\frac{S_1}{S_2}$.

4.1.1 Glossary of Notation

- Contract Parameters

  X the strike price in the domestic currency.

  X^* the strike price in the foreign currency.

  X_i the ith strike price in the domestic currency where $i = 1, 2, \ldots, N$.

  T the maturity.

  E_P the predetermined exchange rate specified in units of domestic currency per units of the foreign currency.

  H the barrier level.

  H_u the upper barrier level.

  H_l the lower barrier level.

  δ_1, δ_2 determine the curvature of the barriers.

  Q a cash amount in domestic currency.

  Q_1 the quantity of the first asset in domestic currency.

  Q_2 the quantity of the second asset in domestic currency.

  t some predefined time such that $t \leq T$.

- Model Parameters

  S the initial underlying asset price in the domestic currency.

  S^* the initial underlying asset price in the foreign currency.

  S_i the initial underlying asset price of the ith asset in the domestic currency where $i = 1, 2, \ldots, N$.

  F the initial forward or futures price in the domestic currency.

  E the spot exchange rate specified in units of the domestic currency per unit of foreign currency.

  E^* the spot exchange rate specified in units of the foreign currency per unit of the domestic currency.

  r the domestic interest rate.

  r_f the foreign interest rate.

  q the continuous dividend rate of the underlying stock.

q_i the continuous dividend rate of the ith underlying stock where $i = 1, 2, \ldots, N$.

σ, σ_S the volatility of the underlying asset.

σ_i the volatility of the ith underlying asset where $i = 1, 2, \ldots, N$.

σ_E the volatility of the domestic exchange rate.

σ_{E^*} the volatility of the foreign exchange rate.

ρ the correlation between the two assets S_1 and S_2.

ρ_{ES^*} the correlation between the asset and the domestic exchange rate.

$\rho_{E^*S^*}$ the correlation between the asset and the foreign exchange rate.

- Numerical Algorithm Parameters

Δt the size of the algorithm time step.

Δx the size of the algorithm asset price step.

Δx_u the size of the algorithm up asset price step.

Δx_d the size of the algorithm down asset price step.

a the upper limit of integration.

b the lower limit of integration.

Ni the number of time steps.

x_j the position of the abscissas.

w_j the weight attached to each abscissa.

N_j the number of abscissas.

M the number of simulations.

4.1.2 Analytical Formulae (Black-Scholes Modeling Assumptions)

All algorithms require the contingent claim contract specifications as input parameters; initial underlying asset level S, strike price X, maturity T, interest rate r, volatility σ and other contract specifications for example, the barrier level H, the number of fixing dates, etc.

Black and Scholes (1973) Prices European stock options. Models one stochastic process for the underlying stock price. Input parameters S, X, T, r and σ.

Merton (1973) Prices European index options. Models one stochastic process for the underlying index price. Input parameters S, X, T, r, q and σ.

Black (1976) Prices European options on futures. Models one stochastic process for the underlying futures price. Input parameters F, X, T, r and σ.

Garman and Kohlhager (1983) Prices European currency options. Models one stochastic process for the underlying exchange rate. Input parameters S, X, T, r, r_f and σ.

Rubinstein (1990) Prices European forward start options. Models one stochastic process for the underlying asset. Input parameters $S, X, T, r, q, \sigma, \alpha$ and t, where α determines by how much the option is in or out of the money and t is when the forward start option commences.

Rubinstein (1991) Prices European simple chooser options. Models one stochastic process for the underlying asset price. Input parameters S, X, T, r, q, σ and t, where t is the time when the option is chosen to be a standard call or a put with maturity T.

Margrabe (1978) Prices European exchange option to exchange one asset for another. Models the two stochastic processes S_1 and S_2 for the underlying asset prices as the single process $\frac{S_1}{S_2}$. Input parameters $S_1, S_2, Q_1, Q_2, T, r, q_1, q_2, \sigma_1, \sigma_2$ and ρ.

Stulz (1982) and Johnson (1987) Models the two stochastic processes S_1 and S_2 for the underlying asset prices as the single process $\frac{S_1}{S_2}$. Prices European options on the maximum or the minimum of two risky assets. Input parameters $S_1, S_2, X, T, r_1, r_2, q_1, q_2, \sigma_1, \sigma_2$ and ρ.

Goldman, Sosin and Gatto (1979) and Garman (1989) Prices European floating strike lookbacks. Models one stochastic process for the underlying asset. Input parameters S, T, r, q, σ and S_{min} or S_{max} where S_{min} is the lowest price observed and S_{max} is the highest price observed.

Conze and Viswanathan (1991) Prices European fixed strike lookbacks. Models one stochastic process for the underlying asset. Input parameters S, X, T, r, q, σ and S_{min} or S_{max}.

Heynen and Kat (1994) Prices European partial time floating strike lookbacks. Models one stochastic process for the underlying asset. $S, \lambda, T, t, r, q, \sigma$ and S_{min} or S_{max} where the time up to t is the lookback period and T is the maturity of the option. λ enables the creation of a fractional lookback option where the strike is fixed at some percentage above or below the extremum.

Heynen and Kat (1994) Models one stochastic process for the underlying asset. Prices European partial time fixed strike lookbacks. Input parameters S, X, T, t, r, q, σ and S_{min} or S_{max} where the time after t is the lookback period and T is the maturity of the option.

Reiner and Rubinstein (1991) Prices European standard barrier options. Models one stochastic process for the underlying asset. Input parameters S, X, H, T, r, q and σ.

Geman and Yor (1996) and Ikeda and Kunitomo (1992) Prices European double barrier options. Models one stochastic process for the underlying asset. Input parameters $S, X, H_u, H_l, T, r, q, \delta_1, \delta_2$ and σ.

Heynen and Kat (1994) Prices European partial time (start or end) single asset barrier options. Models one stochastic process for the underlying asset. Input parameters S, X, H, t, T, r, q and σ, where t is either the beginning or the end of the monitoring period for the barrier depending on whether it is a partial time end barrier option or a partial time start barrier option.

Heynen and Kat (1994) European double asset barrier options. Models two stochastic processes one that models how much the option is in or out of the money and one that is linked to barrier hits. Input parameters S_1, S_2, X, H, T, r, q and σ, where S_1 determines how much the option is in or out-of-the-money, and the other asset, S_2, is linked to barrier hits.

Broadie, Glasserman and Kou (1995) Models one stochastic process for the underlying asset. Prices European discrete barrier options. Input parameters $S, X, r, T, H_D, H, \Delta t$ and σ, where H is the level of the continuous barrier in any continuous barrier options formulae, H_D is the corrected discrete barrier level and Δt is the time between monitoring instants.

Reiner and Rubinstein (1991) Prices gap binary options. Models one stochastic process for the underlying asset. Input parameters S, X_1, X_2, T, r, q and σ where the binary is triggered by X_1 and the size of the payoff is a function of X_2 and S.

Reiner and Rubinstein (1991) Prices cash-or-nothing binary options. Models one stochastic process for the underlying asset. Input parameters S, X, Q, T, r, q and σ.

Heynen and Kat (1996) Two-asset cash-or-nothing binary options. Models two stochastic processes for the underlying assets. Input parameters $S_1, S_2, X_1, X_2, Q, T, r, q, \sigma$ and ρ.

Cox and Rubinstein (1985) Prices asset-or-nothing options. Models one stochastic process for the underlying asset. Input parameters S, X, T, r, q and σ.

Reiner and Rubinstein (1991) Prices binary barrier options. Models one stochastic process for the underlying asset. Input parameters S, X, H, T, r, q and σ.

Kemna and Vorst (1990) Prices geometric average-rate options. Models one stochastic process for the underlying asset. Input parameters S, X, T, r, q and σ.

Turnbull and Wakeman (1991) Prices arithmetic average-rate options (approximation). Models one stochastic process for the underlying asset. Input parameters S, X, T, r, q and σ.

Levy's Approximation (1992) Prices arithmetic average-rate options (approximation). Models one stochastic process for the underlying asset. Input parameters S, X, T, r, q and σ.

Curran's Approximation (1992) Prices arithmetic average-rate options (approximation). Models one stochastic process for the underlying asset. Input parameters $S, X, T, t, \Delta t, n, r, q$ and σ, where t is the time to the first averaging point, Δt is the time between averaging points and n is the number of averaging points.

Reiner (1992) Foreign equity options struck in domestic currency. Input parameters $S^*, X, T, r, q, E, E^*, \sigma_{E^*}, \sigma_{S^*}, \sigma_E, \rho_{ES^*}$ and $\rho_{E^*S^*}$.

Reiner (1992) Fixed exchange rate foreign equity options. Input parameters $S^*, X^*, T, r, r_f q, E_p, E^*, \sigma_{S^*}, \sigma_E$ and ρ.

Reiner (1992) Equity linked foreign exchange options. Input parameters S^*, X, $T, r, r_f q, E, E^*, \sigma_{S^*}, \sigma_E$ and ρ.

4.1.3 Analytical Formulae (Alternative Processes to GBM)

Merton (1976) Models a diffusion and Poisson process for the underlying asset. Input parameters $S, X, T, r, \sigma, \lambda$ and γ, where λ is the expected number of jumps per year and γ the percentage of the total volatility explained by the jumps.

Rubinstein (1983) Displaced diffusion model. Input parameters $S, X, T, r, q, \sigma_R, \alpha$ and β, where σ_R is the volatility of the risky assets, α is the proportion of the total value of the firm invested in the risky asset and β is the firm's debt to equity ratio.

Cox and Ross (1976) Constant elasticity of variance model. Input parameters S, X, T, r, σ and α, where α is the degree of inverse proportionality of the stock price volatility to the stock price.

4.1.4 Analytical Approximations

These algorithms price contingent claims which have payoffs that are the functions of choices and are therefore free boundary problems. A Newton-Raphson algorithm is used to determine the location of the free boundary and thus the optimal choice.

Geske and Johnson (1984) Prices European and American options. Models one stochastic process for the underlying asset. Input parameters S, S^*, X, T, r, q and σ where S^* is the critical commodity price which is determined to an acceptable tolerance by the Newton-Raphson algorithm.

Barone-Adesi-Whaley (1987) Prices European and American call and puts. Models one stochastic process for the underlying asset. Input parameters

S, S^*, X, T, r, q and σ where S^* is the critical commodity price which is determined to an acceptable tolerance by the Newton-Raphson algorithm.

Rubinstein (1991) Prices European complex chooser options. Models one stochastic process for the underlying asset. Input parameters $S, X_1, X_2, T_1, T_2, I, r, q, \sigma$ and t, where t is the time when the option is chosen to be a standard call with maturity T_1 and strike X_1 or a standard put with maturity T_2 and strike X_2. I is the critical price which is determined to an acceptable tolerance by the Newton-Raphson algorithm.

Geske (1977) Prices European compound options. Models one stochastic process for the underlying asset. Input parameters $S, X_1, X_2, I, T, r, q, \sigma$ and t, where X_1 is the strike price of the underlying option, X_2 is the strike of the option on the option, T is the time to maturity of the underlying option and t is the time to maturity of the option on the option. I is the critical commodity price which is determined to an acceptable tolerance by the Newton-Raphson algorithm.

Longstaff (1990) Prices European options with extendible maturities. Models one stochastic process for the underlying asset. Input parameters $S, X_1, X_2, A, I_1, I_2, T, r, q, \sigma$ and t, where t is the time where the holder of the option can choose to extend its maturity to T for an additional premium A. The strike price can be adjusted from X_1 to X_2 at t. I_1 and I_2 are critical values of S which are determined by Newton-Raphson and control whether the put will be extended or exercised.

Carr (1988) Prices European option to exchange one option for another. Models the two stochastic processes S_1 and S_2 for the underlying asset prices as the single process $\frac{S_1}{S_2}$. Input parameters $S_1, S_2, Q, I, t, T, r, q_1, q_2, \sigma_1, \sigma_2$ and ρ, where t is the maturity of the original option and T is the maturity of the underlying option. Q is the quantity of asset S_2 if exercised. σ_1 and σ_2 are the volatilities of the two assets with correlation ρ. I is a unique critical price ratio determined by Newton-Raphson.

4.1.5 Numerical Approximations

These algorithms price contingent claims that are too complicated to be priced using analytical or analytical approximation algorithms. For example, the complication can be that the contingent claim has a free boundary or is modeled using multiple stochastic processes. The input parameters are the contract specifications, the model specifications and the numerical approximation algorithm specifications.

Trees / Lattices These algorithms can price both European and American options. They can be generalized to model ≤ 3 stochastic processes. The number of time steps N_i and the maturity of the contingent claim T imply the size of the time steps $\frac{T}{N_i} = \Delta t$. Each binomial algorithm requires the parameters u, d and p to be specified. The initial boundary condition is given by the contingent claim payoff at maturity.

- **Cox, Ross and Rubinstein** $u = exp\left(\sigma\sqrt{\Delta t}\right)$, $d = exp\left(-\sigma\sqrt{\Delta t}\right)$ and $p = \frac{1}{2} + \frac{r-\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}$. The $E[S_T]$ is correct only as $\Delta t \to 0$.
- **Jarrow and Rudd** $u = exp\left(\left(r - 1/2\sigma^2\right)\Delta t + \sigma\sqrt{(\Delta t)}\right)$, $d = exp\left(\left(r - 1/2\sigma^2\right)\Delta t - \sigma\sqrt{(\Delta t)}\right)$ and $p = 1/2$. The $E[S_T]$ is correct only as $\Delta t \to 0$.
- **Hull and White** $u = exp\left(\sigma\sqrt{\Delta t}\right)$, $d = exp\left(-\sigma\sqrt{\Delta t}\right)$ and $p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$. The $E[S_T]$ is correct only as $\Delta t \to 0$.
- **Trigerogis** $\Delta x_u = \Delta x_d = \sqrt{\sigma^2\Delta t + (r - \frac{1}{2}\sigma^2)^2\Delta t^2}$ and $p_u = \frac{1}{2} + \frac{1}{2}\frac{(r-\frac{1}{2}\sigma^2)\Delta t}{\Delta x}$. The $E[S_T]$ is correct for any Δt.
- **Equal Probabilities** $\Delta x_u = \frac{1}{2}(r-\frac{1}{2}\sigma^2)\Delta t + \frac{1}{2}\sqrt{4\sigma^2\Delta t - 3(r - \frac{1}{2}\sigma^2)^2\Delta t^2}$, $\Delta x_d = \frac{3}{2}(r-\frac{1}{2}\sigma^2)\Delta t - \frac{1}{2}\sqrt{4\sigma^2\Delta t - 3(r - \frac{1}{2}\sigma^2)^2\Delta t^2}$ and $p = 1/2$. The $E[S_T]$ is correct for any Δt.
- **Trinomial Trees** $p_u = \frac{1}{2}\left(\frac{\sigma^2\Delta t + (r-d-\frac{\sigma^2}{2})^2\Delta t^2}{\Delta x^2} + \frac{(r-d-\frac{\sigma^2}{2})\Delta t}{\Delta x}\right)$, $p_m = 1 - \frac{\sigma^2\Delta t + (r-d-\frac{\sigma^2}{2})^2\Delta t^2}{\Delta x^2}$ and $p_d = \frac{1}{2}\left(\frac{\sigma^2\Delta t + (r-d-\frac{\sigma^2}{2})^2\Delta t^2}{\Delta x^2} - \frac{(r-d-\frac{\sigma^2}{2})\Delta t}{\Delta x}\right)$. For stability the choice of Δx cannot be made independently of Δt, $0 < \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$.

Finite difference methods The number of time steps N_i and the maturity of the contingent claim T imply the size of the time steps $\frac{T}{N_i} = \Delta t$. The number of asset price steps N_j and the position of the boundary conditions imply the size of the asset price steps Δx. The boundary and initial conditions are required for the contingent claim.

- **Fully explicit finite difference** Prices European and American options. Models 1 stochastic process. $p_u = \frac{1}{2}\left(\frac{\sigma^2}{\Delta x^2} + \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$, $p_m = 1 - \Delta t\frac{\sigma^2}{\Delta x^2}$ and $p_d = \frac{1}{2}\left(\frac{\sigma^2}{\Delta x^2} - \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$. For stability $0 < \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$.
- **Fully implicit finite difference** Models 1 stochastic process. Prices European and American options. $p_u = -\frac{1}{2}\left(\frac{\sigma^2}{\Delta x^2} + \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$, $p_m = 1 + \Delta t\frac{\sigma^2}{\Delta x^2}$ and $p_d = -\frac{1}{2}\left(\frac{\sigma^2}{\Delta x^2} - \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$. The algorithm is unconditionally stable.

- **Crank-Nicolson finite difference** Models 1 stochastic process. Prices European and American options. $p_u = -\frac{1}{4}\left(\frac{\sigma^2}{\Delta x^2} + \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$, $p_m = 1 + \Delta t \frac{\sigma^2}{2\Delta x^2}$ and $p_d = -\frac{1}{4}\left(\frac{\sigma^2}{\Delta x^2} - \frac{(r-d-\frac{1}{2})}{\Delta x}\right)\Delta t$. The algorithm is unconditionally stable.
- **Alternating direction implicit finite difference (2-Dimensional)** Prices European and American options. It can model ≤ 3 stochastic processes. The six probabilities (3 probabilities for each of the 2 planes representing the stochastic processes S_1 and S_2) are $p_u = \frac{\lambda_1}{2\Delta y_1^2}$, $p_m = -\frac{2}{\Delta t} - \frac{\lambda_1}{\Delta y_1^2}$, $p_d = \frac{\lambda_1}{2\Delta y_1^2}$, $p_u = \frac{\lambda_1}{2\Delta y\prime_2^2}$, $p_m = -\frac{2}{\Delta t} - \frac{\lambda_1}{\Delta y\prime_2^2}$ and $p_d = \frac{\lambda_1}{2\Delta y\prime_2^2}$ where λ_1 is the eigenvalue of the first stochastic process S_1. $y_1 = e_{11}x_1 + e_{12}x_2$ where e_{11} and e_{12} is the first eigenvector, $x_1 = ln(S_1)$ and $x_2 = ln(S_2)$. $y\prime_2 = \sqrt{\frac{\lambda_1}{\lambda_2}}y_2$ where λ_2 is the eigenvalue of the second stochastic process S_2 and $y_2 = e_{21}x_1 + e_{22}x_2$ where e_{21} and e_{22} is the second eigenvector. The algorithm is unconditionally stable.

Quadrature (numerical integration) Prices European options. Models ≤ 3 stochastic processes. All the quadrature algorithms require the lower a and upper b limits of integration for the function $f(x)$ to be integrated (which is the product of the probability density function of the underlying asset and option payoff at maturity).

- **Trapezoidal Rule** The number of abscissas N_j or alternatively a desired accuracy threshold are required. If a threshold of accuracy is stated then the number of abscissas is increased until the desired accuracy is attained.
- **Simpson's Rule** The number of abscissas N_j or alternatively a desired accuracy threshold are required. If a threshold of accuracy is stated then the number of abscissas is increased until the desired accuracy is attained.
- **Romberg Integration** The number of abscissas N_j or alternatively a desired accuracy threshold are required. If a threshold of accuracy is stated then the number of abscissas is increased until the desired accuracy is attained.
- **Gaussian Quadrature** For Gaussian quadrature the position x_j of the abscissas as well as the weights w_j must be computed. The input parameters remain the lower a and upper b limits of integration for the function $f(x)$ to be integrated and the number of abscissas N_j (for N_j-point quadrature).

Monte Carlo Simulation Prices European options. Models N stochastic processes. The number of time steps N_i and the maturity T of the contingent claim imply the size of the time steps $\frac{T}{N_i} = \Delta t$. The number of simulations M or the statistical threshold of accuracy (standard error) are required.

- **Antithetic Variates** The algorithm requires a good set of uniform random deviates (i.e. a pseudo random number generator that is capable of producing n uncorrelated statistically random numbers where $n > N_i \times M$ that is to say the number of deviates required $N_i \times M$ is less than the period or maximal length n of the pseudo random number generator). The uniform deviates are transformed into normal (Gaussian) deviates ϵ_i using the Box-Muller algorithm. The antithetic deviate is then just the negative $-\epsilon_i$ of the deviate ϵ_i.
- **Control Variates** A suitable hedge is required as a control variate. This can be a delta, gamma and / or vega hedge but it can also be a static style hedge. For example, if one wanted to use Monte Carlo simulation to price a European arithmetic Asian option then a good approximate hedge would be a European geometric Asian option for which a closed form solution exists (an algorithm to price the hedge is required.) The algorithm also requires a good set of uniform random deviates, as above.
- **Low Discrepancy Sequences** A quasi random number (low discrepancy sequence) generating algorithm requires the length of the deterministic sequence $N_i \times M$ as an input. The deterministic sequence is then drawn from at random, the uniform deviates are transformed into normal (Gaussian) deviates ϵ_i and used to construct the Monte Carlo trajectories as normal.

Non-Parametric Methods Prices European and American options. Models N stochastic processes. It requires a large number of contract prices C for different market input parameters S_0, K, T, r, σ. Neural networks and other non-parametric pricing methods essentially involve non-linear regressions or various interpolation methods such as splines to recover an unknown pricing function $y = f(\bar{x})$ given a historical data set $(\bar{x}_t, y_t)$ and some smoothness constraint. Once the network has been "trained" or fitted to the data it is applied to out of sample data in order to determine the unknown price.

4.2 Calibration as an Implicit Algorithm Parameter

It is common for practitioners to use models which are calibrated to the implied volatility of liquid vanilla options trading on the same underlying asset when pricing exotic equity, equity index and currency options. The model then has the desirable feature that it can reproduce the prices of the vanilla options. The price of the exotic option is thus determined relative to the vanilla options. The implied binomial and trinomial trees of Derman and Kani (1994), Rubinstein (1994) and Dupire (1994) are the best examples of this kind of model. With this type of calibrated model the distinction between the model and the algorithm is blurred.

Similarly, interest rate contingent claims are normally priced with models which are calibrated to the yield curve and sometimes also its volatility, these are implied from bond and swap prices (the later being a very liquid market). Examples of this kind of model are Ho and Lee (1986), Hull and White (1990), Black, Derman and Toy (1990), Black and Karasinski (1991), Brennan and Schwartz (1982), Fong and Vasicek (1991), Longstaff and Schwartz (1992) and Heath, Jarrow and Morton (1992). Recent so called market models are calibrated to indices like LIBOR as this avoids the problem of infinitesimal short rates which can occur with HJM, an example of this kind of model is Brace, Gatarek and Musiela (1997). Furthermore, credit models and convertible bond models may be calibrated to the spread between the yield curve of a risky bond and a riskless bond or to the classifications of the credit rating agencies like Standard and Poor's or Moody's.

Calibration is thus an implicit parameter of these hybrid model-algorithms. However, an issue arises as implied volatilities and interest rate yields are often required for strikes and maturities which are not traded. This then becomes a problem of interpolation and extrapolation and there is little consensus amongst academics and practitioners about which is the best method to use. Thus it is possible for two banks to price an identical contingent claim using the same model and the same input data but to reach different prices as a result of different methods of calibration. Our contingent claim prices have thus become conditional on the data we calibrate to and the method of calibration, this is not robust theoretically or empirically. Moreover, if the calibration data contains stale prices then one may calibrate on noise and thus over-fit the date.

The historical procedure for devising new models has tended to involve looking at the available market data and then choosing a model with a parameterization such that the model spans all the data without redundancy or arbitrage. Derman (1996) [31] summarizes this methodology succinctly when he states the following "recipe" for pricing derivatives:

- Model the deterministic and stochastic behavior of the hedging instruments as realistically as you can.
- Within the model, adjust any unknown parameters that represent the user's views of the future (growth rates, dividend yields and so on) to ensure that the model reproduces the market values of all the known hedging instruments.
- Finally, use the same model, with these parameters fixed, to value the derivative security.

Tables 1 and 2 list common hybrid model-algorithms by their author, the algorithm they are based upon, the types of contingent claim they price and the instruments on which they are calibrated.

Table 1: Algorithms / 1-factor models which have calibration data as an implicit input

Derivative Security	*Model*	*Algorithm*	*Hedging Instruments Whose Values Constrain the Model Parameters*
Exotic equity options	Derman and Kani (1994)	Binomial tree.	Equity, S, riskless bond r and standard options of all strikes and expirations $c(K,T)$.
Exotic equity options	Rubinstein (1994)	Binomial tree.	Equity, S, riskless bond r and standard options of all strikes and one expiration date $c(K)$.
Exotic equity options	Dupire (1994)	Trinomial tree.	Equity, S, riskless bond r and standard options of all strikes and expirations $c(K,T)$.
Interest rate options contingent on the level of the yield curve.	Ho and Lee (1986)	Binomial tree.	Bonds, swaps, etc. that form the yield curve $r(t)$.
Interest rate options contingent on the level of the yield curve.	Hull and White (1990)	Trinomial tree.	Bonds, swaps, etc. that form the yield curve $r(t)$.
Interest rate options contingent on the level of the yield curve.	Black, Derman and Toy (1990)	Binomial tree	Bonds, swaps, etc. that form the yield curve $r(t)$.
Interest rate options contingent on the level of the yield curve.	Black and Karasinski (1991)	Trinomial tree	Bonds, swaps, etc. that form the yield curve $r(t)$.

Table 2: Algorithms / N-factor models which have calibration data as an implicit input.

Derivative Security	*Model*	*Algorithm*	*Hedging Instruments Whose Values Constrain the Model Parameters*
Interest rate options contingent on different parts of the yield curve.	Brennan and Schwartz (1982)	2-Dimensional tree or alternating direction finite difference.	Bonds, swaps, etc. that form the yield curve $r(t)$ and the consol bond long rate l.
Interest rate options contingent on different parts of the yield curve	Fong and Vasicek (1991)	2-Dimensional tree or alternating direction finite difference.	Bonds, swaps, etc. that form the yield curve $r(t)$ and the square root of the volatility of the spot rate $\sqrt{\xi}$ (not observable).
Interest rate options contingent on different parts of the yield curve.	Longstaff and Schwartz (1992)	2-Dimensional tree or alternating direction finite difference.	Bonds, swaps, etc. that form the yield curve $r(t)$.
Interest rate options contingent on different parts of the yield curve	Heath, Jarrow and Morton (1992)	It can be modeled in one-factor form in a bushy tree or as an N-factor form using Monte Carlo simulation.	Bonds, swaps, etc. that form the yield curve $r(t)$ and their volatilities $\nu(t)$ (obtained via principal component analysis).
Interest rate options contingent on different parts of the yield curve	Brace, Gatarek and Musiela (1997)	It can be modeled in one-factor form in a bushy tree or as an N-factor form using Monte Carlo simulation.	An observable index for different maturities $r(t)$ e.g. LIBOR. and their volatilities $\nu(t)$ (obtained via principal component analysis)
Convertible bonds	Brennan and Schwartz (1977)	2-Dimensional tree or alternating direction finite difference.	Equity S, bonds, swaps, etc. that form the yield curve $r(t)$ for the corporate bond.

4.3 Model and Algorithm Isomorphism

"Do we arrive at very different contingent claim prices and Greek values for the same instrument as we increase model complexity?" It is a waste of computer resources using a rich multi-factor model if a simple one-factor model gives the same price. In a hypothetical trading system when one is faced with a new contingent claim which one wishes to model then the above question can be answered in a number of ways:

- Do the sums on the run and compare performance of different models at the end. This is a simple approach but computationally expensive.
- Use prior experience stored in an expert system to select the model. This just requires a large computer lookup table but assumes that all possible contingent claims have been previously characterized. There is therefore little scope for novel situations without human intervention.
- Use answers from expert knowledge for a related situation. This is a very flexible solution but requires the computer to have genuine artificial intelligence which is a non-trivial software task.

The complexity of the model (and therefore algorithm) one chooses to use should be a function of the contingent claim being priced and the assumptions one makes about the world.

An interest rate contingent claim, for example a European bond option, has a payoff which is only a function of one point on the term structure so it is superfluous to use a 2 or 3 factor term structure model to capture the correlation between different interest rates. The contingent claim is not sensitive to this correlation and therefore a one factor model, which assumes all rates are perfectly correlated, will suffice. Single factor and multi-factor models may well produce isomorphic results for this contingent claim. However, if one has a spread option on different maturities of the term structure then although the single factor model will be fast it will also be mis-specified for this contingent claim. The single and multi-factor models are not likely to be isomorphic in this case.

If both an implied volatility tree and Reiner and Rubinstein (1991) are used to price a European barrier option then as long as one makes similar modeling assumptions for both i.e. volatility is constant, the monitoring of the barrier is continuous and that the tree has a large number of time steps (large enough to approximate continuity) then both models should be isomorphic. In which case Reiner and Rubinstein should be the model of choice as it is less computationally intensive. However, if one assume that volatility is not constant across strikes and maturity and if monitoring of the barrier is weekly then, as barrier options are sensitive to the underlying volatility, one would not be surprised if the models were not isomorphic. In this instance given the assumptions about the nature of the world it would be wiser to use the implied tree prices as Rubinstein and Reiner is not correctly specified. An expert system, designed to select the optimal algorithm (in terms of computational efficiency) for a given contingent claim, would need to be "taught" that the fastest algorithm (the closed form solution) is not the most accurate solution in this case. That is to say that the continuous closed form models are not isomorphic with the discrete numerical models.

Whether two algorithms are isomorphic or not will be a function of the contingent claim being priced and whether it is sensitive to a modeling factor or assumption that is present in one algorithm and absent in the other.

5 Conclusions

Analytical formulae can typically only model one stochastic process except where a change of numeraire is employed to transform two stochastic processes S_1 and S_2 into one stochastic process $\frac{S_1}{S_2}$. As input parameters they require the contract specifications (such as whether the option is a call or put, the maturity and the strike price) and model specifications (such as the volatility, the interest rate and the dividend rate). Analytical formulae can only value fixed boundary problems but the algorithms require little memory and are fast.

Analytical approximations have similar properties to the analytical formulae except that they can price free boundary problems. They require at least one input parameter for the numerical method. This is typically an accuracy level or tolerance for a Newton-Raphson root finding algorithm. The root finding makes these algorithms slower than the analytical formulae although still a lot quicker than the numerical methods.

Lattices and PDEs can price contingent claims with free boundary problems and up to 3 stochastic processes. They require the contract and valuation model input parameters. Lattices require the initial conditions, the number of time steps and also a decision as to whether the up and down probabilities or jump sizes will be equal. PDEs require the initial and boundary conditions, the number of time steps and the number of asset price steps. Trinomial trees and explicit finite difference methods also have stability and convergence criteria which must be satisfied. Lattices and PDEs can require a lot of computer memory especially if they are being used to model multiple stochastic processes or if the lattices are non-recombining. They are slower than analytical solutions and analytical approximations but considerably faster than Monte Carlo simulations.

Quadrature algorithms can only price fixed boundary problems and are only feasible for low numbers of dimensions (i.e., stochastic processes), for higher dimensions Monte Carlo simulation is superior. Quadrature algorithms require the contract parameters, and the valuation model parameters. Additional inputs are the upper and lower limits of integration and either the number of abscissas or the desired accuracy threshold.

Monte Carlo simulation can only be used with fixed boundary problems but is the best algorithm for high dimensional situations. The algorithm inputs are the contract parameters, the valuation model parameters, the number of time steps and either the number of simulations or the statistical threshold of accuracy (the standard error). Monte Carlo simulations tend to be the slowest of all the algorithms as it is often necessary to have tens of thousands of simulations to achieve the pricing accuracy required.

The greatest difficulty in algorithm selection comes with contingent claims which are free boundary problems and where more than 3 stochastic processes must be modeled. This is a non-trivial set of problems and there appears to be little consensus amongst academics and practitioners as to what algorithms to employ. Probably the most important work on this to date is that of Broadie, Glasserman and Jain(1997) [19] in which they extend the Monte Carlo framework to accommodate free boundary problems. However, contingent claims of this complexity remain difficult to price accurately. Often practitioners make model simplifications to reduce the dimensionality of the problem so that it can be priced using one of the above algorithms. This is still a rich area for future algorithm development.

Calibration to market date creates hybrid model-algorithms. It can be argued that because of different interpolation methods and different calibration data sets these model-algorithms do not produce very robust results. The development of new models is often a pragmatic process based on the creation of a framework which is arbitrage free conditional on the available data as opposed to a theoretical framework initially independent of the market data. Tests of the robustness of results from calibrated models like BGM are much needed as are comparisons of model and algorithm isomorphism.

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