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Static versus Dynamic Hedging of Exotic Options: An Evaluation of Hedge Performance via Simulation

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ABSTRACT

A number of pricing models have been derived for exotic options that rely upon many of the perfect market assumptions made in the original derivation of European call options by Black & Scholes (1973). Such assumptions include continuous rebalancing of the replicating portfolio, no transaction costs when dealing and constant volatility (and correlation). A keystone of the Black & Scholes (1973) derivation is that a hedge can be constructed in continuous time that will allow risk neutral evaluation techniques to be employed.

Due to product specific characteristics for some exotic options, these assumptions may be inappropriate when dynamically hedging these products. This paper relaxes three of the perfect market assumptions including continuous rebalancing of the hedged portfolio, no transaction costs and constant (and known) volatility. Hedging simulations were run for a wide variety of exotic options with these assumptions relaxed and comparisons were made between the theoretical value of the option from the "perfect markets" pricing model and the estimated cost of hedging the option. As a benchmark for comparison the same price paths and assumed market frictions were used to determine the hedge performance of a dynamically hedged European call option.

In almost all cases, it is shown that the inclusion of transaction costs significantly increases the cost to sellers of hedging the option and the introduction of discrete time hedging and stochastic volatility increases the variability in the hedge performance. Given that a number of static hedging approaches have been proposed in the literature for a number of these exotic options, we also examine the relative performance of these strategies. Our findings suggest that neither dynamic nor static hedging strategies are universally superior for all exotic options. However, for the majority of the exotic options tested, the static hedging strategies seem to have less impacts from transaction costs, discrete time monitoring and stochastic volatility.

JEL classifications: C15, G13

Keywords: Exotic Options, Static vs. Dynamic Hedging, Imperfect Markets, Stochastic Volatility, Transaction Costs.

1. INTRODUCTION

Recently, the depth and breadth of the market for exotic contingent claims has expanded dramatically [see Moessner (2001) for a discussion of the current size of the exotic derivatives markets]. Concern has been voiced by regulators (see General Accounting Office [1994]) that these products are dangerous and could cause risks to the financial system. This is due to the supposed difficulty in effectively hedging the risks of these products. More recently, Moessner (2001) suggests that these products are particularly sensitive to liquidity risk and that market makers use alternative hedging strategies compared to European options.

For all of the exotic options examined in this paper, either analytic or numerical solutions to their pricing have appeared in the literature. A keystone in the derivation of these pricing solutions is the creation of a hedge portfolio that will under ideal market conditions exactly replicate the payoffs of the derivative security. Following the lead of Black & Scholes [1973] the derivative can be exactly replicated by the construction of a portfolio of the underlying asset (stock) and riskless bonds. This concept of dynamically rebalancing the hedging portfolio is fundamental to the derivation of the theoretical results. Unfortunately, these solutions stretch the perfect markets assumptions to their breaking points when exotic options are considered. Many authors have relaxed the perfect market conditions for European options and examined the inclusion of transaction costs [Hodges and Neuberger (1992), for example] and others [Hull and White (1987), among others] have relaxed the assumption of constant volatility. Recent work by Gondzio, Kouwenberg and Vorst (1999) considers both. The aim of our paper is to provide insights into hedging problems associated with exotic options. We will consider hedging performance under market imperfections. Such frictions will include discrete time hedging, transaction costs and stochastic volatility and non-constant correlation.

A number of authors have previously examined the hedging of exotic options and have proposed two general approaches to the solution. The first requires the products to be dynamically hedged in the same manner as a standard European option. Kat (1996) examined the dynamic hedging of two types of exotic options (Asian and Lookback calls) and found that significant hedging errors occurred when hedge ratios from models based upon the Black & Scholes (1973) assumptions were used. His approach was to examine hedge results of the exotic options using a stochastic simulation model. This was extended from the approach first introduced by Figlewski [1989] and later used by Hull [1999].

Another approach to this hedging problem has been the static solution. This approach requires the construction of a portfolio of standard options (or other products) and this portfolio is maintained either until the expiration of the exotic contingent claim (if European) or until some event occurs prior to expiration. This could include early exercise (if an American option) or a barrier option being knocked in or out. At that point, the hedge is removed. Recent papers, which propose static hedging (using standard European options) for barrier and Lookback options include Derman, Ergener & Kani (1994, 1995), Carr, Ellis & Gupta (1998), Chou & Georgiev (1998), Toft & Xuan (1998) and Carr & Pichon (1999).

For other exotic options, static hedges have also been proposed by Rubinstein (1991) for Chooser options, for Digital/Binary options by Chriss & Ong (1995), for Asian options by Levy (1996) and by Thomas (1996) for compound options.

While a number of authors have proposed such static hedges, the effectiveness of these proposed strategies has rarely been tested. One paper by Thomsen (1998) has compared the relative benefits of such static strategies to traditional dynamic approaches for barrier options using an approach similar to that proposed in this paper. Recently, work by Davis, Schachermayer & Tompkins (2001) considers proposed static hedges for compound and installment options and compares the performance to dynamic hedging of these products. So far nothing has appeared in the literature regarding an examination of the suitability of both approaches for most major categories of exotic options and the establishment of a benchmark for comparison.

In this paper, we will examine hedge performance via Monte Carlo simulations [similar to Kat (1996)]. We will estimate the difference between the cost of hedging and the theoretical value of the claim when alternative hedging strategies are employed. The strategies we will consider include the traditional dynamic hedging approach and various static hedging strategies that have been suggested in the literature or are claimed to be used by practitioners who deal in these products. Of particular interest is the expected shortfall (for writers) when transaction costs are introduced and the variability of this difference when hedging is discretely monitored and the volatility (and correlation) is stochastic.

It is well known that the introduction of transaction costs will cause the option price to deviate from the theoretical value [see Leland (1985) and Hodges & Neuberger (1992)] and for the writers of the claims, the cost of hedging will exceed the theoretical value. Boyle & Emanuel (1980) demonstrated that the variability of the hedging portfolio increased when hedge rebalancing was done discretely. The inclusion of stochastic volatility will also increase hedge variability, with the expected price of the claim equal to the integrated volatility realised over the time horizon of the hedge. Neuberger (1994) and Anagnou & Hodges (2001) show that option hedging errors are directly related to the difference between the expected and realised quadratic variability (volatility) in the underlying asset. The latter paper shows that the major source of error is due to the incorrect estimation of the delta.

Another important issue is the establishment of a benchmark for the comparison of hedging performance. A number of authors have proposed a value at risk (VAR) approach. Given the lack of consensus on which variant of a VAR approach would be most suitable, we chose a more direct method of comparison. Given the extensive evidence in the literature that market frictions exist, we know that European options will deviate from their Black & Scholes (1973) values. Institutions dealing in these products recognise this and accept a margin of error when pricing and hedging these products. Notwithstanding these errors, most institutions choose to hedge these products using the dynamic hedging approach proposed by Black & Scholes (1973) [see Moessner (2001) for a recent survey of how market makers in options markets hedge these products]. Therefore, we assume that these hedging errors are accepted as the nature of the business. Under this assumption, we will establish as our benchmark for comparison the case of a dynamically hedged European call option. Using the same sample paths (and assumed market frictions) for both European calls and proposed hedging strategies for exotic options, we can compare whether the relaxation of perfect market conditions is more or less important for exotic options (compared to European calls). Simply said, if the proposed hedging strategy for the exotic options performs as well as (or better than) the European call benchmark example, it is logical that the institution would find this hedging strategy acceptable. Furthermore, the better the performance of the hedging strategy, the more likely it is that the firm would choose to implement this approach.

In the first section of this paper, we will outline the simulation process and evaluate the dynamic hedging of a European call option on an underlying asset that pays no dividends. We will relax the assumptions of perfect market conditions and introduce discrete time hedging, proportional and fixed transaction costs and stochastic volatilties (and correlations) to assess the impact on hedge efficiency.

In the following sections, we will examine most of the major categories of exotic options. Section 3 will examine single factor exotic options (based on a single underlying asset) include compound options, simple digital/binary options, range binary options, Chooser options, average rate options, a variety of barrier options. For these instruments, we will examine both a dynamic hedging approach and a static hedging solution for a variety of market imperfections. In Section 4, we consider the dynamic hedging of multi-factor exotic options (based upon multiple underlying assets). As the valuation of these products depends upon the correlation between assets, we will introduce stochastic correlation and examine the impact on hedging performance. Given that there are no securities that allow hedging of correlations, static hedging strategies can not be used and we restrict our analysis solely to dynamic hedging approaches. We will consider two such multi-factor options: an option to exchange one asset for another and a currency translated (Quanto) option. For all these examples, we will compare the results ultimately to our benchmark, which is the hedging example of the European call to evaluate the effectiveness of the approach.

Finally, conclusions and suggestions for future research appear.

2. EVALUATION OF OPTION HEDGING PERFORMANCE BY SIMULATION

One of the key principles in pricing options is that a hedge portfolio can be created which will allow the seller of the option to achieve a riskless portfolio. Based on the analytic approach to pricing options, the partial differential equation describes the factors that comprise this portfolio. If we assume that the only stochastic variable that influences the options price is the price of the underlying asset, then we are interested in determining the sensitivity of the option's price to a change in the price of the underlying asset. This parameter is known as the delta ($\partial f/\partial S$). By maintaining a continuously rebalanced position in the underlying asset that is equal to the delta, a riskless hedge can be obtained. This was an important insight of the Black & Scholes (1973) options pricing model, which is the foundation of option pricing in general.

For the derivation of the Black & Scholes (1973) model, certain perfect market assumptions were made. As discussed previously, three of the more contentious assumptions are that there are no transaction costs, volatility is constant and that rebalancing of the hedged portfolio occurs continuously. For the last constraint, Hull (1999) examined the impact of discretely rebalancing the hedge portfolio [this has been examined by a number of the authors, the first major contribution was by Boyle & Emanuel (1980)]. Hull examined via simulation the deterioration in the performance from delta hedging as the time between the hedge rebalancing was increased. He found that the hedge performance deteriorated as the period between hedge rebalancing was increased. As with Hull (1999), we will relax the assumptions of continuous and frictionless financial markets and will assess how the cost of hedging deviates from the theoretical values derived under perfect (continuous) financial markets.

To evaluate the hedging performance of European and exotic options, hedging was done by simulation. Underlying stock price series were simulated for 1000 paths of 180 days. These simulations used the anti-thetic approach suggested by Boyle (1977) and the control variate method suggested by Hull & White (1988) to increase efficiency. In a spirit similar to Hull (1999) [pages 314-316], we assumed that the stock did not pay dividends and compared the cost to the writer of hedging the option to the theoretical value. The price process for the asset was assumed to conform to equation (1).

$$dS_t = rS_t dt + \hat{\sigma} S_t dw_t \tag{1}$$

where r is the riskless rate and w_t a standard Brownian motion; thus (1) is the price process under the risk-neutral measure. The price series were generated using the usual Euler approach:

$$S_t = S_{t-1} \cdot e^{\mu dt + \hat{\sigma}_{t-1} dw_t} \tag{2}$$

Apart from the discrete increments (once per day) also assumed by Hull (1999), we included for the same set of simulated underlying prices, transactions costs when dealing in the underlying asset (or options in the static hedge). For this simulation, we assumed that the spread between the bid and offer price (of both the underlying and options) was fixed at 1/8th. In addition, a proportional cost reflecting a commission of 0.5% was charged (of both the underlying values of the stock purchase or of the option) whenever a transaction took place. [1]

The hedging in discrete time with transaction costs was further distinguished by the examination of two cases. The first case assumed that the volatility (σ) is a fixed constant (the Black-Scholes model) and the second case allowed the volatility to be stochastic. The stochastic process assumed for modelling the volatility $(\sigma_t)_{0 \le t \le T}$ was a mean reverting process described by equation (3.1) [Hull & White (1987) model]. When this case was considered 1000 paths of stochastic volatility were simulated [using the Euler discrete approximation in equation (3.2)] and 1000 new price paths were determined with $\tilde{\sigma}$ replacing the constant volatility $\hat{\sigma}$ in equations (1) and (2). For both cases, the same random draws from the Wiener Process (w_t) were used to determine the price paths. The stochastic volatility model chosen can be expressed as:

$$d\sigma = k\sigma(\theta - \sigma)dt + \xi\sigma \cdot d\tilde{w}_t \tag{3.1}$$

and the path of volatilities was determined using an Euler approximation of the form:

$$\hat{\sigma}_t = \hat{\sigma}_{t-1} + \Delta\sigma \tag{3.2}$$

In equation (3.1), κ represents the rate of mean reversion, which was set to 16 for this simulation. The term ξ reflects the volatility of volatility input and this was set to 1.0. The term θ is the long-term level of the instantaneous volatility and this was set to 20% per annum. These parameters were similar to ones reported by Hull & White (1987) for stocks. The term σ is the instantaneous volatility realised by the stochastic process and given that we are examining this model in discrete time, is replaced by $\hat{\sigma}$. In equation 9.1, the volatility inputs in the right-hand side of the equation are replaced by $\hat{\sigma}_{t-1}$. The term $\tilde{w}_t$ reflects draws from a Wiener Process independent of the draws (w_t) used to determine the price paths.

2.1 DYNAMIC HEDGING OF A EUROPEAN CALL: SETTING A BENCHMARK

Following the customary dynamic hedging approach (with daily rebalancing of the portfolio), an arbitrary notional amount of 100000 shares was assumed and the delta amount of the stock was purchased. This was partially paid for by the receipt of the premium and the remaining funds required to purchase the delta amount of the stock was borrowed. The interest rate used in the simulation was 5% per annum and was assumed to be constant. The starting value of the stock price (S_0) was 98 and the strike price of the European call was 100

(K). Each subsequent day, the new estimated delta was determined and the hedge portfolio was revised. As with Hull (1999), the costs of rebalancing were accumulated until the option expired.

In the absence of transaction costs, the expected cost of hedging the option will exactly equal the theoretical value. In the stochastic volatility case, Merton (1973) and Hull & White (1987) demonstrate that the theoretical value of the option will be the Black & Scholes (1973) value evaluated using the average realised volatility of the holding period. The stochastic volatility process in equation (3.1) will yield an expected realised volatility of θ (which in this instance is 20%) and the expectation of a zero hedging cost is retained. Across the 1000 simulations, we determined the average cost of hedging and divided this by the theoretical value of the underlying call option at the inception. This can be expressed as:

$$AD = HC / C(s,t,K) \qquad \textbf{(4)}$$

Where AD is the average percentage difference, HC is the average hedging cost and C (*t, S, K*), is the theoretical value of the European call option underlying the compound call. Given that the expectation of the hedging cost in the absence of transactions costs is zero, to reduce the errors introduced by the selection of the 1000 random price paths, the average hedging cost was standardised to zero. This control variate approach [suggested by Hull & White (1988)] assured that for the first simulation scenario (solely discrete time rebalancing with no transactions costs and constant volatility) the average hedging cost would be exactly equal to the theoretical value.[2] This can be found in Table 1 (under the column labelled *Discrete Dynamic Hedging No Transaction Costs*) with an average hedging cost equal to 0.0%. In addition, we were also interested in assessing the variability of this hedge result. We measure this variability (or hedge performance) by dividing the standard deviation of the cost of hedging the option by the theoretical value of the underlying call option. This can be expressed as:

$$HP = \frac{\sigma_{HC}}{C(s,t,K)} \qquad \textbf{(5)}$$

This is similar to the measure of hedging performance proposed by Hull (1999).

In this table, the reader can see that this standard deviation was equal to 6.1% of the theoretical value of the option in the case of constant volatility and no transaction costs.[3]

	Constant Volatility		**Stochastic Volatility**	
	Discrete Dynamic Hedging, No Transaction Costs	**Discrete Dynamic Hedging, Transaction Costs**	**Discrete Dynamic Hedging, No Transaction Costs**	**Discrete Dynamic Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-8.6%	0.0%	-8.2%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	6.1%	6.4%	20.5%	20.4%

Table 1, Results of 1000 Simulation Runs for Delta Hedging a European Call

Using the same paths of 180 days (1000 trials), transaction costs were included in the dynamic hedging strategy. The delta used to construct the hedge portfolio was determined using the simulated price but when purchases of stock were required, the hedger was required to pay $1/16^{th}$ (units) more and when selling would receive $1/16^{th}$ less. In all instances, a 0.5% commission was charged relative to the value of the transaction. In the table, under the column *Discrete Dynamic Hedging Transaction Costs*, one can see that the average cost of hedging the call option for 1000 simulations has risen to 8.6% of the theoretical premium value. The standard deviation (hedge performance) is essentially unchanged at 6.4% (from 6.1% for discrete hedging without transaction costs). This is to be expected as the inclusion of transactions costs to the same price paths only serves to increase the average cost of hedging and will not affect the deviation of the performance.

Of further interest was the impact of stochastic volatility on the dispersion of hedge costs. In Table 1, the reader will find the results of this hedging simulation under the column *Stochastic Volatility*. As stated previously, the averaged expected hedging cost will be zero [Merton (1973) and Hull & White (1987,1988)]. This allows the use of the control variate method to increase the efficiency of the simulation. As would be expected, the standard deviation of the hedging costs has risen to 20.5%. Clearly, the variability of the hedging costs indicates that when volatility changes the price path that could occur for the underlying asset, the dispersion of hedging results increases three-fold.

Finally, to assess both the impacts of transaction costs and stochastic volatility on the hedge costs the compound option; the 1000 simulation runs included both conditions. The results of this simulation can be seen under the column *Stochastic Volatility Discrete Dynamic Hedging Transaction Costs.* As one would expect, both the hedging costs and the variability of the difference between the costs and the theoretical value rise. The average hedging cost was 3.14% higher than the theoretical value of the option (similar to the constant volatility discrete hedging with transaction costs) and the standard deviation of hedging costs was 20.4% (similar to the stochastic volatility discrete hedging without transactions costs).

This approach will be used throughout the paper when we evaluate the hedging of exotic options. At each point, we will evaluate a dynamic hedging strategy for these products using the deltas from the pricing model and hold the appropriate amount of the underlying asset.[4] Rebalancing will be done on a daily basis and the term of the option will be 180 days. These results will be compared to the European call option example provided above, with this example serving as a benchmark. For some exotic options, we will also examine a static hedging approach that entails the construction of a portfolio of standard products that once they are established remain unchanged throughout the life of the exotic option. The goal of this analysis is to provide the reader with a means to evaluate which hedging strategies are more appropriate for the various types of exotic options. By using the hedging simulation for the European call option as a benchmark, we will also be able to compare the hedging approaches between the exotic option categories.

3. HEDGING EXOTIC OPTIONS VIA SIMULATION

3.1 COMPOUND OPTIONS

The pricing model determined the compound or the installment theoretical price and the respective delta (relative to the underling stock price) using a binomial approximation to the Geske (1987) model. The prices of the instruments and their derivatives required for dynamic hedging were estimated from the Monis software system. 180-day compound call and put options (with the right to purchase or sell a 3 month 100 European call) was considered. As with the benchmark European call option example, the starting value of the underlying stock was set to 98, with a fixed interest rate of 5%, no dividends and a starting volatility of 20%. Using these inputs, the initial theoretical value of both compound options ($\hat{p}$) was equal to 4.56 with the striking price equal to this amount (p_1 = 4.56).

3.1.1 DYNAMIC HEDGING OF COMPOUND OPTIONS

Table 2 provides the hedge results for a call on the call and a put on a call for the alternative market scenarios.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-4.0%	0.0%	-7.7%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	14.9%	14.7%	44.4%	43.3%

Dynamic Hedging of a Compound Call on a European Call

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-13.7%	0.0%	-12.7%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	23.1%	22.7%	46.6%	44.9%

Dynamic Hedging of a Compound Put on a European Call

Table 2, Results of 1000 Simulation Runs for Delta Hedging a Compound Call and Put

As with the European Call benchmark, the use of the control variate method yields no average difference between the theoretical value and hedging cost when transaction costs are ignored. However, the standard deviations of the hedging performance are greatly increased. For the call on the call hedge, this standard deviation was equal to 14.9% of the theoretical value of the option. This is almost 3 times the variation observed for the hedge of the European call option in the (solely) discrete time hedging scenario (at 6.1%). For the put on the call hedge, the results are even more variable. For the put on the call hedge, the standard deviation of this difference is almost 4 times the variation observed for the European call results at 23.1%. Clearly in both simulations, the dynamic hedging of compound options is more variable than for European call options even when only the assumption of continuous hedging is relaxed.

As with the European call option example, the inclusion of transaction costs increases the average cost of hedging. For this benchmark, the average cost is 8.6% higher than the theoretical value of the option. For the call on the call hedging simulation, the average difference is 4.9%. A worst result was found for the put on the call where the inclusion of transaction costs caused the costs of dynamic hedging to exceed the theoretical value of the compound by 13.7% (-13.7% in the table). As with the example for the benchmark European call, the standard deviation of the difference between the hedging costs and the theoretical value had not materially changed from the non-transaction cost scenarios.

The process for determining the stochastic process for the volatility remains the same as for the European call simulation with an additional factor. The difference is that at the expiration of the compound, the same variable volatility factor is applied to the estimation of the standard call the compound option can be exercised into. In Table 2, the reader will find the results of this hedging simulation under the columns **Stochastic Volatility**. For the benchmark European call, the standard deviation of the hedging error was found to be 20.5%. For the simulated hedge of the call on the call, the standard deviation of the hedging error is double that (at 44.4%). This is expected as the hedger is directly exposed to the level of volatility at the point the compound expires. For the put on the call, the standard deviation of hedging performance is also double that of the European benchmark (46.6%). Clearly, when dynamically hedging any of these products, changes in volatility levels substantially increase the variability in the hedge performance. However, compound options are even more sensitive to this than standard European options.

When both transaction costs and stochastic volatility are jointly considered, the call on the call simulation had an average cost of hedging which was 7.7% higher than the theoretical value and a much higher standard deviation of this difference of 43.3%. Finally, for the put on the call simulation, the costs of dynamic hedging were 12.7% higher than the theoretical value with a 44.9% standard deviation.

From these simulations, we can draw some conclusions regarding the effectiveness of dynamic hedging for compound options. For compound calls, the average difference between the hedging costs and the value of the options compares well to that of European calls. The problem is the much higher variability of this result. However, it appears that dynamic hedging for compound calls would be an acceptable strategy. For compound puts, in all scenarios except perfect market conditions, the average difference between the hedging costs and premium value is significantly worse than for the benchmark case. It is even worse that this variation in the performance is extremely high regardless of the simulation scenario. Thus, dynamic hedging of compound puts appears to be less viable than for standard European calls or compound calls.

This is most probably due to the fact that when hedging the exposure to another option (which is in itself a derivative of some underlying asset) and using the underlying asset itself to dynamically hedge, a second order risk is introduced which remains uncovered. Furthermore, compound options are much more exposed to changes to implied volatility levels at their expiration since the underlying option at that point is a function of both the underlying asset price and the level of implied volatility. For the European call option at expiration, the payoff is solely a function of the underlying price at that date. Thus, when one dynamically hedges compound options extra care must be taken, especially in regards to the future level of implied volatility that could occur.

3.1.2 STATIC HEDGING OF COMPOUND CALL OPTIONS – PART 1

Thomas (1996) has previously proposed that writers of compound call options could hedge the option with a static hedge consisting of the purchase of a European call option with a term to expiration of $T\text{-}t_0$ and a strike price of $K + e^{r(T-t_1)}p_1$. Davis, Schachermayer and Tompkins (2001) provide proof of this upper bound on the compound price

For the simulation of this hedge, exactly the same price paths as examined for the dynamic hedge were used. In addition, the same assumptions regarding the transaction costs and the stochastic volatility process were made. Based on the previous section on the upper boundary of the compound price, a 270 day European call with a striking price of 104.56 was purchased upon the sale of the 180 day compound call (p_1 = 4.56 and the underlying European call K=100). To fund the purchase of the static hedging call, borrowing at a constant rate of 5% was required in addition to the receipt of the compound premium.

If at the expiration of the compound option, the value of the underlying call (strike price K=100) was less than the compound payment (of 4.56), the purchased static call was sold at the current market price. The value of the European call was determined using the simulated underlying price (and volatility) at that point and the Black & Scholes (1973) formula was used. Otherwise, the compound payment was made to the seller who invested it into a cash account, ultimately the terminal payoffs of the two European options were examined. Table 3 displays the hedging results for the four alternative scenarios of the 1000 simulated price paths.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.4%	0.0%	-0.3%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	12.1%	12.1%	20.6%	21.5%

Table 3, Results of 1000 Simulation Runs for Static Hedging a Compound Call – Part 1

In this table, the almost negligible impact of the transaction costs is due to the fact that these costs only apply when the call is purchased and at expiration when it is sold. Regarding the average performance due to changes in volatility levels, the reader may recall that the expected level of volatility was 20%. Therefore, even though a stochastic volatility process was introduced, that would not change the expected (or average) result.

As opposed to the previous dynamic hedging simulations, the standard deviation of the hedging costs has increased but is not substantively impacted when volatility levels are allowed to vary. This is to be expected as the purchase of the European call allows hedging against stochastic volatility. Essentially, the static hedge losses are bounded and are equal to the initial borrowing required to establish the hedge. The potential gains are unbounded as profits exist when the compound fails to be exercised and the static hedger sells the static European call in the market.

3.1.2 STATIC HEDGING OF COMPOUND CALL OPTIONS – PART 2

Thomas (1996) suggests an alternative static hedge for compound calls. He proposes the purchase of a European call with the same term as that of the compound option and with the same theoretical value at the beginning of the period. For this hedge, we considered a compound call on a call with 180 days to expiration at, which point the underlying call option that could be bought, would also have 90 days to expiration. The strike price of the underlying option was once again $100 and the strike of the compound option was $4.56 which remained the expected value of the underlying option at the expiration of the compound. The volatility was fixed at 20% and the interest rate was a constant 5% for the term of the compound and the underlying option. For this hedge, it was assumed that a 180-

day call was purchased with a strike price of $102.5. At this strike price, the theoretical value of the standard option was exactly equal to the premium cost of the compound call. Table 4 provides the hedge results for a call on the call hedged in this manner both for all four market scenarios.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.3%	0.0%	-0.3%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	10.1%	10.2%	30.2%	29.8%

Table 4, Results of 1000 Simulation Runs for Static Hedging a Compound Call – Part 2

As with the previous hedging example, the average hedging performance is close to zero and is unaffected by transaction costs and changes in the level of the volatility. The impact of the transaction costs is again negligible because these costs only apply when the call is purchased and at expiration when it is sold. For the first scenario (solely discrete time hedging), the static hedge has a standard deviation of hedge performance of only 10.1%. This compares to the dynamic hedge (in Table 2) that had a standard deviation of 14.9%. Similarly, the static hedge also displays less variability when transaction costs are considered.

As with the previous hedging simulations, the standard deviation of the difference between the hedging costs and the premium received tends to increase dramatically when volatility levels are allowed to vary. However, it is clear that the variability of the static hedge result in Table 4 (at 30.2%) is lower than the variability of the dynamic hedging approach presented in Table 2 (at 44.4%). This is probably due to the fact that the purchase of the option covers the risks of changing volatility during the life of the compound (but provides no coverage of the volatility risk at the point both options expire). As the previous static hedge in Table 3 has less variability, this provides evidence that the nature of this error is substantially due to the impact of volatility on the underlying European call option at the expiration of the compound call.

Finally, when we combine both transaction costs and varying volatility (which can be seen under the columns **Transaction costs and Stochastic Volatility**), we see the overall benefits of this static approach to hedging a compound call. As one would expect, the average hedge performance is unaffected by transaction costs. The standard deviation of the hedge performance equal to 29.8% implies that this approach may also be a better approach to dynamically covering these products (when compared to the 43.3% standard deviation for the dynamic hedge). Even so, the previous static hedging example would be preferred as it also covers the implied volatility risk at the expiration of the compound.

3.1.3 STATIC HEDGING OF COMPOUND PUT OPTIONS

Recent work by Schilling (2001) extends Davis, Schachermayer and Tompkins (2001) and derives static hedging bounds on the price of compound put options on European calls. In this paper, Schilling (2001) proves the put-call parity relationship between European

Compound call and put options. Given that the compound call on a call is bounded by a standard European Call and that a put-call parity relationship holds for compound options, the put on a call can be expressed as:

$$Put_{Call} = (Call_{European} - p_1) - Call_{call} \quad \textbf{(6)}$$

This suggest that the purchase of a European call with 270 days to expiration with a strike price of $100 minus $4.56 (or a strike price of $95.44) and the sale of a call on a call with a strike price of $104.56 and a term of 270 days would provide an upper boundary on the value of the put on a call. Given put-call parity for European call and put options, this equivalent can be constructed with 270-day puts with the same strike prices. Since the seller of the compound put would receive a premium (at the onset of the transaction), they would reverse the strategy with put options to achieve the static hedge.

If the call on the call is bounded by a standard European call with a striking price of $K + e^{r(T-t_1)} p_1$, equation (6) suggests that a European put at this strike price would be purchased. The additional term on the right-hand side of equation suggests the sale of a standard European put minus the compound strike. This could be interpreted as selling a European put with strike price $K - e^{r(T-t_1)} p_1$. Subsequently, it will be shown that this simple vertical spread will provide an upper boundary on the value of the put on the call.

When this strategy was attempted, the bounds are no longer as tight as for the compound call and the cost of the hedge was much higher than the theoretical value of the compound put. Therefore, we considered a simpler strategy with 180-day put options. The rationale was that at the expiration of the compound, the holder would decide whether or not to sell the European call and receive $4.56 or let the option lapse and receive nothing. Thus, we constructed a vertical spread that matured at the same point in time as the compound put. For this hedge, we assumed that a 180-day European put was purchased with a strike price of $100 and a 180-day European put was sold with a strike price of $95.44 was sold. As with the previous static hedging example, the hedge portfolio was self-financing. Table 5 provides the hedge results for the put on the call hedged in this manner under all four market scenarios.

	Constant Volatility		**Stochastic Volatility**	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.2%	0.0%	-0.2%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	32.9%	33.1%	50.5%	50.8%

Table 5, Results of 1000 Simulation Runs for Static Hedging a Compound Put

As the hedge construction is self-financing, it is not surprising that the average difference is either zero or close to it. As with all static hedges, the impacts of transaction costs are minimal on this average result. What is more worrisome is the extremely high standard deviation of the hedging performance. When this is compared to the dynamic hedging results for the put on the call, the average results are better when transaction costs are included, the standard deviation of the hedging performance is worse.

This indicates that the simple upper boundary hedging approach will not provide an adequate static hedge for compound put options. Schilling (2001) also reports this and suggests the construction of an alternative static hedging portfolio with one long put and several short puts. This hedging portfolio is similar to the easy static hedges derived from the upper no-arbitrage bounds but consists of more standard options to gain a more accurate replication of the compound option's payoff function. While he fails to provide summary statistics for the hedging performance, it would appear that his alternative static hedge should be examined in preference to the one proposed here. Such an evaluation of this alternative static hedge remains for future research.

3.2 DIGITAL/BINARY OPTIONS

The most common form of this product is a straight bet that an underlying market will finish above or below the strike price at the expiration of the contract. Given that the terminal asset price is above (or below) a strike price, a fixed amount is paid. When cash is paid, this is referred to as a Cash or Nothing digital/binary option.

3.2.1 DYNAMIC HEDGING A CASH OR NOTHING DIGITAL OPTION

In this section, we will examine only the simplest European Digital/Binary options where the final price of the underlying asset determines whether a payout occurs or not. Specifically, we examine a cash or nothing digital option with 180 days to expiration, a strike price of \$100 and a payout of \$10 if the underlying market finishes above \$100. The pricing of these digital options used the variant of the Black Scholes (1973) model proposed by Reiner & Rubinstein (1991b). Table 6 provides the hedge results for this product.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-4.8%	0.0%	-5.3%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	21.2%	22.0%	36.0%	35.3%

Table 6, Results of 1000 Simulation Runs for Dynamically Hedging a Cash-or-Nothing Binary Call

The average results of dynamic hedging are similar to those found in Table 1 for the European call option. However, the variability is much higher for all market scenarios. The most likely reason for the increase in the standard deviation of hedging performance is that the delta of the digital option is not as well behaved as the delta of the European call. Reiner & Rubinstein (1991b) point out that the price of the cash or nothing digital call is equal to:

$$DigitalCall = Payout \cdot N(d2) \cdot e^{-rt} \quad \textbf{(7)}$$

where N(d2) is the second element in the Black & Scholes (1973) formula for the European call option. As the digital option approaches expiration, N(d2) approaches the value of N(d1). As N(d1) is the delta of a European call (assuming no dividends), the price of the digital option will approximately be equal to the delta of a standard call option times the payout of the digital. This, in turn, suggests that the delta of the digital option will be intimately related to the gamma of the European call. Furthermore, the gamma of the digital option will be related to the third derivative of the change in the European call price for a change in the underlying stock price. This derivative has been termed "Speed" by Garman (1995). Given that the behaviour of a European call gamma displays more extreme dynamics relative to the delta, it is not surprising that the delta hedging approach displays more variability. It is well known that the poor performance of discretely rebalanced delta hedges is directly related to the gamma of the option [see Boyle & Emanuel (1980)]. In a previous examination of dynamic replication of digital options, Chriss & Ong (1995) report that this approach breaks down in discrete intervals.

3.2.2 STATIC HEDGING A CASH OR NOTHING DIGITAL OPTION

Given the difficulty in dynamically hedging the higher order risks associated with digital options, a simple static approach has been proposed to address these risks. Chriss & Ong (1995) show that the common vertical spread constructed with European options will replicate the digital payoff at expiration. This can be achieved by the purchase of a standard option (call or put) at a lower strike price and the sale of a standard option (again call or put) at a higher strike price. When replicating the Cash or Nothing Digital/Binary option with standard option vertical spreads, one must solve three variables simultaneously. The first two are the strike prices of the standard options to be employed and the third is the quantity of spreads that will produce the same cash payoff if the underlying price is above the strike price at expiration. More formally:

$$\{0, P \mid S_t > K\} = Q\{[MAX(S_t - K_1),0] - [MAX(S_t - K_2),0]\} \quad \textbf{(8)}$$

where:
- P is the payoff from the exotic option (given that S_t is greater than K)
- Q is the quantity of the vertical spreads required
- S_t is the price of the underlying asset at expiration
- K is the strike price of the Digital/Binary option
- K_1 is the lower strike price European option
- K_2 is the higher strike price European option, and $K_2 > K_1$ & $K_2 = K$

Consider a Cash or Nothing Digital/Binary call that has a strike price of 100 and a fixed payment of 10 if the market finishes above 100 at expiration. This position could be hedged statically by the purchase of one 90 call and the sale of one 100 call for the same expiration. At and above 100, both the vertical spread and the Digital/Binary have a value of 10 and would be a perfect replication. Likewise, at and below 90, both structures would have no value. However, between 90 and 100, the vertical spread is always worth more than the Digital/Binary. Therefore, this vertical spread is an over-replicating portfolio and must represent an upward boundary on the value of the Cash or Nothing Digital/Binary call. However, tighter boundary conditions exist if it is possible to deal in European options in greater numbers and at strike prices approaching the strike price of the Cash or Nothing Digital/Binary call option. Chriss & Ong (1995) show that in the limit as the difference in the strike prices converges to zero and the quantity of the spreads required increases to infinity, the Digital/Binary has been exactly replicated. In practice, the establishment of the static vertical spread hedge is constrained by the liquidity of the European options markets.

For this hedging strategy, we purchased a 90 strike price call and sold a 100 strike price call both with 180 days to expiration. As this over-replicates the cash or nothing call paying 10 if the market finishes above 100 at expiration, the hedge requires borrowing to be self-financing. Once the strategy was constructed, we examined solely the terminal prices of the underlying stock at expiration and consolidated the payoffs of the static hedge deducting the costs of borrowing. The results of this hedge appear in Table 7.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging, Transaction Costs	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.4%	0.0%	-0.2%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	26.8%	26.4%	24.7%	24.8%

Table 7, Results of 1000 Simulation Runs for Static Hedging a Cash or Nothing Call with a Vertical Call Spread.

As expected, the average hedging result is zero with minimal impact from the addition of transaction costs or stochastic volatility. These results make sense when one considers that the only possible transaction costs occur when the vertical spread is established and unwound at expiration. As with previous hedge simulations, the average result should also be insensitive to stochastic volatility as long as the average of the volatility distribution remains at 20% (which was our assumption).

One significant difference for these series of simulations compared to previous hedging simulations, is the fact that the standard deviation of the difference between the hedging costs and the premium is similar across the four simulation scenarios. This result was between 24% and 26%. While this is higher than the standard deviations for dynamically hedging a European call option (in Table 1), it is lower than the dynamic hedging of the digital when we introduce varying levels of volatility. Even so, we must interpret these results carefully. The reader should consider that this static hedge would never be worth less than the cash or nothing digital option at expiration. If the final underlying price finishes between 95 and 100, it will be worth more. The standard deviation in this case indicates the dispersion of profits that would occur using this strategy less the fixed borrowing costs. In the ranges below 95 and above 100, the loss on this strategy would be equal to this fixed borrowing cost. Nevertheless, this approach compares favourably to the dynamic approach in Table 6.

While this hedging approach displays greater variability in hedge performance for the constant volatility simulations compared to the dynamic case, the static hedge reduces hedging variability in the simulations where volatility is stochastic. Furthermore, the hedging variability is remarkably stable for all the simulations. This result suggests that the introduction of non-constant volatility (which increases the standard deviation of the hedge performance for the dynamic hedging approach) makes no substantive difference to the static hedge. Therefore, this static approach to hedging digital options will address two of the major problems associated with hedging these products: the substantial transaction costs required when rebalancing the hedge and uncertain levels of market volatility. It is certain that choosing strike prices, K_1 and K_2 (and Q) such that these are closer to the actual digital payoff would reduce the variability. The choice of these values would be a simple optimal control problem with the constraint that the variability of the hedging portfolio must be equal to or less than the variability of the European call hedge in Table 1. This remains for future research.

3.2.3 DYNAMIC & STATIC HEDGING A RANGE BINARY OPTION

A particularly popular version of the binary option is a range binary option. This product is an extension of the Super-Share first proposed by Hakansson (1976). When these products appeared in the capital markets, they were offered in the interest rate field, where they were embedded within interest bearing securities. The holder of this security receives a payout at maturity that depends upon the behaviour of a specified underlying asset price or interest rate during the term of the option. Often, a typical structure evaluates the underlying asset price every day (typically at a fixed point in time such as the closing price) and counts the number of days the price remained with the range. For many of these structures, for each day the underlying price is in the range, some amount of the final payout accrues. At the expiration of the option, the number of days that the underlying price remained within the range is counted and divided by the total number of days in the period. This is multiplied by some fixed amount to provide the final payout to the holder. Essentially, these products are cash or nothing Digital/Binary options. However, when these products appear in this guise, they are known as a ***Wall option***. For our research, we will not consider products that accrue over time, but restrict our analysis solely to a structure where the payout is solely a function of the terminal price of the stock.

To examine how alternative hedging approaches would work for these types of binary options, we will consider a Wall option on the DAX index. As with the previous examples, we examined a range binary option with 180 days to expiration, 2000 from below and 2200 from above, with a payout of 100 bounding the range if the underlying market finishes in this range at expiration. The starting price for the DAX index was 2100, no dividends were paid by the DAX for this period, volatility was fixed at 20% and the interest rate was set at a constant 5% for the term of the option[5]. As before, the Reiner and Rubinstein (1991b) approach was used for the pricing of the range binary option and for the determination of the deltas required for dynamic hedging.

The static hedging strategy entailed combining two vertical spreads on both sides of the range. Thus, this hedge required four 180-day European options. We sold a 2000 put and a 2200 call and purchased a 1975 put and 2225 call. This strategy is commonly known as a condor spread. The number of condors was established such that the payout between 2000 and 2200 would be equal to 100.[6] Table 8 provides the hedge results for binary range option for the four market scenarios. For the sake of comparison, the dynamic hedging approach is presented first.

	Constant Volatility		**Stochastic Volatility**	
	Discrete Dynamic Hedging, No Transaction Costs	**Discrete Dynamic Hedging, Transaction Costs**	**Discrete Dynamic Hedging, No Transaction Costs**	**Discrete Dynamic Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-15.2%	0.0%	-14.0%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	48.6%	43.5%	87.0%	89.7%

Dynamic Hedging of a Range Binary Option (Wall Option)

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging, Transaction Costs	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.5%	0.0%	-1.0%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	46.6%	47.5%	39.3%	38.9%

Static Hedging of a Range Binary Option (Wall Option)

Table 8, Results of 1000 Simulation Runs for Alternative Hedging Strategies for a Range Binary Option

For the dynamic hedging strategy, the hedge results are similar to those observed for the cash or nothing binary option in Table 6. With the proviso that since two sides of the range must be hedged as opposed to only one side for the simple binary, this will increase both the cost of hedging and the variability. As is expected, this doubles the expected cost of hedging (due to transaction costs) and the variability in the hedge result is also doubled. Relative to the European call benchmark, hedge variability is four times as great for the final scenario (stochastic volatility). Therefore, this hedging approach would probably not be acceptable. On the other hand, the static hedge performs much better.

As expected, the impact of transaction costs on the static hedge is minimised. The relatively high level of variability of the hedge result (although relatively similar for constant and stochastic volatility) is of more concern. This variability is twice as high as that found for the benchmark European call in Table 1. This is because the hedge results will vary between a constant amount below 1975 and above 2225 by the cost of the borrowing (to make the hedge self-financing). This amounts to only 5% of the theoretical value of the Wall option. Between the two strike prices in both vertical spreads, a substantial profit accrues to the writer (approaching 100). Thus, the majority of the variability in the hedging performance is due to those price paths where the terminal underlying price finished in this range. Given that this hedging strategy displays less impact from transaction costs and less variability, the static approach would seem to be a superior strategy compared to the dynamic approach for this product.[7]

3.3 DYNAMIC & STATIC HEDGING A CHOOSER OPTION

This exotic option allows the holder to choose at a predetermined period whether the option that was purchased is a call or a put at a pre-specified exercise price. After this choice is made, the option remains a standard European call or put for the remaining term of the contract. These products are also known as *'As you like it' options* and *'Pay now, choose later' options*. They come in two varieties: simple Choosers and complex Choosers.

Simple Choosers allow the holder the choice between a standard call or put option with the same strike price and maturity date. Rubinstein (1991) first discussed these types of securities. The choice must be made on a predetermined date prior to the expiration of either of the two options. A complex Chooser allows the holder the choice between a standard call or put with different strike prices and potentially different maturity dates. Again there will be a single choice date where the holder has to decide which option they want and for both types of Choosers it is assumed the holder will choose the option which has a greater value at that point.

For this simulation, we will restrict our analysis to the simple Chooser. Rubinstein (1991) suggests a static hedging strategy for the simple Chooser (as a direct result of the derivation of the pricing formula). He states: "the pay-off from a Chooser will be the same as the pay-off from: (1) buying a call with an underlying asset price S, striking price K and time to expiration T, [and] (2) buying a put with an underlying asset price $S \cdot e^{-d(T-t)}$, striking price $K \cdot e^{-r(T-t)}$, and time to expiration t". Where T is the penultimate expiration date for the Chooser and t is the time of choice. Therefore, the simple Chooser has often been compared to a straddle, with the only difference that the standard European call and put option (with the same strike price) have different terms to expiration. Using this as our guide, we examined this static hedge and compared it the dynamic hedging approach.

For hedging a Chooser option, we purchased a longer maturity call and a shorter maturity put. The Chooser we examined had 180 days to the date of choice and at that point the holder could choose between a $100 strike price call or put with 90 further days to expiration. The starting price for the non-dividend paying stock was $100. The volatility was fixed at 20% and the interest rate was set at a constant 5% for the term of the analysis. The static hedge required the purchase of a 270-day call with a strike price of $100 and the purchase of a 180-day put option with a strike price of $98.78. Table 9 provides the hedge results for both the dynamic and static hedging strategies for the Chooser option across the four market scenarios.

	Constant Volatility		**Stochastic Volatility**	
	Discrete Dynamic Hedging, No Transaction Costs	**Discrete Dynamic Hedging, Transaction Costs**	**Discrete Dynamic Hedging, No Transaction Costs**	**Discrete Dynamic Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-3.5%	0.0%	-3.7%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	8.3%	8.8%	19.1%	19.7%

Dynamic Hedging of a Chooser Option

	Constant Volatility		**Stochastic Volatility**	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-0.1%	0.0%	-0.2%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	2.4%	2.4%	2.3%	2.4%

Static Hedging of a Chooser Option

Table 9, Results of 1000 Simulation Runs for Alternative Hedging Strategies for a Chooser

For the dynamic hedging approach, the hedge results compare well to the European call option. Not only is the impact of transaction costs less, but the variability of the hedge is also less than that found for the European call (especially under stochastic volatility). This result is expected as the Chooser is simply a combination of standard European options. Therefore, we would expect the dynamic hedging performance to be similar. The reduction in the average cost and hedging variability is due to the fact that the joint holding of a call and put option means that the delta exposure is less than for a standard European call. Therefore, less of the underlying stock needs to be purchased. Secondly, the gamma exposure is spread over time. Instead of a maximum gamma exposure at expiration, the Chooser has two periods where the gamma exposure rises. For one-half of the structure, this occurs at the time of choice and for the other half of the structure at expiration. Therefore, this spreading of the gamma risk over time reduces the variability relative to a European option, where the entire maximum gamma exposure occurs at expiration. Given this, it would appear that if institutions accept the European call option case as a benchmark for hedging performance, they would accept dynamic hedging of Chooser options as an viable approach.

As Rubinstein (1991) proves the existence of the static hedge, it is not surprising that this performs well. This superior performance is based on the immunity of the strategy to the addition of transaction costs or stochastic volatility. The standard deviation of the difference between the hedging costs and the premium is the lowest yet observed in these hedging simulations, at between 2.3% to 2.4%. This result is even better than the standard deviations for our benchmark case of dynamically hedging a European call option (in Table 1) or the dynamic hedging approach, which appears above. Therefore, those institutions that sell these products can either hedge the products dynamically (or simply include these in their European option trading books) or create a static hedge.

3.4 AVERAGE RATE (ASIAN) OPTIONS

An Average Rate (or *Average Price*) option is a simple standard call or put option but where the underlying price is replaced by the averaged prices observed during the prescribed interval or the entire life of the option. This average is estimated on an arithmetic basis using either the closing or settlement prices of the underling asset that occurred on every business day during the period. The average rate option is almost always cash settled at its expiration.

3.4.1 DYNAMIC HEDGING AN ARITHMETIC AVERAGE PRICE OPTION

To assess the effectiveness of delta hedging average rate options, we examined an arithmetic average price option with 180 days to expiration, a strike price of $100 and the averaging of the underlying asset price is done once per day. The volatility was fixed at 20% and the interest rate was constant at 5% for the term of the option. The price (and deltas) of the Asian option was determined using the analytic approximation for an option on the arithmetic average suggested by Levi (1992). It should be noted that the delta used for the dynamic hedging was expressed relative to the underlying spot price of the stock and not the average stock price to that date. Table 10 provides the hedge performance for dynamically hedging this product.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-5.0%	0.0%	-5.1%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	8.3%	8.3%	20.1%	20.9%

Table 10, Results of 1000 Simulation Runs for Dynamically Hedging an Asian Option

The results of the simulations compare well to that of the European call (in Table 1). However, some important differences are found. Specifically, the impact of transaction costs is slightly less for the Asian option relative to the European call. This is not surprising given that Levy (1996) points out that the delta and gamma dynamics of average price options are much more benign relative to European options. The lower impact of transaction costs simply reflects that less rebalancing is required as the average rate becomes seasoned. When stochastic volatility is introduced, the hedge variability is almost identical to that observed for the European call. Therefore, it appears that standard dynamic hedging of Asian options would perform as well as the dynamic hedging of European options.

3.4.2 STATIC HEDGING AN ARITHMETIC AVERAGE PRICE OPTION

Levy (1996) also suggests a static hedging strategy for Asian options. He states that it is common market practice for sellers of average rate call options to purchase a European call option with a different term to expiration but with the same theoretical value. This approach entails: "[choosing] a European option position that offsets the gamma and volatility exposures. A rule of thumb is to choose a European option with a similar strike to the [average rate option] but with expiration $(T-t_0)/3$, that is one-third of the averaging period of the average rate option" (page 92).

For this hedge, we considered the same arithmetic average price option as before and the hedge consisted of the purchase of a European call option with a strike price of $100 and a term to maturity of 60 days. In this example, the premium cost charged for the Asian call was equal to the cost of 60-day European call option. Table 11 provides the hedge results for this hedging strategy across the alternative market scenarios.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging, Transaction Costs	Discrete Static Hedging, No Transaction Costs	Discrete Static Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	-0.0%	-3.0%	-0.0%	-3.7%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	33.2%	32.9%	38.6%	37.3%

Table 11, Results of 1000 Simulation Runs for Static Hedging an Average Price call with a Shorter Maturity European Call.

In this table, the average difference between the cost of the hedge and the value of the average option is zero in the absence of transaction costs. This is to be expected, as the theoretical prices of both options were equal at the inception of the period. The inclusion of transaction costs results in a much higher cost of hedging compared to other static hedges previously considered. This is due to the fact that the European option only covers the risk of the Asian option for the first 60 days of the period. After the expiration of the European option, the hedger must resort to dynamic hedging for the remaining life of the Asian option. This introduces the relatively high level of average hedging costs.

While these simulations show that the average hedging result is close to the theoretical value of the average price option, there is considerable variability in the hedge results. The standard deviation of the difference between the hedging costs and the theoretical value is between 33% and 39%. This is considerably higher than the hedge variations that occur when dynamically hedging the average price option presented in Table 10. This hedge variation is most likely due to the variability in the payoff of the European option. Furthermore, since the entire average price option premium is used to pay for the European option any costs of dynamically hedging incurred from that point forward must be covered at a loss and this will clearly affect the variability of the hedge results.

While the results of static hedging average price options with standard options is somewhat disappointing, the dynamic hedging approach appears to work as well for average price options as for European options. Thus, even though the pricing of average price options requires extensive modifications to standard option pricing theory to determine their value, the hedging of these exotics appears to be no worse than for European options.

3.5 BARRIER OPTIONS

A barrier option, like any option, promises a predefined payoff that is related to the price of some underlying asset. Compared to standard European options, the payoff of barrier options is determined by two functions of the underlying price. The first function is the terminal value of the underlying. This is conditioned by a second function: a critical level for the underlying asset specified at the beginning of the contract where if the underlying price breaches this level prior to the expiration of the option the nature of the option contract changes. This critical level is known as the barrier level or the trigger level. Either the option immediately ceases to exist or is replaced by a standard option. If the option is cancelled, it is commonly referred to as a *Knock Out option* or if it is replaced by another option, it is known as a *Knock In option.*

In this research we will restrict our analysis to the simplest Barrier option structures. We will examine those barriers where the monitoring of the underlying spot price (to determine whether the option has been created or destroyed) occurs discretely (once per day). We further assume that upon a barrier event being triggered, the price of both the underlying asset and of any options involved in a static hedge would be dealt at the price recorded at that time (with a provision for bid/offer prices).

3.5.1 DYNAMIC HEDGING OF THREE TYPES OF BARRIER CALL OPTIONS

Specifically, we examined three types of barrier call options with 180 days to expiration. The strike price of each barrier option was $100 with the down barrier at $90 and the up barrier at $110. We considered the dynamic and static hedging of (1) a $100 down and out call, knocked out at $90, (2) a $100 down and in call, knocked in at $90 and (3) an up and out $100 call knocked out at $110. The determination of the theoretical prices and hedge ratios (deltas) for these options were determined using the formulae presented by Reiner and Rubinstein (1991a). All the parameters remain the same as the previous simulations, with the exception that the interest rate was set to zero.[8] Table 12 provides the hedge results for all three types of barrier options assuming the dynamic hedging approach was used.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-8.7%	0.0%	-7.3%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	5.0%	6.4%	16.6%	18.4%

Dynamic Hedging of a Down & Out Call Option

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-9.4%	0.0%	-9.1%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	28.9%	30.6%	84.4%	82.1%

Dynamic Hedging of a Down & In Call Option

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-67.2%	0.0%	-77.3%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	195.3%	258.4%	370.8%	385.3%

Dynamic Hedging of an Up & Out Call Option

Table 12, Results of 1000 Simulation Runs for Dynamically Hedging Three Barrier Call Options

These three simulations display considerable variation in their performance. For the down-and-out call, the hedging results compare well to that of the European call benchmark. It appears that the average cost of hedging and the variability of the hedging approach is actually better than for the European option. According to Thomas (1996), "As normal knock-outs have less gamma than European options, there is a strong tendency to rely on delta hedging" (page 121). Furthermore, the functional form of the knockout call is everywhere smooth; monotonically approaching a value of zero as the barrier is approached.

Essentially, this behaviour is that of a European call with the additional benefit that if the barrier is breached, any further requirement for dynamic hedging is ended.

For the down-and-in call, much greater hedge variability was observed, four times as great as observed for the European call. This may seem counter-intuitive given that there is the well-known parity relationship between knock-out, knock-in and European options: knock-out call + knock-in call = European call. Since both the knock-out and European call can be successfully dynamically hedged, it would be logical to assume that the knock-in would also be a candidate for dynamic hedging. However, as this simulation shows, this is not the case. The dynamic hedging approach will work well when the option is far from the barrier and the underlying call option has not been triggered to that point. When the barrier is breached the delta immediately changes sign. Hedgers are exposed to an extreme change in the hedged portfolio at this point and significant hedging errors result [see Thomas (1996, page 121) for a deeper discussion of these problems].

Easily, the most variable of the dynamic hedging strategies occurs for the up-and-out knock out call. The problem with this product is that when the barrier is breached, the option must by design be worthless (assuming no rebate). Given that the option is deeply in the money immediately prior to breaching the barrier, the dynamic hedging portfolio must also be constructed such that the substantial amount of intrinsic value is eliminated for a small change in the price of the underlying asset. To achieve this, an enormous position in the underlying asset must be held (approaching infinity as the option is close to expiration and the price of the underlying is immediately below the barrier level). If the underlying breaches the barrier, the entire holding of the underlying asset must be liquidated. If the underlying price falls, the hedging portfolio will realise a loss equal to the intrinsic value. Therefore, it is not surprising that when transaction costs are considered, the average cost of hedging is substantial (in this case an average loss of 77%). The variability of the hedging strategy is 385% of the option premium. This extreme result is tempered by the fact that the theoretical value of the barrier option is extremely low. Nevertheless, this simulation indicates the extreme danger associated with a dynamic hedging strategy for these types of products. This risk is well known and has led a number of authors [see Derman, Ergener & Kani (1994, 1995)] to propose alternative static hedges for these type of barrier options.

Overall, it appears that dynamic hedging approaches are suitable for "out" options, where the underlying option is out-of-the-money when the barrier is breached. Hedging performance is better than for a European call. For "in" options (where the underlying option is out-of-the-money when the barrier is breached), dynamic hedging is more problematic due to the discontinuity existing at the barrier level. For "out" options, where the underlying option is in-the-money when the barrier is breached, dynamic hedging is extremely dangerous.

3.5.2 STATIC HEDGING OF DOWN & IN AND DOWN & OUT BARRIER CALLS

For these hedges, we considered the same three barrier options examined above and applied static hedges that have been suggested in the literature.

For down-and-in call hedge, an out-of-the-money put (at a strike price of 81) was purchased in the quantity (1.11) suggested from the put-call symmetry property discussed by Carr (1994) and implemented by Thomas (1996). This approach can be used only when the net cost of carry (interest rates minus dividends) is zero. This is because the spot price determines whether or not the barrier has been breached, while the expected spot price (futures price) determines the expected terminal payoff of the option. While latter work by Carr & Chou (1996) and El Karoui & JeanBlanc-Piqué (1997) consider the case when the cost of carry is non-zero, we have chosen to examine the simplest ideal case (thus, the reason for the assumption of zero interest rates for these simulations). The cost of the put was exactly met by the sale of the down-and-in call. The simulation monitored the price path of the underlying asset and if the barrier was breached, the put was immediately sold and the proceeds used to buy a $100 call. This was then provided to the holder of the knock-in option and all further hedging ceased. If the barrier was not breached during the simulation, the hedge remained unchanged.

For the down-and-out call, a similar static approach was used. Due to the parity relationship between European barrier options and European calls discussed above, it is trivial to show that: 100 Call Knocked Out @ 90 = 100 European Call – 100 Call Knocked in @ 90 **(9)**

Since a knock in call can be hedged statically by the purchase of the appropriate quantity of equidistant out-of-the-money puts, the appropriate static hedge for the knock out call is to purchase a European $100 call (for the same term as the barrier option) and to sell (1.1 times) the 81 put. The cost of the hedge portfolio was exactly equal to the premium receipt of the down-and-out call. If the price path breached the barrier, the $100 call was sold and the $81 put was purchased. From the put-call symmetry principle, these two options should have the same value and the expected hedge cost (at any time) would be zero. Finally, if the simulated price path did not breach the barrier, the payoffs of the long $100 call were compared to the knock-out option at expiration.

Table 13 provides the hedge results both barrier calls hedged in this manner for the four simulated market scenarios.

	Constant Volatility		**Stochastic Volatility**	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	-0.0%	-0.1%	-0.0%	-0.1%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	0.0%	0.0%	0.0%	0.0%

Results of the Simulation for a $100 Down-and-In Call with a Barrier at $90.

	Constant Volatility		**Stochastic Volatility**	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	-0.0%	-0.2%	-0.0%	-0.2%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	0.0%	0.1%	0.0%	0.1%

Results of the Simulation for a $100 Down-and-Out Call with a Barrier at $90.

Table 13, Results of 1000 Simulation Runs for Static Hedging Down-and-In and Down-and-Out Calls.

In this table, the average difference between the cost of the hedge and the value of the barrier options was exactly zero. This demonstrates (via simulation) the effectiveness of the put-call symmetry principle when the cost of carry is zero. There is a negligible impact of the transaction costs as to be expected. However, these amounts are trivial. They only impact the initial establishment and unwinding of the static option portfolio (if the barrier is breached). Also, there seems to be little impact from stochastic volatility. This makes sense as the option (or package of options) has exactly the same volatility sensitivity as the barrier option sold. This analysis assumes that the put-call symmetry relationship holds for actual option markets. Given that implied volatilities for actual option markets display the well-documented smile or skew patterns, this is not necessarily the case. However, the principle behind put-call symmetry can still provide guidance as to the construction of an appropriate static hedge even when implied volatility patterns are neither flat nor symmetrical about the barrier level. The principle is to determine the appropriate out-of-the-money put option, which is *expected* to be equal to the value of the call when the barrier is breached. This is examined by Carr, Ellis & Gupta (1998) and Aparicio & Clewlow (1997), who consider the nature of static hedges when the underlying price process is non-lognormal (and implied volatility patterns would not necessarily be flat or symmetrical).

The standard deviation of the difference between the hedging costs and the premium received is also zero (or extremely close to zero) for all the simulation runs. The only variability that is introduced is when transaction costs are considered. These results not only confirm the effectiveness of the put-call symmetry principle for these barriers but also imply that this approach is a vastly superior approach to dynamically covering these products.[9]

3.5.3 STATIC HEDGING OF AN UP & OUT BARRIER CALL OPTION

Thomas (1996), Carr, Ellis and Gupta (1998) and Carr and Picron (1999) have considered static hedging strategies for barrier options with discontinuities. Carr, Ellis and Gupta (1998) extend the approach of Derman, Ergener and Kani (1994, 1995) to create a portfolio of standard options that matches the boundary conditions of Up & Out call options. Initially, the terminal boundary condition will be considered. The first step is the purchase of a standard European call option with a strike price equal to that of the barrier option. This will match the terminal payoff of the barrier call from zero up to the barrier level. At the barrier level, the terminal payoff of the barrier will be zero. To achieve this a quantity of digital options must be sold such that the payoff of the cash or nothing digital option eliminates the intrinsic value of the call option that would be extinguished. The replication of the digital option follows the static replicating strategy discussed in the section on Digital/Binary options (section 3.2.2). As a review, a cash or nothing digital can be replicated by a vertical spread of standard European options. Expressing the replicating portfolio in the notation we have chosen for this article yields:

$$U \& O \cdot Call = C(S_t, K) - (B - K)DC(B) \qquad \textbf{(10a)}$$

where $C(S_t,K)$ is the value of a standard European call at strike price K, and $DC(B)$ is the value of a digital option which pays \$1 at the barrier level, B.

Next the barrier boundary condition must be considered. This means that if at any time prior to the terminal date of the option, if the underlying price breaches the barrier level, the barrier option and the replicating portfolio must be equal to zero. Two problems are introduced by the inclusion of this path dependent boundary condition. The first is that the purchase of the standard European call will also contain time value if the barrier is triggered prior to expiration. The digital option will only cover the intrinsic value.

To eliminate the time value of the standard European call, an Up & In put option must be sold such that when the barrier is triggered a short position in a standard European put with the same striking price as the call is established. The combination of a long position in a standard European call and a short position in a standard European put (with the same term to expiration and striking price) results in a long forward position. At the point the barrier is breached, this long forward contract with a striking price below the barrier level will be worth solely the difference between the striking price and the barrier level. This requires the hedging portfolio to be modified. This can be expressed in terms of standard European options and digital options as:

$$U \& O \cdot Call = C(S_t, K) - (K/B)C[S_t, (B^2/K)] - (B-K)DC(S_t, B) \qquad \textbf{(10b)}$$

where the middle term in equation (10b) relates the value of the Up & In put to a standard European call through Put-Call Symmetry.

The second problem relates to the construction of the digital option, which in the case of the barrier option is path dependent. Carr, Ellis and Gupta (1998) recognise this fact by restricting their analysis to a path dependent digital option, they refer to as an Up & In Bond. This security pays $1 at expiration if at any time during the life of the option the barrier K has been breached. Using our notation, they show that such a security can be replicated by the purchase of two European digital calls struck at the barrier level, K and the purchase of 1/K of a standard European call also with a strike price equal to the barrier level. This can be expressed as:

$$U \& I \cdot Bond = 2 \cdot DC(S_t, K) + (1/K) \cdot C(S_t, K) \qquad \textbf{(10c)}$$

Finally, as discussed in section 3.2.2, Chriss and Ong (1995) show that the digital call can be replicated by a infinite number of vertical spreads of standard calls with the strike prices of the two calls converging in the limit. This can be expressed as:

$$DC(S_t, K) = \lim_{n \uparrow \infty} n[C(S_t, K) - C(S_t, K + 1/n)] \qquad \textbf{(10d)}$$

Combining these four formulas yields the price of the Up & Out call and provides an indication of the appropriate static hedging portfolio. Equations (10a) and (10b) suggest that the first step in the static hedge would be the purchase of a standard European call at strike price K. From equation (10b), additional European calls would be sold at a strike price of B^2 / K and with a strike price at the barrier level B. The quantity required would be K/B of the former and (B-K)*(1/B) of the latter. Finally, 2*(B-K) of a vertical call spread must be sold with the lower strike price equal to the barrier.[10]

Even though Carr, Ellis and Gupta (1998) prove that this static hedging portfolio will statically replicate the Up & Out call, this replicating portfolio is relatively complicated compared to the static hedges previously presented for barrier options. Thomas (1996) states, "In fact the [static hedging] solution [for Up & Out call options] may be considered more difficult to manage than the original problem." (page 122). One reason for this is as the barrier is being breached, the hedger of the barrier option must immediately and simultaneously unwind the entire portfolio of options that comprise the static hedge. Apart from constraints on the simultaneous execution of all option contracts, the success of the hedge depends upon all contracts being transacted at their theoretical values. When one includes transaction costs, bid/offer spreads and illiquidity of out-of-the-money options with smallish nominal amounts, this will reduce the hedge effectiveness. Toft and Xuan (1998) also found that the higher the volatility of volatility, the quality of the static hedge deteriorates. Thomas (1996) suggests an alternative.

The approach of Thomas (1996) also entails the purchase of a standard call option with the strike price equal to X. Given that at the barrier level, K, this standard will have substantial value, while the barrier is worthless; some option must be sold. Rather than construct the static portfolio of Carr, Ellis and Gupta (1998), Thomas sells a single call option. The choice of the strike price and quantity of the call option to be sold is determined by a least-squared-errors decision rule which solves at a particular time to expiration the short position that would be equal to the initial call purchase (at the barrier level).

Rather than follow the suggestion of Carr, Ellis and Gupta (1998), we chose to test a simpler approach inspired by Thomas (1996). For this hedge, a series of 180 day European call options were purchased and sold. First, we bought a $100 European call. We also sold 5.57 $110 calls. The cost of establishing this portfolio was exactly met by the sale of the up-and-in call. The simulation monitored the price path of the underlying asset and if the barrier was breached, the hedging portfolio was immediately unwound. If the barrier was not breached during the simulation, the hedge remained unchanged and the results were evaluated at expiration. Table 14 provides the hedge results for the up-and-out call hedged in this manner.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-32.7%	0.0%	-33.0%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	387.7%	383.9%	420.7%	443.5%

Table 14, Results of 1000 Simulation Runs for Static Hedging an Up-and-Out $100 Call with a Barrier level of $110.

In this table, the average difference between the cost of the hedge and the value of the up-and-out call ranged between -32.7% to -33.0%. This implies that when transaction costs are considered, the seller of this product would on average realise a loss on the hedge. However, since this is measured relative to the theoretical value of the barrier option and this value was only 0.30, the absolute levels of the profit are small. Nevertheless, relative to the dynamic hedging approach in Table 13, the expected costs of dealing are less than expected from the nature of the static portfolio. Even so, it appears that the seller would have to charge a higher price for the product to break even (roughly 0.40).

It is worrisome that the variability of the hedging performance for this static hedging portfolio is even higher than for the dynamic hedging approach. In the final scenario (with stochastic volatility and transaction costs), the static hedge has a standard deviation of performance of 420.7% and the dynamic hedge of 370.8%. However, in both cases, these results are much too variable to be acceptable.

This result confirms the findings of Toft and Xuan (1998), that the static hedge performance deteriorates when volatility is stochastic. However, we find that performance deteriorates even when constant volatility is assumed. The major source of risk is that of discrete time rebalancing. We concur with Thomas (1996) that hedging of these products remains problematic. It remains for future research to uncover more robust hedging strategies for these products, which are insensitive to market frictions.

3.6 LOOKBACK OPTIONS

These products allow the holder to purchase or sell the underlying asset at the best price attained in the option's lifetime. For a call option, this means the minimum price for the period and for a put option this means the maximum price. The original evaluation of this product was done by Goldman, Sosin & Gatto (1979), who show that the pricing of a Lookback call can be thought of as a standard at-the-money European call plus the expected value of being able to reset the strike price as lower underlying prices are realised. This second option was further examined by Garman (1989) and was coined a "strike bonus" option.

3.6.1 DYNAMIC HEDGING OF LOOKBACK OPTIONS

For this experiment, we considered a 180-day Lookback call option where the initial underlying price was $100. The underlying asset price was monitored discretely (once per day) to determine if a minimum had been achieved. Table 15 provides the hedge results.

	Constant Volatility		Stochastic Volatility	
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-5.0%	0.0%	-4.9%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	6.8%	6.7%	21.8%	21.9%

Table 15, Results of 1000 Simulation Runs for Dynamically Hedging a Lookback Call Option

From this series of simulations, it appears that dynamic hedging is as effective for Lookback options as for European calls (see Table 1). In fact, the costs of hedging when transaction costs are included are somewhat less compared to the European benchmark. One possible reason is that this product is, in essence, a European call with the added benefit of a reduced possible range of values of the delta. Consider that for a European call, the range of deltas is between 0.0 and 1.0. For the Lookback call, the option is either at-the-money or in-the-money. When the underlying stock price falls, the strike price is reset to equal that new lower level (therefore at-the-money) and when the underlying price rises above a previously achieved minimum level, the option is in-the-money. Given the truncated range for the possible delta values, the exposure from changing delta levels (gamma) is also truncated and results in less rebalancing to remain hedged.

3.6.2 STATIC HEDGING OF DOWN & IN AND DOWN & OUT BARRIER CALLS

Evaluating the same Lookback call as in section 3.6.1, we implemented the "rolling down" strategy suggested both by Goldman, Sosin & Gatto (1979) and Garman (1989). This approach entailed the purchase of a European call with 180 days to expiration and a strike price equal to the starting value of the simulation ($100). The same price paths as were considered for the previous simulations were monitored. If for a particular price path, a new minimum level was achieved, the previously purchased call option was sold and a new call was purchased with the remaining time to expiration of the Lookback and with a strike price set at the new minimum level. If a new minimum was not achieved, the hedge required no further action. At expiration, the final costs of hedging were estimated by including the initial cost of the standard $100 call, the costs of rolling the call spreads down and the final payout required for the Lookback. This was then compared to the premium initially received for the Lookback call. Table 16 provides the hedge results for this hedging approach for the Lookback call assuming perfect and imperfect market conditions.

	Constant Volatility		Stochastic Volatility	
	Discrete Static Hedging, No Transaction Costs	**Discrete Static Hedging, Transaction Costs**	**Discrete Static Hedging, No Transaction Costs**	**Discrete Static Hedging Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-4.0%	0.0%	-4.7%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	29.0%	28.8%	40.2%	39.1%

Table 16, Results of 1000 Simulation Runs for Hedging a Lookback Call with a Roll Down Strategy.

The average hedging results compare well to the both the European call benchmark and to the dynamic hedging of Lookbacks in Table 15. Compared to other static hedging strategies, the impact of transaction costs is higher as the roll down strategy requires constant monitoring and relatively frequent trading. When comparing the variability of the static hedging approach, the roll down strategy is almost twice as variable compared to either the European call or the dynamic hedging approach. This is due partially to the slippage that occurs when monitoring the underlying price only once a day and partially to the introduction of stochastic volatility. If hedging were implemented in continuous time, the results would converge to the fair value of the Lookback. However, this would not be practical given that it would require continuous monitoring and would generate higher transaction costs. When volatility is not constant, the hedger will be exposed to the levels of volatility at which the options can be bought and sold. By buying an at-the-money call and selling (what is at that point) an out-of-the-money call, this amounts to dealing a vertical spread (bull spread). As volatility is changing the costs of this spreads will vary as well.

It appears that dynamic hedging of Lookback options is an acceptable strategy and superior to the roll down strategy, when the monitoring of the underlying for the achieved minimum is done discretely.

4. DYNAMIC HEDGING OF CORRELATION DEPENDENT OPTIONS

For all the exotic options we have examined so far, the basis for evaluation has been the price and volatility price paths for a single underlying asset. In this final section, we will consider more complicated products that require the simulation of multiple price and volatility paths. These products will typically allow the holder the right to exchange one risky asset for another or will allow the final payoff of the option to be in another currency (numeraire). While there is a rapidly expanding range of possible multi-factor correlation dependent securities, we will restrict our analysis to two such products: the Best-Of (Exchange) Option (also known as a Margrabe Option) and a Quantoed Call on a Stock.

As with the previous simulations, this will require us to generate 1000 price and volatility paths for the second asset. This was done using formula (2) for the new asset price process and formula (3.2) for the new volatility process. The assumed parameter values for these new processes remain the same as were used previously in the single market case. We assumed that the underlying price processes displayed dependence and that the volatility processes were independent. To simulate dependent processes, the stochastic process for the second asset was assumed to be a modified Wiener process, $\tilde{w}_t$ (in formula 2), using:

$$\tilde{w}_t = \rho \cdot w_t + \sqrt{1-\rho^2} \cdot \dot{w}_t \qquad \textbf{(11)}$$

where w_t is the draw from a Wiener process for the first asset (identical to the draws used for the previously examined simulations), $\dot{w}_t$ is an independent draw from a new Wiener process and ρ is the correlation between the two processes.

As correlations are critical to the valuation and hedging of these types of products, we needed to introduce stochastic correlation. To do this, we assumed an Ornstein-Uhlenbeck type process for the correlation similar to that proposed for volatility by Stein & Stein (1991) and Vasicek (1977) for interest rates.[11] The stochastic correlation model chosen can be expressed as:

$$d\rho = k\rho(\theta - \rho)dt + \xi \cdot d\tilde{w}_t \qquad \textbf{(12.1)}$$

and the path of correlations was determined using an Euler approximation of the form:

$$\hat{\rho}_t = \hat{\rho}_{t-1} + \Delta\rho \qquad \textbf{(12.2)}$$

In equation (12.1), κ represents the rate of mean reversion, which was set to 4 for this simulation. The term ξ reflects the volatility of volatility input and this was set to 0.2. The term θ is the long-term level of the instantaneous volatility and this was set to either +50% per annum for the Best-Of option or +20% for the Quanto.[12] As with the estimation of stochastic volatility in (3.2), The term $\tilde{w}_t$ reflects draws from a Wiener Process independent of the draws used to determine either the price or volatility paths.

4.1 BEST-OF (EXCHANGE) OPTIONS

One of the most common multi-asset options is the option to exchange one risky asset for another. Margrabe (1978) first considered this with a generalisation of the Black & Scholes (1973) model. When these options are combined with a holding in one of the underlying assets, the option is referred to as an Outperformance Option. These products have also been referred to as a Relative Performance Option, a Multiple or Dual-Strike Option and the Best-Of [Two Assets] Option.

This type of option has been referred to as a first order correlation dependent option as the hedging solution by Margrabe (1978) requires that delta hedging be done for both assets. As in Black & Scholes (1973), N(d1) represents the delta for the first risky asset and N(d2) represents the delta for the second risky asset. Maintaining a position in both assets suggests that we are interested in the correlation between these asset prices. Therefore, the effect is of order one. This will not be the case for the Quanto considered later. For this type of option, the correlation effect is of order two: that is, the correlation between a risky asset price and the value of the option premium.

As correlation is not directly traded in the markets, we are lacking a candidate for the creation of a static hedge. Clearly, European options on a single asset would not allow such a risk to be hedged. Therefore (and given that), since no static hedging strategies appear to exist for such products using European (single asset) options, we will restrict our analysis solely to dynamic hedging strategies.

4.1.1 DYNAMIC HEDGING OF A BEST-OF (EXCHANGE) OPTION

We examined an exchange option with 180 days to expiration where the holder has the right to exchange one risky asset for another (see Margrabe [1978]). Both assets start at prices of $100 and both had volatilities fixed at 20%. Neither asset paid dividends during the period. The correlation between the assets was initially assumed to be +50%. For the pricing of this option and the relevant hedging ratios (deltas), we used the Margrabe model. Table 17 provides the hedge results for the exchange option for five market scenarios (including stochastic correlation).

	Constant Volatility & Correlation		Stochastic Volatility & Correlation		
	Discrete Dynamic Hedging, No Transaction Costs	Discrete Dynamic Hedging, Transaction Costs	Stochastic Volatility No Transaction Costs	Stochastic Correlation No Transaction Costs	Stochastic Volatility & Correlations Transaction Costs
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-15.4%	0.0%	0.0%	-24.1%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	6.5%	9.5%	25.4%	40.6%	24.9%

Table 17, Results of 1000 Simulation Runs for Dynamically Hedging a Best-Of Option

For these simulations, we see divergences from the benchmark European call results. Exchange options are much more sensitive to transaction costs than the single asset European call. This is to be expected as two assets must be dynamically rebalanced and transaction costs are accumulated for both. Regarding the variability of the hedging results, the impacts of discrete rebalancing seems to be similar for both the Exchange and European call option. In some ways, this is intuitive as Margrabe (1978) showed that an exchange option is simply a special case of a standard European call. Likewise, when volatility is stochastic, the hedge variability is similar. To assess the sole effect of stochastic correlation, we assumed that the volatility series were constant (as in the first two scenarios) and simply allowed the correlation to vary according to equation 12.2. Here we observe that the variability is twice that of stochastic volatility. In the final scenario, we allow both the correlation and volatility to vary. The hedging variability is less than that observed for stochastic correlation and slightly higher than the stochastic volatility case. This is to be expected as the processes driving the stochastic correlation and volatilities are assumed to be independent. Thus, it appears that some sort of diversification effect may explain this result.

Overall, from the standpoint of hedge variability, it would appear that the dynamic hedging of Exchange options would be approximately as efficient as the dynamic hedging of European call options. The fact that when transaction costs are included, the average hedging cost is higher than that of the European call is also expected as two assets must be traded.

Therefore, if institutions accept as their benchmark for hedging efficiency the European call, they would accept the risks of dynamic hedging Exchange options. The fact that this structure is more sensitive to transaction costs, would suggest that the price would exceed that of the theoretical value by a higher margin than for the European call.

4.2 QUANTO OPTIONS

The term Quanto is an abbreviation for "quantity adjusted option". The terminal payout of a European option depends jointly on the final price of the underlying asset and some other variable that determines the quantity this terminal payoff will be multiplied by. While these products have been examined in a variety of contexts [see Jamshidian (1994) and Reiner (1992)], the most popular version of this product is as a Currency Translated Option.

In this guise, the product allows the payoff of a standard (typically a European) option to be expressed in an alternative 'numeraire', which is another currency. To make the exposition clearer and aid in our description of the dynamic hedging strategy, we will consider a concrete example. Consider an at-the-money call option on a particular equity, for example the French stock Michelin with a strike price of 250 FF. An American investor considers the purchase of this option. The Quanto allows the terminal payoff of this product to be expressed at an exchange rate fixed at the beginning of the transaction. For example, let us say 5 FF/US Dollar. The ultimate underlying the investor is interested in remains the foreign shares in French Francs. The function of the Quanto is to simply allow the expression of this contingent claim in his domestic currency. As the exchange rate of US Dollars for French Francs is also a stochastic variable, these products are also multi-factor options sensitive to the correlation between the stock price and the exchange rate. As with the Exchange option, these options allow the exchange of one risky asset for another, with the difference that the Exchange option exchange risky underlying assets, while the Quanto exchanges one risky underlying asset for an option. The fact that the option is the exchange asset means that the effect of the correlation for this product is of second-order.

4.2.1 DYNAMIC HEDGING OF A QUANTO OPTION

Reiner (1992) was the first to detail the hedging of Quanto options. For this simulation, we follow his lead. The first step is to hedge the standard contingent claim, in our case a European call on Michelin shares traded at the Paris Bourse. Clearly as this the same European call examined as our benchmark, we can simply modify the simulations reported in Table 1 (and dynamically hedge the option).[13] Using the formula proposed by Reiner (1992), we determined the US Dollar price of the option. In this example, we assumed the agreed exchange rate (and spot rate) was 5 FF/US$, the interest rates in both the US and France were 5%, the volatility of both the stock and the exchange rate were 20% and the correlation between the FF/US$ and Michelin shares was +20%. The US Dollar investor pays the initial premium to the seller, who will place this amount into a US Dollar deposit account and borrows in French Francs to fund the purchase of the Michelin shares required to implement the dynamic hedge. Intuitively, this can be seen as equivalent to a forward foreign exchange contract on the current value of the option premium.

To maintain the fixed exchange rate hedge of the stock option, everyday the seller must evaluate the value of the Michelin call in French Francs and assess the current holding in the US Dollar deposit account. Consider that on the day after the Quanto has been sold, the Michelin call price has risen on the Paris Bourse to 60 FF. The seller has an unrealised profit of 10 FF. This amount is the discounted present value of the expected additional terminal payoff and can be logically thought of as a FF deposit of 100 FF. At the agreed exchange rate of 5.0 FF/US$, the US Dollar account should hold 12 US$ (times 100 shares). This will not be the case as the account only has 10$ plus one day's interest. The seller must *borrow* 2 US$ to top up the account. This will amount to a reversal of the original portfolio that was equivalent to a foreign exchange forward contract: now the seller will be holding a FF deposit and a US Dollar borrowing.[14]

As the price of the option changes overtime, the hedge simply maintains the value of this option contract in the US Dollar deposit account. If the option finishes in the money at expiration, the final holding in the US Dollar account is simply paid to the US investor. By definition, this means that the Michelin call will also be in-the-money in Paris and the proceeds of the purchased call will be applied to covering the initial borrowing required to purchase it. For the 1000 sample price paths, we accumulated the daily transactions required to maintain the US $ value of the Michelin call at the fixed exchange rate for each day. The results in Table 18, indicate the overall results of this hedging approach.

	Constant Volatility & Correlation		**Stochastic Volatility & Correlation**		
	Discrete Dynamic Hedging, No Transaction Costs	**Discrete Dynamic Hedging, Transaction Costs**	**Stochastic Volatility No Transaction Costs**	**Stochastic Correlation No Transaction Costs**	**Stochastic Volatility & Correlations Transaction Costs**
Average Difference (%) Hedging Cost/ Theoretical Value	0.0%	-9.0%	0.0%	0.0%	-5.6%
Standard Deviation (%) of Hedging Cost/ Theoretical Value	15.4%	17.6%	26.4%	31.9%	26.6%

Table 18, Results of 1000 Simulation Runs for Dynamically Hedging a Quanto Option

The results of the five hedging simulations seem to suggest that dynamic hedging of Quantos is a viable hedging approach. The average cost of hedging is much lower than that observed for the Exchange option. This is because the main contributor to the transaction costs is the dynamic hedging of the European call (on the Paris Bourse). Therefore, the difference to the cost of hedging relative to the theoretical value of the option compares closely to that of the European call benchmark case and are much less than the Exchange option case. The variability of the hedging performance is higher for the Quanto compared to the European call in the first four scenarios. However, for the final (all encompassing scenario), the difference is only slightly higher.

One final and interesting point is that the risk solely due to stochastic correlation is much less extreme for the Quanto relative to the Exchange option. This is due to the fact that the quantity of foreign exchange risk (the alternative risky asset) is much less. It is to be expected that given the option premium is a relatively small percentage of the value of the underlying (alternative risky) asset, impacts of changes in the correlation are much more subdued.

For both varieties of multi-asset options considered here, it appears that dynamic hedging strategies are suitable methods for risk management. When market frictions such as discrete time hedging, transaction costs and stochastic volatility & correlation are included, the hedge performance is close to that observed for the hedging performance of European call options. It remains for future research to examine empirically whether the assumptions made in these simulations actually hold. The possibility of interdependence in the volatility and correlated processes may very well exist and could yield alternative conclusions to those found here.

5 CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

In this paper, we have examined the effectiveness of dynamic and static hedging approaches in addressing the risks of exotic contingent claims. Hedging performance was evaluated via a Monte Carlo approach for a wide variety of the popular exotic options and for a variety of market frictions and imperfections. As a benchmark for comparison, a European call option was dynamically hedged and the same price paths and market friction assumptions were applied to each category of exotic options.

The results varied among the types of exotic securities. Neither dynamic nor static hedging approaches were found to be universally superior. For many exotic options, the dynamic hedging approach performed as well as or better than the European call option benchmark case. This includes compound call options, Chooser options, average rate options, down-out-calls, exchange options and Quantos. Nevertheless, for many of these same exotic options, some appropriate static hedge was identified that would perform even better than dynamic hedging. It was found that both approaches failed to adequately address the risks of up-and-out calls and further research into how these products are hedged is warranted.

Finally, in the examination of the two major categories of correlation dependent exotic options (exchange options and Quantos) it was pointed out that correlation variability is a substantial risk for these products. Given that correlation is not directly traded in the capital markets, an important finding is the fact that for these products dynamic hedging approaches cannot cover this risk. This has important implications for risk management of this increasing important class of contingent claims.

Future lines of research could examine richer sources of market imperfections such as jump processes and liquidity (feedback) risks. In addition, an important contribution would be the development of a theory to explain the hedging variability identified for the products examined in this research. Finally, the evaluation of hedging performance was based upon an assumption that hedging errors are normally distributed. Davis, Schachermayer and Tompkins (2001) point out this is not the case for dynamic and static hedging strategies for compound and installment options. Evaluation of hedging performance may need to be redefined to take into account asymmetries in the distribution of hedging errors. From their work, it seems to suggest that the distribution of hedging errors for static hedging is positively skewed (more variability in gains) while for dynamic hedging, it is negatively skewed (more variability in losses). It remains for future research to consider this result for a wider range of exotic options. If such research were attempted it is likely that static hedging may prove to be an even more preferable alternative for the risk management of exotic contingent claims.

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ENDNOTES

[1] Thus, when price paths were simulated, the determination of the delta was based upon the simulated price. However, when the quantity of stock was purchased (sold), the dealing price was fixed at 1/16th more (less) than the simulated price levels (the units in which the price was quoted). As an example, if the simulated price were 100, the buying price was assumed to be 100-1/16 and the selling price was 99-15/16. This was assumed for the entire amount of the asset purchased or sold. The percentage commission of 0.5% was charged on the total amount purchased or sold of the stock and in this simulation we assumed the number of shares in the contracts was 100,000 shares. The same spread and commission rate applied whenever options were purchased or sold.

[2] Without the control variate method, a slight difference in the average hedging cost was found but this was not significantly different from 0.0.

[3] Using this standard deviation it is possible to use a T-test to assess if the average cost of hedging is significantly different from zero. In this case, the use of the control variate approach assures that the average hedging cost is zero. When a T-test is applied to the other cases, the inclusion of transactions costs significantly increases the costs of hedging and for the case with stochastic volatility with no transaction costs, the difference is by design equal to zero. Even without the use of the control variate adjustment this difference was not significantly different from zero.

[4] For all the exotic options we will consider, the prices of the instruments and their derivatives required for dynamic hedging were estimated from the Monis software system (associated with the London Business School).

[5] The evaluation of this structure used different strike price and underlying price levels compared to those used for the hedge evaluation of the European call in Table 1. To allow comparison, the same price paths needed to be used. The principle of homogeneity of option prices allows us to use the same price paths with a constant adjustment for the levels.

[6] As with the previous example for binary options, the cost of the condor exceeded the cost of the Wall option by 5%. To allow the hedge to be self-financing, this additional amount was borrowed.

[7] The use of the standard deviation of hedging performance assumes that the distribution of hedging errors is normally distribution. In Davis, Schachermayer & Tompkins (2001), the distribution of hedging errors is split into gains and losses. It was found that for dynamic and static hedging of compound options, the distribution is not symmetric. For dynamic hedging, there was more variability in hedging losses, while for static hedges there was more variability in hedging gains. They further point out that the very nature of static hedging means that there is a minimum loss potential, which does not necessary occur for dynamic hedging strategies.

[8] By assuming a zero interest rate, the price paths of the simulations will no longer be the same as the previous examples. However, to allow some limited comparison, the same draws of random numbers and the same stochastic volatility processes were used. In addition, the starting underlying price level was set to 100 from 98. This would result in a similar series of price paths with the drift adjusted and a different starting value but would yield approximately the same terminal prices. The reason for changing the interest rate was to simplify the use of the Put-Call symmetry result of Carr (1994) when static hedging. For all other simulations considered in this paper, the interest rate was assumed to be 5%.

[9] Carr & Picron (1999) commenting on an earlier draft of this paper suggest that the superiority of the static hedge for these barrier options reported here might be overly optimistic. However, Thomsen (1998) also found that the static hedging approach was superior to the dynamic hedging approach, although not as perfect as presented here.

[10] Of these standard option strategies, only the replicating strategy is really problematic. The Chriss and Ong (1995) approach implies an infinite number of vertical spreads must be established. To remedy this, Carr, Ellis and Gupta use a Richardson extrapolation technique suggested by Geske and Johnson (1984) for option pricing. For an Up & Out call similar to the one examined here, they were able to achieve adequate replication (convergence to five-decimal-place accuracy using a three-point Richardson extrapolation) with call options at only four strike prices.

[11] While the Stein & Stein (1991) and Vasicek (1977) have been criticised for potentially producing negative volatilities and interest rates, respectively, it is possible for correlations to assume negative values. Of more potential concern is that this model proposed in equation (12.1) could yield correlations outside the admissible range of ± 1.0. While this remains possible, the choice of parameter values was made that this possibility was remote. Furthermore, an additional condition was imposed such that $-1.0 \leq \hat{\rho} \leq +1.0$.

[12] As the model in equation 12.1 has not been proposed (to our knowledge) in the literature, we do not know what feasible (or likely) parameter values would be. Therefore, the choice of parameter values was arbitrary based upon minimising the possibility that the resulting simulated correlations were outside the admissible range of ± 1.0.

[13] A simpler alternative would be to simply borrow French Francs and purchase the call option from a market marker on the Paris Bourse.
[14] In practice, the seller does not enter the US Dollar money markets to borrow or lend to maintain the value of the premium in Dollars. This is achieved by foreign exchange contracts to continually converted back (from) into the US Dollar to maintain the value of the payout in dollars