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Real options problems have recently attracted much attention worldwide. One such problem is how to deal with claims on 'untraded' assets. Often there is another traded asset which is correlated to the untraded asset, and this traded asset is used as a proxy for hedging purposes.

We introduce a second (untraded) log Brownian asset into the well known Merton investment model with power-law utility. The investor has a claim on units of the untraded asset and the question is how to price and hedge this random payoff. The presence of the second Brownian motion means that we are in the situation of incomplete markets. We propose an approximation to the solution for the 'optimal' reservation price and hedge which is accurate when the position is small in comparison to wealth. The resulting loss when a suboptimal proxy strategy is followed is shown to be approximately quadratic in $1-\rho$.

JEL D52, G11, G13

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1 Introduction

In the Merton investment model (Merton [30]) the agent seeks to maximise expected utility of terminal wealth, where utility is constant relative risk aversion, $U(x) = \frac{x^{1-R}}{1-R}$. Under the assumption of a log-Brownian share, the optimal behaviour for an agent in the model is well known: a constant proportion of wealth is invested in the risky asset.

Now introduce a second log-Brownian asset into this scenario, on which no trading is allowed. Suppose the investor has an option on this second asset, payable at time T. The problem is how to price and hedge this random payoff when trading in the second asset is not permissible. The presence of the second Brownian motion means that we are in an incomplete markets situation and replication is not possible. The risk that arises from being unable to hedge perfectly in this situation is often referred to as 'basis risk'.

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This is a mathematical idealisation of a so-called 'real options' problem. Of course in practice, the two assets are specified in reverse; the agent expects to receive an unhedgeable claim on an asset, and chooses a correlated asset with which to hedge. We are particularly interested in the case where the correlation between the two asset price processes is close to 1. An example given by Davis [10] is an option on Dubai oil, where the liquid market is in Brent crude. Other examples are a portfolio of illiquid shares hedged with index futures, or a five year futures contract hedged with a one year futures contract.

In finance, the term 'real options' covers a wider range of problems, see the recent article by Dunbar [19] and the book by Dixit and Pindyck [14]. Busby and Pitts [1] describe real options as focusing on the value of managerial flexibility in handling real asset investments. Some examples of real options problems include extraction rights to an oil reserve or the option to start up an R&D venture. Often development can be immediate, delayed or abandoned, until either a fixed date, or an open period in the future. More pertinent to our analysis of real options is the study by Johnson and Tian [24] of executive stock options. These are options on the stock of the company, and are given to executives as part of their compensation package. However, frequently executives are not permitted to trade away the risk using the stock or derivatives on the stock, so that they are essentially receiving options on an untraded asset.

This paper will consider the specific real options problem of an option on an untraded asset and will use the modelling framework outlined in the opening paragraphs. A starting point in solving this problem might be to investigate pricing using only the assumption of no arbitrage. However, Huberlek and Schachermeyer [23] showed that this gives no information about the price of the claim, leading us to consider a utility based approach.

For a general utility function it is possible to characterise optimal hedging behaviour in terms of the value function, see for example Duffie [15, Chapter 8]. Duffie et al [16] attempt to characterise solutions in a Markovian model and Cuoco [4] considers a more general model. Zariphopoulou [37] uses separable utility to model prices with additional stochastic factors. In order to determine the value function it is necessary to specify a particular utility. Duffie and Jackson [17] and Svensson and Werner [34] each consider a number of simple examples and Duffie and Richardson [18] find explicit solutions under a quadratic utility.

A second method used to attack utility maximisation problems is to convert via the 'dual approach' into a minimisation over measures. Papers using such an approach include Pliska [31], Karatzas et al [26], Cox and Huang [3], Karatzas et al [27], Xu and Shreve [36], Cvitanic and Karatzas [5], [6], Teplá [35] and in a more abstract setting, Kramkov and Schachermeyer [29], Delbaen et al [13], Schachermeyer [33] and Cvitanic et al [7].

An attractive utility for which explicit solutions can sometimes be found is the exponential utility which has constant absolute risk aversion. The choice of exponential utility yields a separation of the value function into wealth and trading components which makes it particularly tractable. Davis [10] applies the dual approach to untraded assets with this utility. With lognormal asset prices he obtains an expression for the optimal hedge involving the solution to a non-linear pde. Hobson [21] took the primal approach to the same problem and also obtained the hedge as a solution to a non-linear pde.

In this paper we consider agents with constant relative risk aversion or equivalently a Cobb-Douglas or power-law utility. It seems there is no closed form solution for the general utility maximisation problem facing the agent in our model. Consequently we make two simplifying assumptions. Firstly we assume that the option on the untraded asset is a multiple λ of units of the share, and secondly we assume that λ is small, or rather that the position in the untraded asset is small compared with wealth. Under these assumptions, we obtain a series expansion for the optimal hedge and reservation price. See Rogers [32] for a recent use of such expansion techniques in a liquidity problem. Since our payoff is proportional to the asset price, we may use scaling in the style of Davis and Norman [11] to reduce the dimension of the problem.

Although our series expansion is only an approximation to the true solution we can use it both to value claims and more importantly to examine the sensitivity of options prices to changes in parameter values. Without a series expansion we would be forced to examine the problem numerically, and it would be much more difficult to interpret our results.

The remainder of the paper is organised as follows. Section 2 outlines the original Merton problem for later reference and to establish notation. The next section sets up our model with an additional untraded asset and defines the value function for the problem. We consider a power law utility of the form $U(x) = x^{1-R}/(1-R)$. It should be noted that the case $R = 1$ corresponds to logarithmic utility. All our pricing and hedging results extend to this utility function on substitution of $R = 1$. The complete markets case when the untraded asset can be perfectly replicated is treated in Section 4. Section 5 considers the incomplete case and we give an expansion for the value function of the agent as well as for her reservation price and the optimal strategy. In Section 6 we give a discussion of the results obtained from the model and in Section 7 we compare our results with exponential utility. In the penultimate section we consider the effect of hedging with a suboptimal strategy. Section 9 concludes, and an appendix contains the proofs relating to the expansion and approximation.

2 The Merton Investment Problem

The classical Merton wealth problem involves an agent investing in a risky share with price P, growth rate μ and volatility σ, and a riskless bank account with constant interest rate r. Here σ and μ are constants and we assume $r = 0$ for simplicity. This is equivalent to using discounted variables.

The agent chooses to invest the cash amount θ_t in the risky share and starting with initial wealth x, his wealth evolves as

$$dX_t = \theta_t \frac{dP}{P} = \theta_t(\sigma dB_t + \mu dt) \tag{1}$$

where B is a standard Brownian motion. The agent's aim is to maximise expected utility of terminal wealth

$$\sup_{\{\theta_t, 0 \le t \le T\}} \mathbb{E} U(X_T) \tag{2}$$

with $T > 0$, a fixed horizon. A closely related problem is to maximise expected utility of consumption over a finite or infinite horizon, see Svensson and Werner [34] and the references therein.

In (2) the supremum is taken over a suitable class of trading strategies. In this case we mean that θ is previsible with respect to the agent's filtration, or equivalently the filtration generated by the Brownian motion (Brownian motions in later sections). Further θ is such that $\mathbb{E}[\int_0^T \theta_s^2 ds] < \infty$; See Karatzas and Shreve [28, Chapter 5.8].

We will primarily consider utilities with constant relative risk aversion of the form $U(x) = \frac{x^{1-R}}{1-R}$ for $R > 0$, $R \neq 1$. (The corresponding utility for $R = 1$ is logarithmic utility.) For this choice of family of utility functions, utility is only defined for positive wealth. Hence, the class of admissible strategies θ is the set of adapted strategies for which the corresponding wealth process, defined via (1) is non-negative.

Let

$$V(t,x) = \frac{x^{1-R}}{1-R} \exp\left\{ \tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{(1-R)}{R}(T-t) \right\}. \tag{3}$$

Then, applying Itô's formula to $V(t, X_t)$ we find, with $m = \mu^2/2\sigma^2 R$,

$$\begin{aligned} dV &= \left(\frac{dX}{X} - \frac{R}{2}\frac{(dX)^2}{X^2} - \frac{\mu^2}{\sigma^2 R}dt \right) X^{1-R} e^{m(1-R)(T-t)} \\ &= X^{1-R} e^{m(1-R)(T-t)} \left(\frac{\theta_t}{X_t}\sigma dB - \frac{R}{2}\left(\frac{\mu}{\sigma R} - \frac{\theta\sigma}{X_t}\right)^2 dt \right) \end{aligned} \tag{4}$$

so that V is a supermartingale for any strategy θ and a martingale for the optimal strategy $\theta = \frac{\mu}{\sigma^2 R} X_t$. In particular,

$$V(t,x) = \sup_\theta \mathbb{E}_t V(T, X_T) = \sup_\theta \mathbb{E}_t U(X_T)$$

so that V as given in (3) is the value function for the utility maximisation problem.

Note that if $\pi_t = \frac{\theta_t}{X_t}$ is the proportion of wealth invested in the risky asset, then

$$\pi_t = \frac{\mu}{\sigma^2 R} \tag{5}$$

which is constant, the so called 'Merton proportion', see Merton [30]. In particular in a complete market an agent with constant relative risk aversion has a simple optimal strategy.

In the special case where $R = 1$, and utility is logarithmic, we find

$$V(t,x) = \log x + \tfrac{1}{2}\frac{\mu^2}{\sigma^2}(T-t).$$

3 Merton with an additional untraded asset

The purpose of this section is to introduce another risky share with price S on which no trading is allowed. We seek the answer to the question: how best to price and hedge a random payoff on this untraded asset? In this article we are interested in the problem where the random payoff is units of the underlying asset itself.

Assume S follows

$$\frac{dS}{S} = \nu dt + \eta dW^0$$

where W^0 is a Brownian motion and ν, η are constants. We assume W^0 is correlated to the Brownian motion B, the process driving P, with correlation ρ. It is convenient to think of W^0 as a linear combination of two independent Brownian motions B and W. Thus

$$W_t^0 = \rho B_t + \sqrt{1-\rho^2} W_t.$$

For $|\rho| < 1$ the presence of a second Brownian motion W, and the fact that no trading is allowed on S, means that we are in an incomplete market situation.

In this problem, our agents' aim is to maximise expected utility of wealth, where, in addition to funds generated by trading, the agent is to receive λ units of the share S. The value function of the agent is given by

$$V(t, X_t, S_t; \lambda) = \sup_{\theta_u} \mathbb{E}[U(X_T + \lambda S_T) | \mathcal{F}_t] \tag{6}$$

which is a modification of (2) to include the additional payoff. Here $X_T = X_t + \int_t^T \theta_u (dP_u/P_u)$ for some adapted θ which is constrained to ensure that $X_T + \lambda S_T > 0$ almost surely. The definition of the wealth process X reflects the fact that the only strategies available to the agent involve investment in the share P alone, and these strategies must be used to hedge the position in S. Again θ_u is the cash amount invested in the asset P at time u.

When λ is positive it is not too difficult to show that V exists in $(-\infty, \infty)$. Firstly note that, for $\lambda > 0$, we have $V(t, X_t, S_t; \lambda) \geq V(t, X_t, S_t; 0)$, which we determined in the previous section. Secondly we can get a simple upper bound for V by considering the dual problem; we show later that

$$V(t, X_t, S_t; \lambda) \leq V(t, X_t, S_t; 0) \left(1 + \frac{1-R}{x} \lambda \mathbb{E}^0(S_T)\right)$$

where $\mathbb{P}^0$ is the minimal martingale measure which makes the price process P into a martingale without affecting the Brownian motion W. See the appendix.

However when $\lambda < 0$ (and $|\rho| < 1$) we have that V is identically minus infinity. (This problem is common to many utility functions.) This is because the potential obligation λS_T is unbounded, and no hedging strategy can completely remove this risk. Henceforth we will assume that $\lambda > 0$; this corresponds to an investor who is receiving a positive number of units of the claim S_T.

The main purpose in finding the value function is that it can be used to find the price that the agent is prepared to pay for the claim λS_T. Given an initial, time 0, wealth of x, the reservation price is the solution to the equation $V(0, x-p, S_0; \lambda) = V(0, x, S_0; 0)$, see Hodges and Neuberger [22], Davis [8], Davis [9], Davis et al [12], Constantinides and Zariphopoulou [2] and Hobson [21]. In other words the reservation price is the price which leaves the agent indifferent (in the sense that his expected utility is unchanged) between paying p to receive the claim λS_T, and doing nothing.

4 The complete markets case

If $\rho = 1$ then $W^0 = B$ and we have a complete market, with only one source of risk. In this case, the price and hedge may be computed directly, as there will be a unique martingale measure. We discuss this case as a useful point of comparison for the next section, the incomplete case. If $\rho = 1$ then $dS/S = \eta dB + \nu dt$ and the share prices S and P are related by

$$\frac{dS}{S} = \frac{\eta}{\sigma}\frac{dP}{P} + \left(\nu - \frac{\mu\eta}{\sigma}\right) dt$$

or equivalently

$$S_t = S_0 e^{ct} \left(\frac{P_t}{P_0}\right)^{\frac{\eta}{\sigma}} \tag{7}$$

where $c = \nu - \frac{\eta}{\sigma}\mu + \frac{1}{2}\eta(\sigma - \eta)$. Further, since S can now be replicated using P, it must follow that the change of measure which makes P into a martingale, must also make S into a martingale. Hence $\nu = \eta\mu/\sigma$.

Now consider the utility maximisation problem (6). By writing X and S in terms of the Brownian motion B and considering the new wealth variable $Y_t = X_t + \lambda S_t$ we can solve the problem explicitly in this case. Y solves

$$dY = \left(\theta + \frac{\lambda S_t \eta}{\sigma}\right) \sigma dB + \left(\theta + \frac{\lambda S_t \eta}{\sigma}\right) \mu dt = \tilde{\theta}_t(\sigma dB + \mu dt),$$

where $\tilde{\theta} = (\theta + \lambda S_t \eta/\sigma)$, and the agent seeks to maximise $\mathbb{E}U(Y_T)$. This is exactly the usual Merton problem as stated in Section 2, with a modified strategy. So, from the results of Section 2, the optimal $\tilde{\theta}$ is $\frac{\mu}{\sigma^2 R} Y_t$ and θ^*, the optimal amount of cash invested in P, is

$$\theta^* = \frac{\mu}{\sigma^2 R} Y_t - \frac{\eta}{\sigma}\lambda S_t = \frac{\mu}{\sigma^2 R} X_t + \lambda S_t \left(\frac{\mu}{\sigma^2 R} - \frac{\eta}{\sigma}\right). \tag{8}$$

We can also write down the value function for this problem:

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= \mathbb{E}[U(Y_T)|\mathcal{F}_t] = \frac{Y_t^{1-R}}{1-R} e^{(1-R)m(T-t)} \\ &= \frac{X_t^{1-R}}{1-R} e^{(1-R)m(T-t)} \left(1 + \frac{\lambda S_t}{X_t}\right)^{(1-R)} \end{aligned}$$

and use this formula to find the price the investor is prepared to pay to receive λS_T at time T. Following the arguments at the end of the previous section, the price is $p = \lambda S_0$ independently of the initial wealth x. This is the expected value of the claim under the risk neutral measure, as is to be expected as we are in a complete market.

5 The incomplete case

We return to the case of interest, when $|\rho| < 1$. As described earlier, the market is incomplete as the position in S cannot be replicated with P. To begin with, we look for solutions when the size of the claim λS_T is small relative to current wealth x. Later we will investigate the common practice of attempting to hedge the claim when the true correlation is high using the $\rho = 1$ hedge given in (8).

The reason that we concentrate on the case where the claim is proportional to the share price S_T is that this allows us to exploit scalings within the problem to reduce the dimensionality by one. Specifically the value function can be written

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= \sup_\theta \frac{1}{(1-R)} \mathbb{E}_t \left(X_t^{1-R} \left(\frac{X_T}{X_t} + \frac{\lambda S_t}{X_t} \frac{S_T}{S_t} \right)^{1-R} \right) \\ &= \frac{X_t^{1-R}}{(1-R)} g\left(T-t, \frac{S_t}{X_t}\right) \\ &= \frac{X_t^{1-R}}{(1-R)} g(T-t, Z_t) \end{aligned} \tag{9}$$

where $g(0, z) = (1+\lambda z)^{1-R}$ and $Z_t = S_t/X_t$. Using Itô on V and the fact that V is a supermartingale under any strategy θ, and a martingale under the optimal strategy, gives

$$\begin{aligned} \frac{X_t^{1-R}}{(1-R)} \Bigg[& -\dot{g} + \nu z g_z + \tfrac{1}{2}\eta^2 z^2 g_{zz} \\ & +\theta \left(\frac{\mu}{X_t}(1-R)g - \frac{z}{X_t} g_z(\mu + R\sigma\eta\rho) - \frac{z^2\sigma\eta\rho}{X_t} g_{zz} \right) \\ & +\theta^2 \left(-\tfrac{1}{2}\frac{g}{X_t^2}(1-R)R\sigma^2 + \frac{z g_z R\sigma^2}{X_t^2} + \tfrac{1}{2}\frac{\sigma^2 z^2}{X_t^2} g_{zz} \right) \Bigg] \leq 0. \end{aligned}$$

Optimising over θ and substituting this value into the above equation gives the pde:

$$\begin{aligned} & (-\dot{g} + \tfrac{1}{2} g_{zz} z^2 \eta^2 + g_z z \nu) \\ & -\tfrac{1}{2} \frac{(\sigma\rho\eta(-g_{zz}z^2 + z g_z(-R)) + \mu(-z g_z + (1-R)g))^2}{\sigma^2(2z g_z R + g_{zz} z^2 - g(1-R)R)} = 0 \end{aligned}$$

with boundary condition $g(0, z) = (1 + \lambda z)^{1-R}$.

Given the form of the boundary condition, it is convenient to use the substitution $g(\tau, z) = h(\tau, z)^{1-R}$ where τ is defined to be $\tau = T - t$, giving

$$-\dot{h} + \tfrac{1}{2} z^2 \eta^2 \left(h_{zz} - R\frac{h_z^2}{h} \right) + z\nu h_z - \tfrac{1}{2}\frac{1}{\sigma^2} \left(\frac{E(h, h_z, h_{zz})^2}{D(h, h_z, h_{zz})} \right) = 0 \tag{10}$$

where $E = -z^2\sigma\rho\eta h_{zz} + z^2\sigma\rho\eta R(\frac{h_z^2}{h}) + z h_z(-\mu - R\sigma\rho\eta) + \mu h$ and $D = z^2 h_{zz} - (\frac{h_z^2}{h}) z^2 R + 2Rz h_z - Rh$.

We are interested in solutions for the problem where the exposure is small relative to current wealth, or equivalently when λz is small. Accordingly, we consider solutions of the form

$$h(\tau, z) = A(\tau) + \lambda z B(\tau) + \lambda^2 z^2 C(\tau) + \cdots$$

We show in an appendix that such an expansion exists, at least when $\lambda > 0$.

Proposition 5.1 *The coefficients in the above expansion for the value function $h(\tau, z)$ are:*

$$\begin{aligned} A(\tau) &= e^{m\tau} \\ B(\tau) &= e^{m\tau}e^{\delta\tau} \\ C(\tau) &= -\tfrac{1}{2}R\left(\frac{\eta^2(1-\rho^2)}{\Gamma}\right)e^{m\tau}e^{2\delta\tau}\left[e^{\Gamma\tau}-1\right] \end{aligned}$$

with $m = \frac{1}{2}\frac{\mu^2}{\sigma^2 R}$, $\delta = \nu - \frac{\mu\rho\eta}{\sigma}$, $\Gamma = \eta^2 - \frac{2\mu\rho\eta}{\sigma R} + \frac{\mu^2}{\sigma^2 R^2}$.

Remark 5.2 We can place interpretations on these parameters as follows. $(1-R)m$ is the rate of growth of the the expected utility in the Merton problem with no random endowment. δ is the rate of growth of $\mathbb{E}^0(S_t)$, or equivalently the expected rate of growth of the claim under the *minimal* martingale measure. Finally, Γ is (the square of) the volatility of the process formed by the ratio of S and X^0 where X^0 is the optimal wealth process when $\lambda = 0$. Sometimes a more convenient expression for Γ is $\Gamma = \eta^2(1-\rho^2) + (\eta\rho - \mu/\sigma R)^2$.

Proof of Proposition 5.1: If we substitute $h(\tau, z) = A(\tau)$ into the equation (10) above and neglect terms of order λ and above we find that A solves the differential equation

$$\dot{A} = \frac{\mu^2}{2\sigma^2 R}A.$$

Further, from the boundary condition we want the solution for which $A(0) = 1$.

To find higher order terms we proceed inductively. Letting $h(\tau, z) = A(\tau) + \lambda z B(\tau)$, and ignoring terms of order λ^2 or above, we find that the constant terms cancel, and that the terms of order λ give an equation

$$\dot{B} = \left((\nu - \frac{\mu\rho\eta}{\sigma}) + \frac{1}{2}\frac{\mu^2}{\sigma^2 R}\right)B.$$

The solution with $B(0) = 1$ is $B(\tau) = e^{(m+\delta)\tau}$.

Similarly we use an expansion to order λ^2 to find a differential equation for C:

$$\begin{aligned} \dot{C} &= C(\tau)\left(\frac{1}{2}\frac{\mu^2}{\sigma^2 R} + 2(\nu - \frac{\mu\rho\eta}{\sigma}) + \eta^2 - \frac{2\mu\rho\eta}{\sigma R} + \frac{\mu^2}{\sigma^2 R^2}\right) \\ & \quad -\tfrac{1}{2}\eta^2 R(1-\rho^2)e^{(2\nu - 2\frac{\mu\rho\eta}{\sigma} + \frac{1}{2}\frac{\mu^2}{\sigma^2 R})\tau}, \end{aligned}$$

which, when combined with boundary condition $C(0) = 0$ has solution

$$C(\tau) = -\left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma}\right)e^{m\tau}e^{2\delta\tau}\left[e^{\Gamma\tau}-1\right].$$

Higher order terms can be found in a similar fashion. □

We can now compute the value function. Define

$$\begin{aligned} V_2(t, X_t, S_t; \lambda) &= \frac{X_t^{1-R}}{1-R}e^{(1-R)m\tau}\Bigg(1 + \lambda\frac{S_t}{X_t}e^{\delta\tau} \\ & \quad - \lambda^2\frac{S_t^2}{X_t^2}\left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma}\right)e^{2\delta\tau}(e^{\Gamma\tau}-1)\Bigg)^{(1-R)} \end{aligned} \tag{11}$$

Theorem 5.3 *For $\lambda > 0$ the value function $V(t, X, S; \lambda)$ is given by*

$$V(t, X_t, S_t; \lambda) = V_2(t, X_t, S_t; \lambda) + O(\lambda^3) \tag{12}$$

Proof:
This result follows from Proposition 5.1, provided that the expansion is valid. We provide a full proof of Theorem 5.3 in the appendix.

□

We can also calculate the expansion for the reservation price, p the agent would be willing to pay for λ units of the asset S. This involves solving

$$\begin{aligned} X_t^{1-R} &= (X_t - p)^{1-R} \left[1 + \frac{\lambda S_t}{X_t - p} e^{\delta\tau} \right. \\ & \left. - \frac{\lambda^2 S_t^2}{(X_t - p)^2} \left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma} \right) e^{2\delta\tau} [e^{\Gamma\tau} - 1] + O(\lambda^3) \right]^{1-R}. \end{aligned}$$

Theorem 5.4 *The time t price p for λ units of S delivered at time T, given a current wealth X_t is:*

$$p(t, X_t, S_t; \lambda) = p = \lambda S_t e^{\delta\tau} - \lambda^2 \frac{S_t^2}{X_t} \left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma} \right) e^{2\delta\tau} \left[e^{\Gamma\tau} - 1 \right] + O(\lambda^3) \tag{13}$$

Note that when $\rho = 1$ we recover the price in the complete market case, provided $\delta = 0$. However, as was argued in the previous section, in a complete market $\delta = 0$ is necessary to preclude arbitrage.

Finally in this section we give an expression for the optimal strategy θ^*, the optimal cash amount invested in the traded asset.

Theorem 5.5 *The expansion for θ^* is*

$$\begin{aligned} \theta^*(t, X_t, S_t; \lambda) &= \frac{\mu X_t}{\sigma^2 R} + S_t \lambda e^{\delta\tau} \left(-\frac{\eta\rho}{\sigma} + \frac{\mu}{\sigma^2 R} \right) \\ & \quad - \lambda^2 \frac{S_t^2}{X_t} \frac{2(1+R)}{R} \left(\frac{\eta\rho}{\sigma} - \frac{\mu}{\sigma^2 R} \right) \left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma} \right) e^{2\delta\tau} \\ & \quad \times \left[e^{\Gamma\tau} - 1 \right] + O(\lambda^3) \end{aligned}$$

The first term in the expansion for θ^* is the wealth multiplied by the Merton proportion given in (5). In contrast to the Merton result, for $\lambda \neq 0$, we have that optimal θ is not a constant proportion of wealth. If $\rho = 1$ we recover the expression in (8) which was obtained directly. Whether the agent's strategy is to invest more (or less) than the Merton proportion of her wealth in the traded asset depends on the sign of $(\rho R - \mu/\sigma\eta)$.

6 Discussion

If we consider the reservation price for the random payment of λS_T, as given in Theorem 5.4, and convert it into a unit price, we find

$$\frac{p}{\lambda} = S_t e^{\delta\tau} - \lambda \frac{S_t^2}{X_t} \left(\frac{\frac{1}{2}\eta^2 R(1-\rho^2)}{\Gamma} \right) e^{2\delta\tau} \left[e^{\Gamma\tau} - 1 \right] + O(\lambda^2).$$

The 'marginal' price of a derivative is the price at which diverting a little money into the derivative at time zero, has a neutral effect on the achievable utility. This is given by

$$\lim_{\lambda \downarrow 0} \frac{p}{\lambda} = S_t e^{\delta\tau}. \tag{14}$$

In particular, the marginal price is independent of the risk-aversion parameter R. This is an example of a general result which states that the marginal price is independent of the utility function. See Davis [9], Hobson [21, Theorem 1] or Karatzas and Kou [25]. Further the marginal price is the expectation of the payoff under a risk-neutral measure. Indeed the marginal price is the expected payoff under the minimal martingale measure $\mathbb{P}^0$ of Föllmer and Schweizer [20].

Under the minimal martingale measure, processes contained within the span of the traded assets (such as $\sigma B_t + \mu t$) become martingales, and martingales which are orthogonal to this space are unchanged in law. Thus under the minimal martingale measure $\mathbb{P}^0$

$$\begin{aligned} \frac{dS}{S} &= \nu dt + \eta\rho dB + \eta\sqrt{1-\rho^2}dW \\ &= \left(\nu - \frac{\eta\rho\mu}{\sigma}\right)dt + \frac{\eta\rho}{\sigma}(\sigma dB + \mu dt) + \eta\sqrt{1-\rho^2}dW \end{aligned}$$

where the final two terms in the last expression are both martingales. Hence $\mathbb{E}^0_t(S_T) = S_t e^{\delta(T-t)}$, which is the marginal price of the claim.

Importantly, and unlike in the complete market scenario of Section 4, the marginal price the agent is prepared to pay for S_T depends on the drift μ of the traded asset. Ceteris paribus, if $\mu > 0$, then the marginal price decreases as the correlation ρ increases. Intuitively, converting to a risk neutral measure generally deflates the price P and if $\rho > 0$ this also deflates S. Hence, the expected value of λS_T decreases as ρ increases, see Figure 1. This appears paradoxical, as one might expect the price to drop with lower correlation and perceived higher risk. This effect however, is a second order one. In (13) we see the λ^2 term is an increasing function of ρ.

As is to be expected, the marginal price is increasing in ν, the drift of the untraded asset. Again this contrasts with the complete market situation where the parameter ν must be related to the other drift and volatility parameters to preclude arbitrage. As we remarked above, conclusions about the marginal price the agent is prepared to pay for the asset are independent of the agent's utility. However the reservation price for a non-negligible quantity of untraded asset does depend on the utility as expressed in the λ^2 term in the expansion (13).

Note first that the correction term to order λ^2 is negative. This is because utilities are concave, so that the agent is prepared to pay a lower (unit) price for larger quantities. Note second that the order of the correction term expressed as a proportion of the leading term depends on the variables λ, S_t, X_t only through the ratio $\lambda S_t/X_t$. Thus the key variable is the ratio of the claim value and the current wealth.

We can also note the effect of initial wealth on the price, for various risk aversion levels, R. In Figure 2 we graph the reservation price as a function of initial wealth X, for R equal to 0.1, 0.5, 1.0 and 2.0. As expected, for each R, the larger the proportion of wealth held in the untraded asset, the lower the buying price, ie, the agent is less willing to take on more risk. On the graphs, if wealth increases, with fixed S and λ, then the holding in derivatives is diluted, and the price

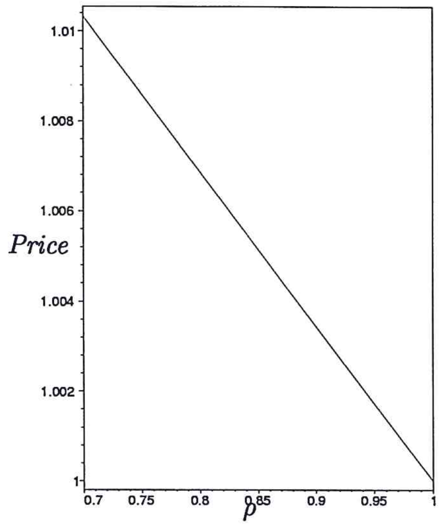

Figure 1: *The reservation price of the claim for* $0.70 \leq \rho \leq 1$. *Parameter values are* $\lambda S_0 = 1.0, T = 1, R = 0.5, \mu = 0.04, \eta = 0.30, \sigma = 0.35$ *and* $\nu = \mu\eta/\sigma = 0.0343$. *Note that since* $\delta = 0$ *in this case, the price when* $\rho = 1$ *is unity.*

larger. This is as expected with our choice of the power-law utility function. For the exponential utility which has constant absolute risk aversion, the price would be independent of wealth. For the Merton utility, the absolute risk aversion $-\frac{U''(x)}{U'(x)} = \frac{R}{x}$ is a decreasing function of wealth and thus the higher wealth, the higher price the agent is willing to pay.

Now we consider the dependence of the reservation price on the risk aversion parameter with surprising results. In Figure 3 we graph the reservation price as a function of R. We see that over most of the parameter range, as risk aversion increases so the reservation price falls. The agent is willing to pay less for the untraded stock as she becomes less tolerant of risk. However, unexpectedly this relationship reverses as R gets very small. For the parameter choices in Figure 3 this happens for R below approximately 0.1. As R decreases below this value the agent is prepared to pay less for the risky untraded asset even though she is becoming more tolerant of risk.

A clue to the cause of this surprising result is given in the expression for the optimal strategy given in Theorem 5.5:

$$\theta = \frac{\mu}{\sigma^2 R} X_t + \lambda S_t e^{\delta(T-t)} \left(\frac{\mu}{\sigma^2 R} - \frac{\eta\rho}{\sigma} \right) + O(\lambda^2).$$

As $R \downarrow 0$, both the first term, the Merton proportion, and the first order correction term become large. Fluctuations in the value of P and S are magnified into large fluctuations in the final wealth. The price an agent is prepared to pay for a random payoff depends on two factors. The first is her level of risk aversion, but the second is the magnitude of the unhedgeable component of the random payoff. Thus even though the agent is only mildly risk averse, the large fluctuations in final wealth have a non-negligible effect on expected utility, and hence price.

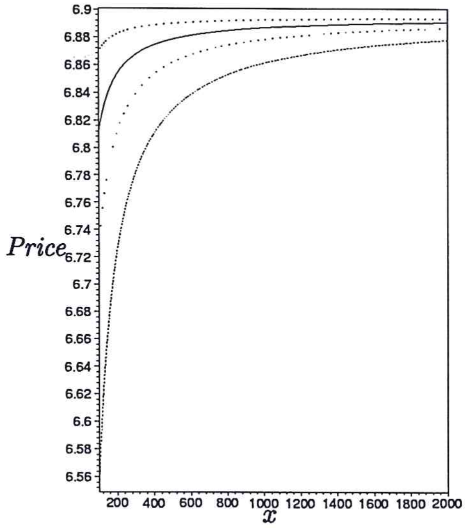

Figure 2: *Price p for initial wealth* $100 \leq x \leq 2000$. *In fact, in order to show clearly the scale of prices the graph shows* $(Price - 1) \times 10^3$. *Parameters are* $\lambda = 0.01, \rho = 0.8, S_0 = 100, \mu = 0.04, \eta = 0.3, \sigma = 0.35, \nu = 0.0343$. *Observe that the reservation price is increasing in wealth, and that it appears to be decreasing in the risk aversion parameter.*

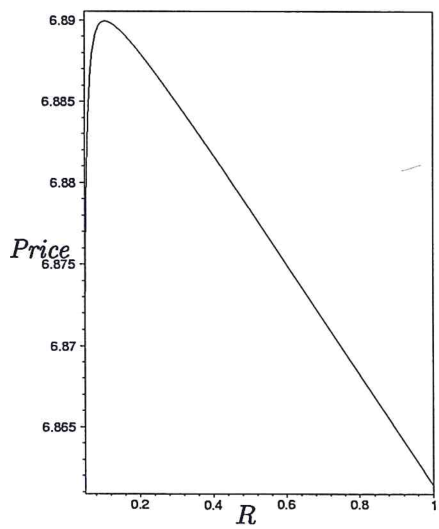

Figure 3: *The quantity* $\{(Price - 1) \times 10^3\}$ *as a function of the risk aversion parameter* R *for* $0.05 \leq R \leq 1$. *Parameters are* $\rho = 0.8, \frac{\lambda S_0}{X_0} = 0.002, T = 1, \nu = 0.0343 = \frac{\eta\mu}{\sigma}, \mu = 0.04, r = 0, \eta = 0.30, \sigma = 0.35$.

7 A Comparison with Exponential Utility

Another popular utility function widely used in the literature is the exponential utility, $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, see for example, Hodges and Neuberger [22], Svensson and Werner [34], Duffie and Jackson [17], Davis [10], Cvitanic et al [7] and Delbaen et al [13]. This utility has constant absolute risk aversion, and its popularity is derived in part from its tractability. In particular, it is separable in the sense that current wealth can be factored out of any problem.

By analogy with previous sections, and for $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, let $V(t, X_t, S_t; \lambda)$ be the value function for the agent who at time t has wealth X_t and who will receive λS_T at time T. Then

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= \sup_\theta \mathbb{E}_t U(X_T + \lambda S_T) \\ &= -\frac{1}{\gamma}e^{-\gamma X_t} \inf_\theta \mathbb{E}_t (e^{-\gamma \int_t^T \theta_u (dP_u/P_u) - \gamma\lambda S_T}) \\ &= -\frac{1}{\gamma}e^{-\gamma X_t} g(T-t, \log S_t) \end{aligned}$$

where $g(0,z) = e^{-\lambda\gamma e^z}$. Using the fact that V is a supermartingale for any strategy, and a martingale for the optimal strategy, we find that g solves the pde:

$$\dot{g} - \nu g_z + \tfrac{1}{2}\eta^2 g_z - \tfrac{1}{2}\eta^2 g_{zz} + \tfrac{1}{2}\frac{(\sigma\eta\rho g_z + \mu g)^2}{\sigma^2 g} = 0.$$

To solve this equation we mimic Hobson [21]. If we set $g(\tau, y) = e^{\alpha\tau} G(\tau, y + \beta\tau)^b$ then we find that G solves

$$\begin{aligned} & b\dot{G} - \tfrac{1}{2}\eta^2 b G_{yy} - \tfrac{1}{2}\eta^2(b(b-1) - \rho^2 b^2)\frac{G_y^2}{G} \\ & \quad + \left[b(\beta + \tfrac{1}{2}\eta^2 - \nu + \frac{\eta\rho\mu}{\sigma})\right] G_y + \left[\alpha + \frac{\mu^2}{2\sigma^2}\right] G = 0. \end{aligned}$$

In particular, if we choose parameter values

$$b = \frac{1}{(1-\rho^2)}, \qquad \alpha = -\frac{\mu^2}{2\sigma^2}, \qquad \beta = \nu - \frac{\eta\rho\mu}{\sigma} - \tfrac{1}{2}\eta^2 = \delta - \tfrac{1}{2}\eta^2$$

we find that G solves

$$\dot{G} = \tfrac{1}{2}\eta^2 G_{yy}.$$

This is the heat equation, with solution

$$G(\tau, y) = \int_{-\infty}^{\infty} G(0, y+z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz$$

so that

$$\begin{aligned} g(\tau, y) &= e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\int_{-\infty}^{\infty} G(0, y + (\delta - \tfrac{1}{2}\eta^2)\tau + z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz\right]^{\frac{1}{1-\rho^2}} \\ &= e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\mathbb{E}(G(0, y + (\delta - \tfrac{1}{2}\eta^2)\tau + \eta\sqrt{\tau}N))\right]^{\frac{1}{1-\rho^2}} \end{aligned}$$

where N is a standard normal random variable. This solution was given in Hobson [21] and extends to general payoffs which are functions of S_T. In particular, using the boundary condition $G(0,y) = e^{-(\lambda\gamma/d)e^y} = e^{-\lambda\gamma(1-\rho^2)e^y}$:

$$V(t, X_t, S_t; \lambda) = -\frac{1}{\gamma} e^{-\gamma X_t - \frac{\mu^2}{2\sigma^2}(T-t)}$$
$$\times \left[\mathbb{E}\left(\exp(-\lambda\gamma(1-\rho^2)S_t e^{\delta(T-t)} e^{\eta\sqrt{T-t}N - \frac{1}{2}\eta^2(T-t)})\right)\right]^{\frac{1}{1-\rho^2}}.$$

It follows that the reservation price for receiving a random payoff λS_T, given as the solution to $V(0, X_0 - p, S_0; \lambda) = V(0, X_0, S_0, 0)$, is

$$p = -\frac{1}{\gamma(1-\rho^2)} \log \mathbb{E}\left[\exp\left\{-\lambda\gamma(1-\rho^2)S_0 e^{\delta T} e^{\eta\sqrt{T}N - \frac{1}{2}\eta^2 T}\right\}\right].$$

We want to find an expansion in terms of small λ which we can compare to our results using the Merton utility. The expansion is

$$p = \lambda S_0 e^{\delta T} - \frac{\gamma}{2}\lambda^2 S_0^2 e^{2\delta T}(1-\rho^2)(e^{\eta^2 T} - 1) + O(\lambda^3).$$

Note that the expansion is only valid for positive λ.

Again we find that to leading order the price is precisely the expected value of the claim under the minimal martingale measure. Hence we concentrate on the correction term. Note that the second order correction is linear in the risk aversion parameter γ. To facilitate comparisons it is convenient to equate the local absolute risk aversion in the Merton and exponential utility models. This involves identifying the parameter γ with R/x. The price becomes

$$p = \lambda S_0 e^{\delta T} - \frac{R}{2}\lambda^2 \frac{S_0^2}{x} e^{2\delta T}(1-\rho^2)(e^{\eta^2 T} - 1) + O(\lambda^3). \tag{15}$$

If we compare this price with Theorem 5.4 we find that the prices under the two models agree if

$$\frac{e^{\eta^2 T} - 1}{\eta^2} = \frac{e^{\Gamma T} - 1}{\Gamma}$$

or equivalently if $\eta^2 = \Gamma$. This happens in the limit as $R \uparrow \infty$, but also, since Γ is non-monotonic as a function of R, when $R = (\mu/2\sigma\eta\rho)$.

From Figure 4 we see that when the risk aversion factor is large, the price correction under the exponential utility model is larger, but that the price corrections under the two models are of comparable size. For small values of the risk aversion parameter the price correction under the Merton model is significantly larger. In particular the remarkable feature of the power-law utility, namely that the price for the random endowment is not monotonic in the risk aversion parameter, is not a feature of the constant absolute risk aversion model.

It is interesting in this context to compare the forms of the exponential and power-law utilities in the limit as the risk aversion parameter tends to zero. In both cases

$$\lim_{\gamma \downarrow 0} \frac{1 - e^{-\gamma x}}{\gamma} = x = \lim_{R \downarrow 0} \frac{x^{1-R}}{1-R},$$

but in the former case the domain of definition is $\mathbb{R}$ whereas in the latter it is $\mathbb{R}^+$. Hence there is no reason to expect identical behaviour in the limit as risk aversion decreases to zero.

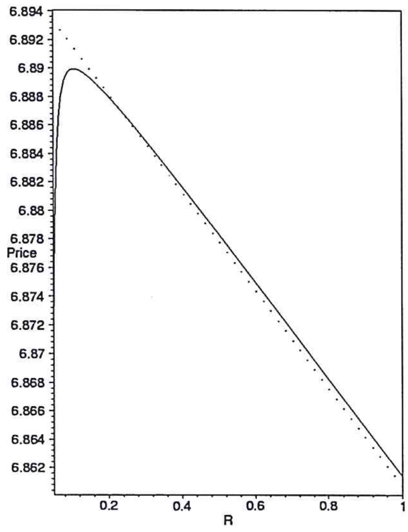

Figure 4: *A comparison of the reservation price for Merton (power law) utility and exponential utility, over a range of values of risk aversion parameters. The solid line represents the price under power law utility, and the dotted line exponential utility. The vertical scale is* $(Price - 1) \times 10^3$. *The parameter* γ *of the exponential utility has been chosen to match the local relative risk aversion to that of the corresponding power law utility. Thus the reservation price for the exponential utility can be presented in terms of the parameter* R*, using the formula (15). Parameters are* $\rho = 0.8, \lambda S_0 = 1, X_0 = 500, T = 1, \nu = 0.0343 = \frac{\eta\mu}{\sigma}, \mu = 0.04, r = 0, \eta = 0.30, \sigma = 0.35$.

8 The effect of using the 'wrong' hedge

In this section we wish to consider the pricing decisions of an agent who uses a suboptimal strategy when hedging the claim. This may be because of a lack on information on parameter values, and especially the correlation coefficient ρ. We are interested in finding the impact this has on the agent's expected utility, and on her reservation price for the random payment.

The strategies we consider have the form

$$\theta(t, X_t, S_t; \lambda) = \frac{\mu}{\sigma^2 R} X_t + \lambda S_t \psi \tag{16}$$

where $\psi = \psi(t)$ is a function of time alone. For example, the choice $\psi = 0$ is equivalent to using the hedging strategy which would have been optimal had there been no random endowment, and the choice

$$\psi(t) = \left(-\frac{\eta}{\sigma} + \frac{\mu}{\sigma^2 R}\right) \tag{17}$$

corresponds to using the strategy which is optimal if the traded asset is perfectly correlated with the untraded claim.

Let $V(\psi) = V(t, X_t, S_t; \lambda; \psi)$ be the expected final utility of an agent with wealth X_t at time t who is due to receive the random payment λS_T at time T and who follows the strategy given by (16). Let $\tau = T - t$.

Theorem 8.1 *The value function of the agent who uses the strategy ψ is given by*

$$V(t, X_t, S_t; \lambda; \psi) = \frac{X_t^{1-R}}{1-R} e^{(1-R)m\tau} \left(1 + \lambda \frac{S_t}{X_t} e^{\delta\tau} - \frac{\lambda^2 S_t^2}{X_t^2} e^{2\delta\tau} f(T-t) + O(\lambda^3)\right)^{1-R}$$

where $f(\tau)$ solves $\dot{f}(\tau) = \Gamma f(\tau) + A(\tau)$ subject to $f(0) = 0$, with

$$A(\tau) = \tfrac{1}{2}\Gamma R - e^{-\delta\tau}\psi(T-\tau)(\mu - R\sigma\rho\eta) + \tfrac{1}{2}e^{-2\delta\tau}\psi(T-\tau)^2\sigma^2 R.$$

This has solution

$$f(\tau) = e^{\Gamma\tau} \int_0^\tau e^{-\Gamma s} A(s) ds.$$

The price the agent is willing to bid for the claim λS_T at time t is

$$p = \lambda S_t e^{\delta(T-t)} - \lambda^2 \frac{S_t^2}{X_t} e^{2\delta(T-t)} f(T-t) + O(\lambda^3).$$

Proof:

The proof proceeds as the proof of Theorem 5.3 except that it is not necessary to optimize over strategies θ. □

Corollary 8.2 *If $\psi \equiv 0$ then the price this agent is prepared to pay is*

$$p = \lambda S_t e^{\delta(T-t)} - \lambda^2 \frac{S_t^2}{X_t} e^{2\delta(T-t)} \frac{R}{2}(e^{\Gamma(T-t)} - 1) + O(\lambda^3).$$

Note first that to order λ the price the agent is prepared to bid for the claim is independent of λ. Again this reflects the fact that perturbations in the strategy of order λ only have a second order effect on pricing decisions.

The function f is minimised by choosing ψ to minimise $A(\cdot)$. With this choice of ψ, f solves $\dot{f}(\tau) = \Gamma f + \frac{1}{2}\eta^2 R(1-\rho^2)$ and we recover the expression in Theorem 5.3. In this way we can deduce the optimal strategy to first order in λ.

We concentrate in the above theorem and corollary on calculating the price the agent will charge because the difference between the price bid by the agent who hedges optimally, and the price bid by an agent who hedges in an alternative fashion represents a loss of initial wealth from hedging suboptimally. By expressing quantities in terms of wealth they can be more readily compared across different utilities.

Corollary 8.3 *Using the strategy given in (16) is equivalent to a reduction in initial wealth of magnitude $\lambda^2(S_0^2/x)e^{2\delta T}L(T,\psi) + O(\lambda^3)$ where*

$$L(T,\psi) = e^{\Gamma T}\int_0^T e^{-\Gamma t}\tfrac{1}{2}R\left\{\left(\frac{\mu}{\sigma R} - \eta\rho\right) - e^{-\delta\tau}\psi\sigma\right\}^2 dt.$$

L is necessarily non-negative.

Consider first the loss that arises from using the Merton hedge which takes no account of the random endowment. Applying Corollary 8.3 with $\psi = 0$ we find a loss in wealth of

$$\lambda^2 \frac{S_0^2 e^{2\delta T}}{x}\frac{R}{2}(e^{\Gamma T} - 1)\left(\eta\rho - \frac{\mu}{\sigma R}\right)^2 + O(\lambda^3).$$

In particular, if $\rho = (\mu/\sigma\eta R)$ then there is no loss to order λ^2. This is to be expected since if $\rho = (\mu/\sigma\eta R)$, the optimal strategy to order λ (as given in Theorem 5.5) is to invest the Merton proportion of wealth in the traded asset.

Now consider the impact of using the strategy presented in (17). This is the strategy of the naive investor who follows the strategy which is optimal in the case when the market is complete, even though in this case the market is incomplete. The resulting loss in wealth is given by $\lambda^2(S_0^2/x)e^{2\delta T}L(T,\psi) + O(\lambda^3)$ where

$$L(T,\psi) = e^{\Gamma T}\int_0^T e^{-\Gamma t}\tfrac{1}{2}R\left\{\left(\frac{\mu}{\sigma R} - \eta\rho\right)(1 - e^{-\delta(T-t)}) + \eta(1-\rho)e^{-\delta(T-t)}\right\}^2 dt. \tag{18}$$

We are interested in the dependence of this quantity on the correlation ρ.

If $\rho = 1$ then arbitrage forces that $\delta = 0$ and we find that $L \equiv 0$. Now suppose $|\rho| < 1$. The dependence of L on ρ is complicated by the fact that δ and Γ both depend on ρ. If $\nu = \eta\mu/\sigma$, then $\delta = (1-\rho)\eta\mu/\sigma$. If δ is small then $(1 - e^{-\delta(T-t)}) \sim \delta(T-t)$ and we can remove a factor

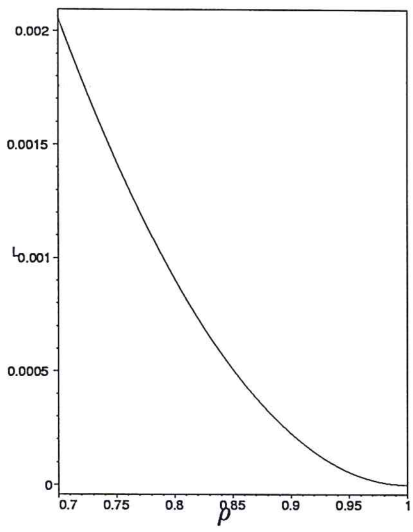

Figure 5: *A plot of the effective loss suffered by an agent who hedges the random endowment S_T using the trading strategy which is optimal in the complete market case. The plot shows the loss (as represented by the function L given in (18)) as a function of the correlation ρ. Parameters are $T = 1, \nu = 0.0343 = \frac{\eta\mu}{\sigma}, \mu = 0.04, \eta = 0.30, \sigma = 0.35, R = 0.5$.*

$(1-\rho)^2$ from the integrand in (18). Essentially, for reasonable sets of parameter values, the loss will grow as the square of $(1-\rho)$, ie $L \sim c_0(1-\rho)^2$. This behaviour is illustrated in Figure 5. This observation supports the use of the naive strategy when the correlation is close to unity, but warns that the performance deteriorates markedly as the correlation decreases. If $\nu \neq \eta\mu/\sigma$ then the loss will not decrease to zero as the correlation increases to unity, except for some special choices of parameter values. However, if $\mu, \nu \ll \eta^2, \sigma^2$, then for ρ not too close to 1, the dominant term in the loss L will be of order $\frac{1}{2}R\eta^2(1-\rho)^2 T$, so that again $L \sim c_1(1-\rho)^2$.

9 Conclusion

In this paper, we have addressed the problem of pricing and hedging a random payoff on an untraded asset by making use of another, correlated, asset which is traded. This situation is very common in the area of real options, where one of the assets involved cannot be traded.

By extending the Merton investment problem [30] to include an additional asset, we formulated a utility maximisation problem. Since the random payoff was units of the untraded asset (rather than, for example, a call option on the untraded asset) we could use a scaling to reduce the dimensionality of the problem. This allowed us to express the solution of the utility maximisation problem as an expansion in the number of units of the untraded asset which formed the claim.

The disadvantage of using a random payoff which was a multiple of the untraded asset, was that this claim is unbounded. As a consequence, if $\lambda < 0$ no expansion exists. However, when $\lambda > 0$, or equivalently when the agent receives units of the untraded asset, an expansion does exist. This expansion was used to derive expressions for the reservation price of the agent, and the optimal strategy she should use.

Our expansion up to $O(\lambda)$ can be used to deduce the marginal price for the untraded asset. As shown by Davis [9] this price is the expected value of the claim under the minimal martingale measure of Föllmer and Schweizer [20]. In particular the marginal price for the asset is independent of the utility function. However, unlike in the complete markets case, the marginal price does depend on the real world measure and the drifts of the two assets.

If we consider the reservation price as a function of λ we find that the unit price of the claim per unit of claim is a decreasing function. This represents the phenomena of diminishing marginal returns, or convexity (in λ) of the value function V.

An unexpected result was that the reservation price is a non-monotonic function of the risk aversion parameter, R. There is a value of R for which the price is maximised. This seems to be a peculiar function of the power-law utility since it does not occur with the exponential utility.

The paper also examined the effect of using a suboptimal hedge on the price the agent is prepared to pay. If the agent uses the strategy which is optimal when the two assets are perfectly correlated, then the effect is that she needs to overcharge for the claim by an amount proportional to $\lambda^2(1-\rho)^2$. This lends some support to the practice of using this naive hedge when correlation is very high, but warns of significant losses as the correlation falls.

In this paper we have concentrated on the linear payoff λS_T. This linearity has greatly simplified some of the analysis, at some technical cost. It is natural to consider option payoffs such as $\lambda(K-S_T)^+$. Since this payoff is bounded some of the technical problems will not arise (but note that the payoff from a call option is not bounded). However, the loss of the scaling property would complicate the computations. This is left for further research.

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References

[1] Busby J.S. and C.G.C. Pitts; Real Options in Practice: An exploratory survey of how finance officers deal with flexibility in capital appraisal, *Management Accounting Research*, 8, 2, June, 1997.

[2] Constantinides G. and T. Zariphopoulou; Bounds on prices of contingent claims in an intertemporal economy with proportional transactions costs and preferences, *Finance and Stochastics*, 3, 3, p345-369, 1999.

[3] Cox J. and C.F. Huang; Optimal consumption and portfolio policies when asset prices follow a diffusion process, *J. Econ Theory*, 49, p33-83, 1989.

[4] Cuoco D.; Optimal consumption and equilibrium prices with portfolio constraints and stochastic income, *Journal of Economic Theory*, 7, 2, 1997.

[5] Cvitanic J. and I. Karatzas; Convex duality in constrained portfolio optimization, *Annals of Applied Probability*, 2, p767-818, 1992.

[6] Cvitanic J. and I. Karatzas; Hedging contingent claims with constrained portfolios, *Annals of Applied Probability*, 3, p652-681, 1993.

[7] Cvitanic J., Schachermayer W. and Wang H.; Utility maximisation in incomplete markets with random endowment, Preprint, 1999, to appear in the 'Proceedings of Symposia in Applied Mathematics'.

[8] Davis M.H.A.; Option Pricing in Incomplete Markets, *in Mathematics of Derivative Securities*, eds M.A.H. Dempster and S.R. Pliska, Cambridge University Press, 1997.

[9] Davis M.H.A.; Option valuation and hedging with basis risk, *in System Theory: Modelling, Analysis and Control*, eds T.E. Djaferis and I.C. Schick, Kluwer, Amsterdam, 1999.

[10] Davis M.H.A.; Optimal Hedging with Basis Risk, *Preprint, Vienna University of Technology*, 2000.

[11] Davis M.H.A. and A.R. Norman; Portfolio Selection with transaction costs, *Math. Operations Research*, 15, p676-713, 1990.

[12] Davis M.H.A., Panas V.G. and T. Zariphopoulou; European Option pricing with transactions costs, *SIAM Journal of Control and Optimisation*, 31, p470-493, 1993.

[13] Delbaen F., Grandits P., Rheinlander T., Samperi D., Schweizer M., Stricker C.; Exponential Hedging and Entropic Penalties, Preprint, TU Berlin.

[14] Dixit A.K. and R.S. Pindyck; Investment Under Uncertainty, *Princeton University Press, Princeton, NJ*, 1994

[15] Duffie D.; Dynamic Asset Pricing Theory, *Princeton University Press, Princeton, NJ*, 1992.

[16] Duffie D., Fleming W.H., Soner H.M. and T. Zariphopoulou; Hedging in Incomplete markets with HARA utility, *Journal of Economic Dynamics and Control*, 21, p753-782, 1997.

[17] Duffie D. and M.O. Jackson; Optimal hedging and equilibrium in a dynamic futures market; *Journal of Economic Dynamics and Control*, 14, p21-33, 1990.

[18] Duffie D. and H.R. Richardson; Mean Variance Hedging in Continuous Time, *Annals of Applied Probability*, 1, 1, p1-15, 1991.

[19] Dunbar N.; The Power of Real Options, *RISK*, Vol 13, No 8, August 2000.

[20] Föllmer H. and M. Schweizer; Hedging of contingent claims under incomplete information, *Applied Stochastic Analysis*, eds. Davis M.H.A. and Elliot R.J.. Gordon and Breach, London, 1990.

[21] Hobson D.G.; Option Pricing in Incomplete Markets, *Unpublished manuscript*, Jan 1994.

[22] Hodges S.D. and A. Neuberger; Optimal replication of contingent claims under transactions costs, *Review of Futures Markets*, 8, p222-239, 1989.

[23] Hubalek F. and W. Schachermeyer; The limitations of No-arbitrage arguments for real options, *Working paper No 58, Vienna University of Technology*, Oct 1999.

[24] Johnson S.A. and Y.S. Tian; The value and incentive effects of nontraditional executive stock option plans, *Journal of Financial Economics*, 57, p3-34, 2000.

[25] Karatzas I. and Kou S.G.; Pricing contingent claims with constrained portfolios, *Annals of Applied Probability*, 6, p321-369, 1996.

[26] Karatzas I., Lehoczky J.P. and S.E. Shreve; Optimal Portfolio and consumption decisions for a 'small investor' on a finite horizon, *SIAM J.Control Optim.*, 25, p1557-1586, 1987.

[27] Karatzas I., Lehoczky J.P., Shreve S.E and G. Xu; Martingale and duality methods for utility maximisation in an incomplete market, *SIAM J.Control Optim.*, 29, 3, p702-730, 1991.

[28] Karatzas I. and Shreve S.E; Brownian Motion and Stochastic Calculus, *Springer-Verlag, New York*, 1987.

[29] Kramkov D. and Schachermayer W.; The asymptotic elasticity of utility functions and optimal investment in incomplete markets, Annals Applied Probability, 9, 1999, p904-950.

[30] Merton R.C.; Lifetime portfolio selection under uncertainty: the continuous time case, *Review of Economics and Statistics*, 51, p247-257, 1969.

[31] Pliska S.; A Stochastic Calculus Model of continuous Trading: Optimal portfolios, *Mathematics of Operations Research*, 11, 2, p371-382, 1986.

[32] Rogers L.C.G.; The relaxed investor and parameter uncertainty, *to appear in Finance and Stochastics*, V, 2, 2001.

[33] Schachermeyer W.; Optimal investment in incomplete markets when wealth may become negative, *Preprint*, March 2000.

[34] Svensson L.E.O. and I.M. Werner; Nontraded assets in incomplete markets: Pricing and portfolio choice, *European Economics Review*, 37, p1149-1168, 1993.

[35] Teplá L.; Optimal portfolio policies with borrowing and shortsale constraints, *Journal of Economic Dynamics and Control*, 24, p1623-1639, 2000.

[36] Xu G. and S.E. Shreve; A duality method for optimal consumption and investment under short selling prohibition, I General market coefficients, II Constant market coefficients, *Annals of Applied Probability*, 2, p87-112, p314-328, 1992.

[37] Zariphopoulou T.; A solution approach to valuation with unhedgeable risks, *to appear in Finance and Stochastics*, V, 1, 2001.

10 Appendix

In Section 5 we derived expansions for the value function to order λ^2. In this appendix we prove that these expansions are valid for positive λ. This is a subtle question, since if $|\rho| < 1$ and $\lambda < 0$ then the expansions do not exist.

We begin by justifying this last remark. If $|\rho| < 1$ then the market is incomplete and S_T is the product of a variable which is measurable with respect to the filtration of B, or equivalently P, and the random quantity $e^{\eta\sqrt{1-\rho^2}W_T}$. This last quantity is independent of B and unbounded above, so that it is impossible to super-replicate S_T using trades in P alone. In particular, for any X_T which can be generated from a finite initial fortune x, and investments in the traded asset P, we have

$$\mathbb{P}(X_T + \lambda S_T < 0) > 0$$

for $\lambda < 0$. Since $U \equiv -\infty$ on the negative real line, we have

$$V(t, X_t, S_t, \lambda) = -\infty, \qquad |\rho| < 1, t < T, \lambda < 0.$$

It remains to show that, for $\lambda > 0$ the expansion in Theorem 5.3 holds. We demonstrate this by exhibiting upper and lower bounds for the supremum of expected utility which agree to order λ^2. For the lower bound we use the fact that $\lambda S_T \geq 0$ almost surely.

The Lower Bound

Consider first the zero endowment problem where (X_t^0, θ_t^0) is the optimal wealth, strategy pair. Then $dX_t^0 = \theta_t^0 dP_t/P_t$ with $\theta_t^0 = (\mu X_t^0)/(\sigma^2 R)$ and

$$X_t^0 = x \exp\left(\frac{\mu}{\sigma R}B_t + \frac{\mu^2}{\sigma^2 R}t - \frac{\mu^2}{2\sigma^2 R^2}t\right).$$

Now consider the problem with a random endowment of λS_T at time T. We would like to consider the strategy in Theorem 5.5 to order λ. However, with this strategy we cannot guarantee that wealth remains positive, so we use a localised version.

Fix K and let

$$H_K = \inf\left\{u : \int_0^u \frac{S_t e^{\delta(T-t)}}{X_t^0}\left(\frac{\rho\eta}{\sigma} - \frac{\mu}{\sigma^2 R}\right)\left(\frac{dP}{P} - \frac{\mu dt}{R}\right) = K\right\}.$$

Suppose $\lambda < \frac{1}{2}K^{-1}$. Consider the wealth process $X^{1,K}$ generated from an initial fortune x using the strategy

$$\theta_t^{1,K} = \frac{\mu}{\sigma^2 R}X_t^{1,K} + \lambda\left(-\frac{\rho\eta}{\sigma} + \frac{\mu}{\sigma^2 R}\right)S_t e^{\delta(T-t)} I_{(t<H_K)}$$

Then $X_t^{1,K}$ is given by

$$X_t^{1,K} = X_t^0\left\{1 + \lambda\int_0^{t\wedge H_K}\frac{S_u}{X_u^0}e^{\delta(T-u)}\left(-\frac{\rho\eta}{\sigma} + \frac{\mu}{\sigma^2 R}\right)\left(\frac{dP}{P} - \frac{\mu}{R}du\right)\right\}.$$

Note that on $H_K < T$ we have $X_T^{1,K} = X_T^0(1-\lambda K)$ and indeed more generally $X_T^{1,K} \geq X_T^0(1-\lambda K)$. In particular the localisation times H_K allow us to bound the wealth process from below.

Now consider the sum of the wealth process and the random endowment. It is convenient to consider $Z_t^{\lambda,K} = X_t^{1,K} + \lambda S_t e^{\delta(T-t)}$. On $t \leq H_K$ we have

$$dZ_t^{\lambda,K} = \frac{\mu}{\sigma^2 R}Z_t^\lambda\frac{dP}{P} + \lambda S_t e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW$$

so that, still with $t \leq H_K$,

$$Z_t^{\lambda,K} = X_t^0\left\{1 + \lambda\left(\frac{S_0 e^{\delta T}}{x} + \int_0^t \frac{S_u}{X_u^0}e^{\delta(T-u)}\eta\sqrt{(1-\rho^2)}dW\right)\right\}.$$

Also, $Z_T^{\lambda,K} = X_T^{1,K} + \lambda S_T \geq X_T^0(1-\lambda K) + \lambda S_T \geq X_T^0(1-\lambda K)$.

From Taylor's expansion we have $U(y+h) = U(y) + hU'(y) + \frac{1}{2}h^2 U''(y+\xi h)$ with $\xi = \xi(\lambda, K, \omega) \in [0,1]$. We will take $y = X_T^0$ and $h = Z_T^{\lambda,K} - X_T^0$, and consider the expected value of this expansion term by term. The first term yields $\mathbb{E}(U(X_T^0)) = V(0, x, S_0; 0)$. For the second term, note that

$$U'(X_T^0) = x^{-R}\exp\left(\frac{\mu^2}{2\sigma^2}\frac{(1-R)}{R}T\right)\frac{d\mathbb{P}^0}{d\mathbb{P}}$$

where $(d\mathbb{P}^0/d\mathbb{P}) = \exp(-(\mu/\sigma)B_T - \frac{1}{2}(\mu^2 T/\sigma^2))$ so that $\mathbb{P}^0$ is the minimal martingale measure. Then since both X^0 and $X^{1,K}$ are martingales under $\mathbb{P}^0$, we have

$$\mathbb{E}[(Z_T^{\lambda,K} - X_T^0)U'(X_T^0)] = x^{-R}e^{(1-R)mT}\mathbb{E}^0(\lambda S_T) = \lambda x^{-R}e^{(1-R)mT}S_0 e^{\delta T}.$$

For the final term in the Taylor expansion we have that for $\xi = \xi(\lambda, K, \omega) \in [0,1]$,

$$X_T^0 + \xi(Z_T^{\lambda,K} - X_T^0) \geq X_T^0(1-\lambda K).$$

This inequality makes crucial use of the fact that $\lambda S_T > 0$. Then, since U'' is increasing,

$$\begin{aligned}
&\frac{1}{\lambda^2}(Z_T^{\lambda,K} - X_T^0)^2 U''(X_T^0 + \xi(Z_T^\lambda - X_T^0)) \\
&\geq (X_T^0)^2\left(\frac{S_0 e^{\delta T}}{x} + \int_0^T \frac{S_t}{X_t^0}e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW\right)^2 U''(X_T^0(1-\lambda K))I_{(H_K\geq T)} \\
&\quad + (S_T - X_T^0 K)^2 U''(X_T^0(1-\lambda K))I_{(H_K<T)}.
\end{aligned}$$

By the dominated convergence theorem, on taking expectations and letting $\lambda \downarrow 0$, we find for each K that $\lambda^{-2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda\mathbb{E}[S_T U'(X_T^0)])$ is bounded below by

$$\mathbb{E}\left[(X_T^0)^2 U''(X_T^0)\left(\frac{S_0 e^{\delta T}}{x} + \int_0^T \frac{S_t}{X_t^0} e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW\right)^2 I_{(H_K \geq T)}\right]$$
$$+\mathbb{E}\left[(S_T - X_T^0 K)^2 U''(X_T^0) I_{(H_K < T)}\right].$$

Letting $K \uparrow \infty$ this expression becomes

$$\mathbb{E}\left[U''(X_T^0)(X_T^0)^2\left(\frac{S_0 e^{\delta T}}{x} + \int_0^T \frac{S_t}{X_t^0} e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW\right)^2\right]. \tag{19}$$

We can interpret $U''(X_T^0)$ as a constant multiplied by a change of measure which affects the drift of dP/P. With this interpretation it is a long but straightforward exercise to show that (19) becomes

$$\begin{aligned}
&-R\mathbb{E}\left[(X_T^0)^{1-R}\left(\frac{S_0 e^{\delta T}}{x} + \int_0^T \frac{S_t}{X_t^0} e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW\right)^2\right] \\
&= -Rx^{1-R}e^{m(1-R)T}\hat{\mathbb{E}}\left[\left(\frac{S_0 e^{\delta T}}{x} + \int_0^T \frac{S_t}{X_t^0} e^{\delta(T-t)}\eta\sqrt{(1-\rho^2)}dW\right)^2\right] \\
&= -Rx^{1-R}\frac{S_0^2 e^{2\delta T}}{x^2}e^{m(1-R)T}\left[1 + \frac{\eta^2(1-\rho^2)}{\Gamma}(e^{\Gamma T} - 1)\right],
\end{aligned} \tag{20}$$

where $\hat{\mathbb{P}}$ is the measure under which both $\hat{B}_t \equiv B_t - (\mu(1-R)/\sigma R)t$ and $\hat{W}_t \equiv W_t$ are Brownian motions.

In conclusion

$$\limsup_{K\uparrow\infty} \lim_{\lambda\downarrow 0} \frac{1}{\lambda^2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda\mathbb{E}[U'(X_T^0)S_T])$$

is greater than the expression (20). Further manipulations yield that

$$\sup_{X_T} \mathbb{E}U(X_T + \lambda S_T) \geq \mathbb{E}U(Z_T^\lambda) \geq V_2(0, X_0, S_0; \lambda) + o(\lambda^2),$$

where V_2 is as given above Theorem 5.3. Hence V_2 is a lower bound to order λ^2. Note that we can extend this result to prove that the correction is $O(\lambda^3)$ rather than just $o(\lambda^2)$ by considering higher order Taylor expansions of the utility function.

The Upper Bound

We find an upper bound on the value function by considering the dual problem. This argument is easier than the lower bound in the sense that no restriction on the sign of λ is necessary.

The problem is to maximise $\mathbb{E}(U(X_T + \lambda S_T))$ over feasible values of the terminal wealth X_T. For a positive random variable Λ consider

$$\begin{aligned}
&\mathbb{E}\left\{U(X_T + \lambda S_T) - \Lambda\left(X_T - \left(x + \int_0^T \theta_t \frac{dP}{P}\right)\right)\right\} \\
&= \mathbb{E}\{U(X_T + \lambda S_T) - \Lambda(X_T + \lambda S_T)\} + \mathbb{E}\{\Lambda(x + \lambda S_T)\} + \mathbb{E}\left\{\Lambda \int_0^T \theta_t \frac{dP}{P}\right\}.
\end{aligned}$$

Suppose Λ is of the form $\Lambda = \alpha d\mathbb{Q}/d\mathbb{P}$ for some change of measure $\mathbb{Q}$. Then, with $\tilde{U}(y) = \sup_x(U(x) - xy)$,

$$\begin{aligned}\sup_{X_T} \mathbb{E}\{U(X_T + \lambda S_T)\} &\leq \inf_{\Lambda} \mathbb{E}\{\tilde{U}(\Lambda) + \Lambda(x + \lambda S_T)\} \\ &= \inf_{\alpha} \inf_{\mathbb{Q}} \left\{ \mathbb{E}^{\mathbb{P}} \left(\tilde{U} \left(\alpha \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right) + \alpha x + \lambda \alpha \mathbb{E}^{\mathbb{Q}}(S_T) \right\}\end{aligned}$$

For the power law utility $U(x) = x^{1-R}/(1-R)$ we have $\tilde{U}(y) = (R/(1-R))y^{(R-1)/R}$. The problem is now to choose Λ in an optimal fashion.

Suppose $R > 1$. Let $M_u = \eta\sqrt{1-\rho^2} \int_0^u (S_t/X_t^0) e^{\delta(T-t)} dW_t$ and set

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma} B_T - \frac{\mu^2}{2\sigma^2} T\right) \exp\left(-R\lambda M_T - \tfrac{1}{2} R^2 \lambda^2 [M]_T\right).$$

Then

$$\mathbb{E}^{\mathbb{P}} \left(\tilde{U} \left(\alpha \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right) = \frac{R}{1-R} \alpha^{(R-1)/R} A,$$

where

$$\begin{aligned} A &= \mathbb{E}\left[\exp\left(\frac{\mu(1-R)}{\sigma R} B_T + \frac{\mu^2(1-R)}{2\sigma^2 R} T\right) \exp\left((1-R)\lambda M_T + \tfrac{1}{2} R(1-R)\lambda^2 [M]_T\right)\right] \\ &= e^{\{m(1-R)/R\}T} \hat{\mathbb{E}}\left[\exp\left(\tfrac{1}{2}(1-R)\lambda^2 [M]_T\right)\right], \end{aligned} \tag{21}$$

and under $\hat{\mathbb{P}}$, both $\hat{B}_t = B_t - (\mu(1-R)/\sigma R)t$ and $\hat{W}_t = W_t$ are Brownian motions. Note that the measure $\hat{\mathbb{P}}$ is the measure which arose in the calculation of the lower bound.

Since $R > 1$, the exponent in the expectation is negative and A can be written as an expansion in λ:

$$A = e^{\{m(1-R)/R\}T} \left[1 + \tfrac{1}{2}(1-R)\lambda^2 \eta^2 (1-\rho^2) \hat{\mathbb{E}} \int_0^T \frac{S_t^2 e^{2\delta(T-t)}}{(X_t^0)^2} dt + O(\lambda^4)\right]. \tag{22}$$

Similarly,

$$\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 e^{\delta T} \tilde{\mathbb{E}}\left[\exp\left(-R\lambda M_T - \tfrac{1}{2} R^2 \lambda^2 [M]_T\right)\right]$$

where $\tilde{\mathbb{P}}$ is a measure under which $\tilde{B}_t = B_t - (\rho\eta - \mu/\sigma)t$ and $\tilde{W}_t = W_t - \eta\sqrt{1-\rho^2}t$ are Brownian motions. We can write this as an expansion involving λ:

$$\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 e^{\delta T} - \lambda S_0 e^{\delta T} R\eta^2 (1-\rho^2) \int_0^T \tilde{\mathbb{E}} \frac{S_t}{X_t^0} e^{\delta(T-t)} dt + O(\lambda^2) \tag{23}$$

The minimisation over α involves finding the minimum of

$$\frac{R}{1-R} \alpha^{(R-1)/R} A + \alpha(x + \lambda \mathbb{E}^{\mathbb{Q}}(S_T)).$$

The minimum is easily seen to be

$$\frac{1}{1-R} A^R (x + \lambda \mathbb{E}^{\mathbb{Q}}(S_T))^{1-R}$$

Substituting the expansions we derived in (22) and (23) above we find

$$\sup_{X_T} \mathbb{E}(U(X_T + \lambda S_T)) \le V_2(0, X_0, S_0; \lambda) + O(\lambda^3)$$

where V_2 is as given at (11). Hence V_2 is an upper bound.

Higher Order Expansions

By combining the upper and lower bounds we conclude that, for $\lambda > 0$, the expansion given to order λ^2 given in Theorem 5.3 is valid. In order to extend this result, and to prove that the expansion can be continued to higher orders, it is necessary to refine the strategy θ^1 used in calculating the lower bound, and the martingale measure $\mathbb{Q}$ for the upper bound. There are no obvious problems with this approach, although the calculations would become very involved.

Extension of the upper bound to $R < 1$.

If $R < 1$ then with $\mathbb{Q}$ as defined above we find that

$$\mathbb{E}\left(\tilde{U}\left(\alpha \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \frac{R}{1-R}\alpha^{(R-1)/R}\mathbb{E}\left(\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{(R-1)/R}\right)$$

is infinite, see (21). However the conclusion that $V_2(0, x, S_0; \lambda)$ is an upper bound (to order λ^2) still holds, since for each $\epsilon > 0$ we can show that $V_2 + \epsilon\lambda^2$ is an upper bound.

For any $K > 0$ define

$$T_K = \inf\left\{u : \int_0^u \frac{S_t e^{\delta(T-t)}}{X_t^0} dt + \int_0^u \frac{S_t^2 e^{2\delta(T-t)}}{(X_t^0)^2} dt = K\right\}.$$

Now choose K large enough so that

$$\tilde{\mathbb{E}}\left[\int_{T_K \wedge T}^T \frac{S_t e^{\delta(T-t)}}{X_t^0} dt\right] < \epsilon.$$

Let $\mathbb{Q}_K$ be given by

$$\frac{d\mathbb{Q}_K}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T\right)\exp\left(-R\lambda M_{T_K} - \tfrac{1}{2}R^2\lambda^2[M]_{T_K}\right).$$

Now

$$\mathbb{E}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right)\right) =: \frac{R}{1-R}\alpha^{(R-1)/R}A_K$$

where

$$A_K = e^{\{m(1-R)/R\}T}\hat{\mathbb{E}}\left[\exp\left(\tfrac{1}{2}(1-R)\lambda^2[M]_{T_K}\right)\right].$$

This time the exponent in the expectation is bounded, so that we have an expansion

$$\begin{aligned} A_K &= e^{\{m(1-R)/R\}T}\left[1 + \tfrac{1}{2}(1-R)\lambda^2\eta^2(1-\rho^2)\hat{\mathbb{E}}\left(\int_0^{T_K}\frac{S_t^2 e^{2\delta(T-t)}}{(X_t^0)^2}dt\right) + O(\lambda^4)\right] \\ &\le e^{\{m(1-R)/R\}T}\left[1 + \tfrac{1}{2}(1-R)\lambda^2\eta^2(1-\rho^2)\frac{S_0^2 e^{2\delta T}}{x^2}\left(\frac{e^{\Gamma T}-1}{\Gamma}\right) + O(\lambda^4)\right] \end{aligned}$$

Similarly

$$\mathbb{E}^{\mathbb{Q}_K}(S_T) \le S_0 e^{\delta T}\left(1 - R\lambda\eta^2(1-\rho^2)\frac{S_0 e^{\delta T}}{x}\left(\frac{e^{\Gamma T}-1}{\Gamma} - \epsilon\right) + O(\lambda^2)\right)$$

Combining these results we find that, for some constant c_0

$$\inf_\alpha \mathbb{E}\left(V\left(\alpha\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right) + \alpha(x + \mathbb{E}^{\mathbb{Q}_K}(S_T))\right) \le V_2(0, x, S_0; \lambda) + c_0\epsilon\lambda^2 + O(\lambda^3).$$