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A recent topical problem is how to deal with claims on 'non-traded' assets. A natural approach is to choose another similar asset or index which is traded to use for hedging purposes. To model this situation, we introduce a second non-traded log Brownian asset into the well known Merton investment model with power-law utility. The investor has an option on units of the non-traded asset and the question is how to price and hedge this random payoff. The presence of the second Brownian motion means that we are in the situation of incomplete markets. Employing utility maximisation and duality methods we obtain a series approximation to the optimal hedge and reservation price. These are computed for some example options and the results compared to those using exponential utility.

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1 Introduction

Valuing claims on non-traded assets presents new challenges in option pricing theory. An agent expects to receive or pay out a claim on a non-traded asset, and must decide how to best manage this risk. One method is to choose another similar asset or index which is traded and use this for hedging purposes. Clearly the higher the correlation between the traded and non-traded assets, the better we expect the hedge to perform. However, there is a need to quantify such statements and to give a framework under which we evaluate the optimal hedge and reservation price using a close asset. This is the objective of the paper.

To model these ideas mathematically, we introduce a second asset into the Merton investment model (Merton [25]) on which no trading is allowed. In the Merton model, the agent seeks to maximise expected utility of terminal wealth, where utility is constant relative risk aversion, $U(x) = \frac{x^{1-R}}{1-R}$. When the asset price follows exponential Brownian motion, the optimal behaviour for an

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agent in the model is well known: a constant proportion of wealth is invested in the risky asset. Now suppose the investor has an option on the second non-traded asset, payable at time T. The problem is how to price and hedge this random payoff when trading in the second asset is not permissible. This is an incomplete markets problem and this type of risk is often called 'basis risk'.

These problems occur often in practice. An example given by Davis [6] is an option on Dubai oil, where the liquid market is in Brent crude. Other examples are a portfolio of illiquid shares hedged with index futures, a basket option hedged with an index, or a five year futures contract hedged with a one year futures contract. In many of these situations, the underlying assets can be traded (eg stocks in a basket option), however, transactions costs may make it preferable to hedge with an index. Another area where claims involving non-traded assets occur frequently is that of real options, see Dunbar [15] and the book by Dixit and Pindyck [9]. Some examples of real options problems include extraction rights to an oil reserve or the option to start up an R&D venture.

A related problem involving stochastic income has been examined in the literature, beginning with He and Pagès [18]. El Karoui and Jeanblanc-Pique [16] and Cuoco [2] both assume the income is spanned by assets but impose a liquidity constraint. Duffie and Zariphopolou [14], Duffie et al [11] and Koo [24] consider infinite horizon optimal consumption and investment with stochastic income imperfectly correlated with the risky asset. Numerical solutions have been given in Munk [26] using a Markov chain approximation. Duffie and Jackson [12] and Svensson and Werner [28] each consider a number of simple examples and Duffie and Richardson [13] find explicit solutions under a quadratic utility.

Detemple and Sundaresan [10] study a non-traded asset model as a special case of a portfolio constraint. Values are obtained numerically in a trinomial model of asset prices. Zariphopoulou [29] studies a related general problem of utility maximisation under CRRA and employs a transformation to reduce the pde to a linear one. The coefficients of the diffusion price process for a traded asset depend on a 'stochastic factor' correlated with the asset price, creating unhedgeable risks. An example in the paper looks at non-traded assets by obtaining price bounds for claims on the traded asset, where the price process is affected by the non-traded asset. This paper and the earlier paper of Henderson and Hobson [19] differ from Zariphopoulou [29] by directly pricing a claim on an non-traded asset by including it in the utility from wealth.

Davis [6] applies the dual approach to non-traded assets with the exponential utility function. Under exponential Brownian motion, he obtains an expression for the optimal hedge involving the solution to a non-linear pde. Hobson [20] took the primal approach to the same problem and also obtained the hedge as a solution to a non-linear pde. We later compare our results to those obtained using exponential utility.

Our model considers agents with constant relative risk aversion or power law utility. As is often the case, it appears there is no closed form solution for the utility maximisation problem in our model, as the pde resulting from the stochastic control problem is highly non-linear. We assume that the money value in the non-traded asset is small compared with wealth and that the payoff is bounded below. Under these assumptions, we "guess" the hedge and prove optimality using a dual approach. From this we obtain a series expansion for the value function and reservation price. The use of an expansion enables us to avoid solving the pde numerically and allows for easier interpretation.

Two examples, a call option and a 'power' payoff are used throughout the paper and prices and hedges are calculated from the general results.

The remainder of the paper is organised as follows. Section 2 sets up our model with an additional non-traded asset and defines the value function for the problem. We use the power law utility of the form $U(x) = x^{1-R}/(1-R)$ where $R = 1$ corresponds to logarithmic utility. The complete markets case when the non-traded asset can be perfectly replicated is treated in Section 3. Section 4 considers the incomplete case and we give an expansion for the value function of the agent as well as for her reservation price and the optimal strategy. The price and strategy are computed for the two example options. In Section 5 we compare our results with exponential utility and specialise to the examples. Section 6 concludes.

2 The Merton Problem with an additional Non-Traded asset

We consider the problem of an agent faced with receiving (or paying) a claim on a risky asset on which trading is not possible, or not allowed. The agent must decide how best to price and hedge this claim. Note we refer to the asset as non-traded, however this can be interpreted in a number of ways. The asset may not be traded at all, or else there may be some restrictions to prevent the agent trading it. One case is when it is illiquid and too expensive to trade, and another may be when the agent is not permitted to trade in the asset, as with executive stock options.

Begin by assuming the non-traded asset S follows an exponential Brownian motion

$$\frac{dS}{S} = \nu dt + \eta dZ \tag{1}$$

where Z is a Brownian motion and ν, η are constants. We will take $r = 0$ throughout the paper for simplicity, although this is equivalent to using discounted variables.

The agent is to receive (or pay) an option with payoff $h(S_T)$ at a future time $T < \infty$. A natural idea to approaching this problem is to look for a close or similar asset which is traded in the market, and use this asset to hedge the position. Introduce a traded asset P

$$\frac{dP}{P} = \mu dt + \sigma dB \tag{2}$$

where B is correlated to the Brownian motion Z, with correlation ρ. The idea is to choose P such that ρ is high, so we are mainly concerned with high, positive ρ. In practice, the asset P may be a related index, or another stock from the same industry group.

It is convenient to think of Z as a linear combination of two independent Brownian motions B and W. Thus

$$Z_t = \rho B_t + \sqrt{1-\rho^2} W_t.$$

For $|\rho| < 1$ the presence of a second Brownian motion W, and the fact that no trading is allowed on S, means that we are in an incomplete market situation.

Our agents' aim is to maximise expected utility of wealth, where, in addition to funds generated by trading, the agent is to receive (or pay out) λ units of the claim $h(S_T)$. The value function of the agent is given by

$$V(t, X_t, S_t; \lambda) = \sup_{(\theta_u)_{u \geq t}} \mathbb{E}[U(X_T + \lambda h(S_T))|\mathcal{F}_t]. \tag{3}$$

We will consider utilities with constant relative risk aversion of the form $U(x) = \frac{x^{1-R}}{1-R}$ for $R > 0$, $R \neq 1$. $R = 1$ corresponds to logarithmic utility.) For this choice of family of utility functions, utility is only defined for positive wealth. Wealth is given by $X_T = X_t + \int_t^T \theta_u (dP_u/P_u)$ for some adapted θ which is constrained to ensure that $X_T + \lambda h(S_T) > 0$ almost surely, see Karatzas and Shreve [23, Chapter 5.8]. Note that θ_t is the cash amount invested in the traded asset P at time t. For this wealth restriction to hold, we need the following assumption on the payoff:

Assumption A: Either:
(i) $0 \leq h \leq b$ (eg. put option) and λ can be positive or negative; or
(ii) $h \geq 0$ but not bounded above (eg. call option) and λ can only be positive.
If h is bounded below by $-c$ then by considering $\tilde{h} = h + c$ we reduce to the above cases.

This assumption allows for three of the four simple option positions. When this assumption does not hold (say a short call, where h is not bounded above but $\lambda < 0$) we have that V is identically minus infinity. (This problem is common to many utility functions.) This is because the potential obligation is unbounded, and no hedging strategy can completely remove this risk. In particular, for any X_T which can be generated from a finite initial fortune x, and investments in the traded asset P, we have

$$\mathbb{P}(X_T + \lambda h(S_T) < 0) > 0.$$

Since $U \equiv -\infty$ on the negative real line, we have

$$V(t, X_t, S_t, \lambda) = -\infty, \qquad |\rho| < 1, t < T.$$

This value function is a modification of the traditional Merton [25] problem to include the additional payoff. We can thus think of this problem as the Merton wealth problem adjusted to include the non-traded asset. In the simple Merton problem, we have

$$V(t, x) = \frac{x^{1-R}}{1-R} \exp\left\{ \tfrac{1}{2} \frac{\mu^2}{\sigma^2} \frac{(1-R)}{R} (T-t) \right\} \tag{4}$$

and if $\pi_t = \frac{\theta_t}{X_t}$ is the proportion of wealth invested in the risky asset, then

$$\pi_t = \frac{\mu}{\sigma^2 R} \tag{5}$$

which is constant, the so called 'Merton proportion'.

Now return to the problem with random endowment, h. We first show that V exists in $(-\infty, \infty)$. Under Assumption A, if (ii) holds so $h \geq 0$ and $\lambda > 0$ then $V(t, X_t, S_t; \lambda) \geq V(t, X_t, S_t; 0)$ where the 'no claim' position is given in (4) above. When (i) holds, $\lambda h \geq -|\lambda| b$ so $X_T + \lambda h(S_T) \geq X_T - |\lambda| b$ and $V(t, X_t, S_t; \lambda) \geq V(t, X_t - |\lambda| b, S_t; 0)$.

Now we can find a simple upper bound for V by considering the dual problem. The problem is to maximise $\mathbb{E}(U(X_T + \lambda h(S_T)))$ over feasible values of the terminal wealth X_T. For a positive random variable Λ consider

$$\begin{aligned}
&\mathbb{E}\left\{U(X_T + \lambda h(S_T)) - \Lambda\left(X_T - \left(x + \int_0^T \theta_t \frac{dP}{P}\right)\right)\right\} \\
&= \mathbb{E}\{U(X_T + \lambda h(S_T)) - \Lambda(X_T + \lambda h(S_T))\} + \mathbb{E}\{\Lambda(x + \lambda h(S_T))\} + \mathbb{E}\left\{\Lambda \int_0^T \theta_t \frac{dP}{P}\right\}.
\end{aligned}$$

Suppose Λ is of the form $\Lambda = \alpha d\mathbb{Q}/d\mathbb{P}$ for some change of measure $\mathbb{Q}$. Then, with $\tilde{U}(y) = \sup_x (U(x) - xy)$,

$$\begin{aligned}
\sup_{X_T} \mathbb{E}\{U(X_T + \lambda h(S_T))\} &\leq \inf_{\Lambda} \mathbb{E}\{\tilde{U}(\Lambda) + \Lambda(x + \lambda h(S_T))\} \\
&= \inf_{\alpha} \inf_{\mathbb{Q}} \left\{\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) + \alpha x + \lambda \alpha \mathbb{E}^{\mathbb{Q}} h(S_T)\right\}
\end{aligned}$$

For the power law utility $U(x) = x^{1-R}/(1-R)$ we have $\tilde{U}(y) = (R/(1-R)) y^{(R-1)/R}$. The problem is now to choose Λ in an optimal fashion. Set

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma} B_T - \frac{\mu^2}{2\sigma^2} T\right) = \frac{d\mathbb{P}^0}{d\mathbb{P}}$$

where $\mathbb{P}^0$ is the minimal martingale measure. It makes the price process P into a martingale without affecting the Brownian motion W. Under the minimal martingale measure of Föllmer and Schweizer [17], processes contained within the span of the traded assets (such as $\sigma B_t + \mu t$) become martingales, and martingales which are orthogonal to this space are unchanged in law. Thus under the minimal martingale measure $\mathbb{P}^0$

$$\begin{aligned}
\frac{dS}{S} &= \nu dt + \eta \rho dB + \eta \sqrt{1-\rho^2} dW \\
&= \left(\nu - \frac{\eta \rho \mu}{\sigma}\right) dt + \frac{\eta \rho}{\sigma}(\sigma dB + \mu dt) + \eta \sqrt{1-\rho^2} dW
\end{aligned}$$

where the final two terms in the last expression are both martingales. Thus S has drift $\delta = \nu - \frac{\mu \rho \eta}{\sigma}$ under $\mathbb{P}^0$. Then

$$\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \frac{R}{1-R} \alpha^{(R-1)/R} A,$$

where

$$A = e^{\{m(1-R)/R\}T}$$

and $m = \frac{1}{2} \frac{\mu^2}{\sigma^2 R}$. The minimisation over α involves finding the minimum of

$$\frac{R}{1-R} \alpha^{(R-1)/R} A + \alpha(x + \lambda \mathbb{E}^0 h(S_T)).$$

The minimum and our upper bound is now easily seen to be

$$\frac{1}{1-R}A^R(x+\lambda\mathbb{E}^0 h(S_T))^{1-R} = V(t, X_t, S_t; 0)\left(1+\frac{\lambda}{x}\mathbb{E}^0 h(S_T)\right)^{1-R}.$$

The value function can be used to find the price that the agent is prepared to pay for the claim $\lambda h(S_T)$. It is common folklore how to price in a utility maximisation framework: the idea is to compare the expected utility for an agent who does not receive any units of the claim to the expected utility of the agent who receives $\lambda h(S_T)$. The adjustment to the initial wealth which makes these values equal gives the so-called reservation price of the option. Equivalently, the investor is indifferent between the investment problem with zero endowment and the problem with the additional opportunity to buy the claim. Mathematically, given an initial (time 0) wealth of x, the reservation price is the solution to the equation $V(0, x-p, S_0; \lambda) = V(0, x, S_0; 0)$, see Hodges and Neuberger [21], Davis [4], Davis [5], Davis et al [7], Constantinides and Zariphopoulou [1], Hobson [20], Rouge and El Karoui [27] and Henderson and Hobson [19].

3 The complete markets case

In a complete market, the asset S can be traded and there is only one source of risk. Mathematically, the correlation is one and Brownian motions Z and B are equivalent. We may compute the price and hedge directly, as there will be a unique martingale measure. With $\rho = 1$, $dS/S = \eta dB + \nu dt$ giving the relationship

$$\frac{dS}{S} = \frac{\eta}{\sigma}\frac{dP}{P} + \left(\nu - \frac{\mu\eta}{\sigma}\right)dt$$

or

$$S_t = S_0 e^{ct}\left(\frac{P_t}{P_0}\right)^{\frac{\eta}{\sigma}} \tag{6}$$

where $c = \nu - \frac{\eta}{\sigma}\mu + \frac{1}{2}\eta(\sigma - \eta)$. The measure under which P is a martingale must also make S into a martingale. Call this measure $\bar{\mathbb{Q}}$ and $\nu = \eta\mu/\sigma$.

We wish to solve the utility maximisation problem in (3). By considering the new wealth variable $Y_t = X_t + \lambda\bar{C}_t$ where $\bar{C}_t = \mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T)$, we can solve the problem explicitly in this case. Define $\bar{C}_t^S = \frac{\partial}{\partial S}\mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T)$, $\bar{C}_t^{SS} = \frac{\partial}{\partial S}\bar{C}_t^S$. By using the pde for $\bar{C}$

$$\dot{\bar{C}} + \tfrac{1}{2}\bar{C}^{SS}S^2\eta^2 = 0,$$

Y solves

$$dY = \left(\theta + \frac{\lambda\bar{C}_t^S S_t\eta}{\sigma}\right)\sigma dB + \left(\theta + \frac{\lambda\bar{C}_t^S S_t\eta}{\sigma}\right)\mu dt = \tilde{\theta}_t(\sigma dB + \mu dt),$$

where $\tilde{\theta} = (\theta + \lambda\bar{C}_t^S S_t\eta/\sigma)$, and the agent seeks to maximise $\mathbb{E}U(Y_T)$. This corresponds to the Merton [25] problem, with a modified strategy. From the results of Section 2, the optimal $\tilde{\theta}$ is $\frac{\mu}{\sigma^2 R}Y_t$ and θ^*, the optimal amount of cash invested in P, is

$$\theta^* = \frac{\mu}{\sigma^2 R}Y_t - \frac{\eta}{\sigma}\lambda\bar{C}_t^S S_t = \frac{\mu}{\sigma^2 R}(X_t + \lambda\bar{C}_t) - \frac{\eta}{\sigma}\lambda\bar{C}_t^S S_t. \tag{7}$$

Using wealth Y, the value function is given by

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= \mathbb{E}[U(Y_T)|\mathcal{F}_t] = \frac{Y_t^{1-R}}{1-R} e^{(1-R)m(T-t)} \\ &= \frac{X_t^{1-R}}{1-R} e^{(1-R)m(T-t)} \left(1 + \frac{\lambda \bar{C}_t}{X_t}\right)^{(1-R)}. \end{aligned}$$

Following the arguments at the end of the previous section, the price the agent will pay for $\lambda h(S_T)$ is

$$p = \lambda \bar{C}_t = \lambda \mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T). \tag{8}$$

This is the expected value of the claim under the risk neutral measure $\bar{\mathbb{Q}}$, as is to be expected in a complete market.

4 The incomplete markets case

As described earlier, if $|\rho| < 1$, the market is incomplete as the position in S cannot be replicated with P. We assume the value of the claim under the minimal martingale measure $\lambda \mathbb{E}_t^0 h(S_T)$ is small relative to current wealth x. Henderson and Hobson [19] concentrate on the case where the claim is proportional to the share price S_T, ie. $h(S_T) = S_T$. This allowed scalings within the problem to be exploited to reduce the dimensionality by one. The resulting non linear pde was approached using a series expansion. If we follow a similar approach here and derive the pde associated with the value function, we have an extra variable. Using the value function in (3), write

$$V(t, X_t, S_t; \lambda) = \sup_{(\theta_u)_{u \geq t}} \frac{X_t^{1-R}}{1-R} \mathbb{E}\left(\frac{X_T}{X_t} + \frac{\lambda h(S_T)}{X_t}\right)^{1-R} = \frac{X_t^{1-R}}{1-R} g(T-t, S_t, X_t) \tag{9}$$

where $g(0, s, x) = (1 + \frac{\lambda h(s)}{x})^{1-R}$.

Using Itô on V and the fact that V is a supermartingale under any θ and a martingale under the optimal strategy gives

$$(-\dot{g} + g_S S \nu + \tfrac{1}{2} g_{SS} S^2 \eta^2) + \frac{[\mu(g_x + \frac{g(1-R)}{X}) + \sigma \rho \eta (g_{xs} S + g_s (1-R) \frac{S}{X})]^2}{4\sigma^2 (\frac{1}{2} g_{xx} - \frac{1}{2} g R(1-R)/X^2 + g_x (1-R)/X)} = 0 \tag{10}$$

This is a non linear pde in three variables, and the method used in the linear case does not seem straightforward. In this paper we take a different approach to treat the general claim $h(S_T)$. Since scaling can no longer be used, we conjecture the form of the optimal strategy θ^* and verify this gives an upper and lower bound on the value function which agree to order λ^2. For this approach, we need to assume $\lambda \mathbb{E}_t^0 h(S_T)/x$ is small.

In the main theorem of the paper, we prove that our conjecture for the optimal strategy is indeed optimal, and derive an expansion for the value function, V_2 up to order λ^2.

Theorem 4.1 *(1) Define $C_t = \mathbb{E}_t^0 h(S_T)$, $C_t^S = \frac{\partial}{\partial S} \mathbb{E}_t^0 h(S_T)$. For h and λ satisfying Assumption A, the optimal strategy θ^* is given by*

$$\theta^*(t, X_t, S_t; \lambda) = \frac{\mu}{\sigma^2 R}(X_t + \lambda C_t) - \frac{\eta \rho}{\sigma} \lambda S_t C_t^S + O(\lambda^2). \tag{11}$$

(2) Using θ^* we define

$$V_2(t, X_t, S_t; \lambda) = \frac{X_t^{1-R}}{1-R} e^{(1-R)mT} \left(1 + \lambda \frac{C_t}{X_t} - \frac{\lambda^2}{2} R\eta^2(1-\rho^2)\hat{\mathbb{E}} \int_0^T \frac{S_u^2 C_u^{S^2}}{X_u^{0^2}} du\right)^{1-R} \tag{12}$$

where $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \exp(\frac{\mu(1-R)}{\sigma R} B_T - \frac{1}{2}(\frac{\mu(1-R)}{\sigma R})^2 T)$ *and* $m = \frac{1}{2}\frac{\mu^2}{\sigma^2 R}$. *Then for λ and h satisfying Assumption A, the value function $V(t, X, S; \lambda)$ is given by*

$$V(t, X_t, S_t; \lambda) = V_2(t, X_t, S_t; \lambda) + O(\lambda^3). \tag{13}$$

We first consider some examples before returning to prove the Theorem.

Example 1: Taking $h(S_T) = (S_T - \bar{K})^+$ gives the important example of a call option. We must have $\lambda > 0$ (since h is not bounded above) thus the agent is long a call option. We can evaluate the price under the minimal martingale measure $\mathbb{P}^0$. Under this measure, recall S has drift $\delta = \nu - \frac{\mu\rho\eta}{\sigma}$ hence

$$\mathbb{E}_t^0 (S_T - \bar{K})^+ = C_t = e^{\delta(T-t)} S_t N(d_+) - \bar{K} N(d_-) \tag{14}$$

where $d_\pm = \frac{\ln \frac{S_t}{\bar{K}} + (\delta \pm \frac{1}{2}\eta^2)(T-t)}{\eta(T-t)}$ and

$$\frac{\partial}{\partial S} \mathbb{E}_t^0 (S_T - \bar{K})^+ = C_t^S = e^{\delta(T-t)} N(d_+)$$

giving an optimal hedging strategy of:

$$\theta_t^* = \frac{\mu X_t}{\sigma^2 R} + \lambda e^{\delta(T-t)} S_t N(d_+) \left[\frac{\mu}{\sigma^2 R} - \frac{\eta\rho}{\sigma}\right] - \lambda \frac{\mu \bar{K}}{\sigma^2 R} N(d_-) + O(\lambda^2). \tag{15}$$

If $\lambda = 0$ we regain the Merton 'constant proportion of wealth' hedge of (5). Taking $\delta = 0$ and $\rho = 1$ we recover the complete case of Section 3 and have $\bar{C}_t, \bar{C}_t^S$ and the optimal hedge.

We can examine the effect of changing ρ on hedge θ^*. To get a comparison, we fix μ and δ, the drift of S under $\mathbb{P}^0$. This means that ν, the real world drift, varies with ρ. Figure 1 shows hedge θ^* net of the Merton hedge in (5). Thus zero represents the Merton strategy. For this choice of parameters, the agent holds less of the asset than the Merton hedge, and this decreases with correlation. When $\rho = 0$, the agent follows a strategy close to the 'no claim' Merton strategy (in (5)) as the traded asset is no use in reducing risk. This strategy deviates from Merton as correlation increases. In Figure 2, the effect of changing S on the strategy is displayed. If we fix S then we recover the behaviour displayed in Figure 1. If S is low, and the option is out-of-the-money, then it is optimal to use the 'no claim' Merton hedge given as zero on the graph. If S is large, and the option is far in-the-money, the hedge differs most from the Merton hedge.

□

Example 2: Taking $h(S_T) = S_T^2$, we have a 'power' payoff. Again we need $\lambda > 0$ for Assumption A to hold. This example is used because the price can be calculated explicitly.

$$\begin{aligned} C_t &= \mathbb{E}^0 S_T^2 = S_t^2 e^{(2\delta+\eta^2)(T-t)} \\ C_t^S &= 2S_t e^{(2\delta+\eta^2)(T-t)} \\ \theta_t^* &= \frac{\mu X_t}{\sigma^2 R} + \lambda S_t^2 e^{(2\delta+\eta^2)(T-t)} \left[\frac{\mu}{\sigma^2 R} - \frac{2\eta\rho}{\sigma}\right] + O(\lambda^2). \end{aligned} \tag{16}$$

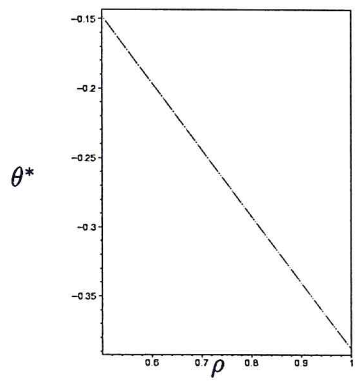

Figure 1: *The optimal hedge* θ^* *for the claim* $(S_T - \bar{K})^+$ *for* $0.5 \le \rho \le 1$. *Note the hedge is net of the Merton hedge (5) which is 326.53 in this example. Parameter values are* $\lambda = 0.01, S = 100, T = 1, \bar{K} = 100, x = 500, R = 0.5, \mu = 0.04, \eta = 0.30, \sigma = 0.35, \nu = \frac{\mu\eta\rho}{\sigma}$ *and* $\delta = 0$.

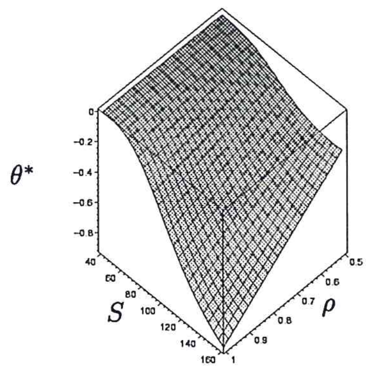

Figure 2: *The optimal hedge* θ^* *for the claim* $(S_T - \bar{K})^+$ *for* $0.5 \le \rho \le 1$ *and* $40 \le S \le 160$. *Note the hedge is net of the Merton hedge (5). Parameter values are* $\lambda = 0.01, T = 1, \bar{K} = 100, x = 500, R = 0.5, \mu = 0.04, \eta = 0.30, \sigma = 0.35, \nu = \frac{\mu\eta\rho}{\sigma}$ *and* $\delta = 0$.

Interestingly, the sign of the λ term in the hedge depends on $(\frac{\mu}{\sigma^2 R} - \frac{2\eta\rho}{\sigma})$ where the power two appears in the second term. For the call example in (15) the same factor decides the sign but the power is one. $\square$

Proof of Theorem 4.1: We demonstrate the strategy θ^* is optimal and derive an expansion for the value function by exhibiting upper and lower bounds for the supremum of expected utility which agree to order λ^2.

The exposition for the lower bound requires the fact that $\lambda h(S_T) \geq 0$ almost surely. Under Assumption A (ii), this is satisfied. Under (i), we have two cases. If $\lambda > 0$ then again $\lambda h(S_T) \geq 0$. If $\lambda < 0$ we write $-|\lambda|h = -|\lambda|b + |\lambda|(b-h)$. Thus the payoff has a positive component minus a constant. In our model, prices for claims are non-linear due to the appearance of $(C^S)^2$. Thus taking two claims h_1 and h_2, we could not simply compute the second order term for the sum $h_* = h_1 + h_2$ by adding components. However, if h_1 is a constant, then $(C_1^S)^2 + (C_2^S)^2 = (C_*^S)^2$ and the sum of the second order terms (only one of which is non-zero) is equal to the second order term for the sum. Hence we can split our claim h into a constant part $(-b)$ and a positive component $(b-h)$.

The Lower Bound

Consider first the zero endowment problem where (X_t^0, θ_t^0) is the optimal wealth, strategy pair. Then $dX_t^0 = \theta_t^0 dP_t/P_t$ with $\theta_t^0 = (\mu X_t^0)/(\sigma^2 R)$ and

$$X_t^0 = x \exp\left(\frac{\mu}{\sigma R} B_t + \frac{\mu^2}{\sigma^2 R} t - \frac{\mu^2}{2\sigma^2 R^2} t\right).$$

Now consider the problem with a random endowment of $\lambda h(S_T)$ at time T. We would like to consider the strategy in Theorem 4.1. However, with this strategy we cannot guarantee that wealth remains positive, so we use a localised version.

Fix K and let

$$H_K = \inf\left\{u : \int_0^u \frac{1}{X_t^0}\left(-\frac{\mu}{\sigma^2 R} C_t + \frac{\eta\rho}{\sigma} S_t C_t^S\right)\left(\frac{dP}{P} - \frac{\mu dt}{R}\right) = K\right\}.$$

Suppose $\lambda < \frac{1}{2}K^{-1}$. Consider the wealth process $X^{1,K}$ generated from an initial fortune x using the strategy

$$\theta_t^{1,K} = \frac{\mu}{\sigma^2 R}(X_t^{1,K} + \lambda C_t I_{(t<H_K)}) - \frac{\eta\rho}{\sigma}\lambda S_t C_t^S I_{(t<H_K)}$$

Then $X_t^{1,K}$ is given by

$$X_t^{1,K} = X_t^0 \left\{1 + \lambda \int_0^{t\wedge H_K} \frac{1}{X_u^0}\left(\frac{\mu}{\sigma^2 R} C_u - \frac{\rho\eta}{\sigma} S_u C_u^S\right)\left(\frac{dP_u}{P_u} - \frac{\mu}{R} du\right)\right\}.$$

Note that on $H_K < T$ we have $X_T^{1,K} = X_T^0(1-\lambda K)$ and indeed more generally $X_T^{1,K} \geq X_T^0(1-\lambda K)$. In particular the localisation times H_K allow us to bound the wealth process from below.

Now consider the sum of the wealth process and the random endowment. It is convenient to consider $Z_t^{\lambda,K} = X_t^{1,K} + \lambda C_t$. On $t \leq H_K$, using the pde satisfied by the option price C, we have

$$dZ_t^{\lambda,K} = \frac{\mu}{\sigma^2 R} Z_t^\lambda \frac{dP}{P} + \lambda S_t C_t^S \eta\sqrt{(1-\rho^2)} dW$$

so that, still with $t \le H_K$,

$$Z_t^{\lambda,K} = X_t^0 \left\{ 1 + \lambda \left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^t \frac{S_u C_u^S}{X_u^0} \eta \sqrt{(1-\rho^2)} dW \right) \right\}.$$

Also, $Z_T^{\lambda,K} = X_T^{1,K} + \lambda h(S_T) \ge X_T^0(1-\lambda K) + \lambda h(S_T) \ge X_T^0(1-\lambda K)$.

From Taylor's expansion we have $U(y+h) = U(y) + hU'(y) + \frac{1}{2}h^2 U''(y+\xi h)$ with $\xi = \xi(\lambda, K, \omega) \in [0,1]$. We will take $y = X_T^0$ and $h = Z_T^{\lambda,K} - X_T^0$, and consider the expected value of this expansion term by term. The first term yields $\mathbb{E}(U(X_T^0)) = V(0, x, S_0; 0)$. For the second term, note that

$$U'(X_T^0) = x^{-R} \exp\left(\frac{\mu^2}{2\sigma^2} \frac{(1-R)}{R} T \right) \frac{d\mathbb{P}^0}{d\mathbb{P}}$$

where $\mathbb{P}^0$ is the minimal martingale measure. Then since both X^0 and $X^{1,K}$ are martingales under $\mathbb{P}^0$, we have

$$\mathbb{E}[(Z_T^{\lambda,K} - X_T^0)U'(X_T^0)] = x^{-R} e^{(1-R)mT} \mathbb{E}^0(\lambda h(S_T))$$

For the final term in the Taylor expansion we have that for $\xi = \xi(\lambda, K, \omega) \in [0,1]$,

$$X_T^0 + \xi(Z_T^{\lambda,K} - X_T^0) \ge X_T^0(1-\lambda K).$$

Then, since U'' is increasing,

$$\begin{aligned}
&\frac{1}{\lambda^2}(Z_T^{\lambda,K} - X_T^0)^2 U''(X_T^0 + \xi(Z_T^{\lambda,K} - X_T^0)) \\
\ge\ & (X_T^0)^2 \left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)} dW \right)^2 U''(X_T^0(1-\lambda K)) I_{(H_K \ge T)} \\
&+ (h(S_T) - X_T^0 K)^2 U''(X_T^0(1-\lambda K)) I_{(H_K < T)}.
\end{aligned}$$

By the dominated convergence theorem, on taking expectations and letting $\lambda \downarrow 0$, we find for each K that $\lambda^{-2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda \mathbb{E}[h(S_T)U'(X_T^0)])$ is bounded below by

$$\begin{aligned}
&\tfrac{1}{2}\mathbb{E}\left[(X_T^0)^2 U''(X_T^0) \left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)} dW \right)^2 I_{(H_K \ge T)} \right] \\
&+ \tfrac{1}{2}\mathbb{E}\left[(h(S_T) - X_T^0 K)^2 U''(X_T^0) I_{(H_K < T)} \right].
\end{aligned}$$

Letting $K \uparrow \infty$ this expression becomes

$$\tfrac{1}{2}\mathbb{E}\left[U''(X_T^0)(X_T^0)^2 \left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)} dW \right)^2 \right]. \tag{17}$$

We can interpret $U''(X_T^0)$ as a constant multiplied by a change of measure which affects the drift of dP/P. With this interpretation it is straightforward to show that (17) becomes

$$\begin{aligned}
&-\frac{R}{2}\mathbb{E}\left[(X_T^0)^{1-R} \left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)} dW \right)^2 \right] \\
=\ & -\frac{R}{2} x^{1-R} e^{m(1-R)T} \hat{\mathbb{E}} \left[\left(\frac{\mathbb{E}_0^0 h(S_T)}{x} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)} dW \right)^2 \right] \\
=\ & -\frac{R}{2} x^{1-R} e^{m(1-R)T} \left[\frac{\mathbb{E}_0^0 h(S_T)^2}{x^2} + \eta^2 (1-\rho^2) \hat{\mathbb{E}} \left(\int_0^T \frac{S_t^2 (C^S)_t^2}{X^{0^2}} dt \right) \right],
\end{aligned} \tag{18}$$

where $\hat{\mathbb{P}}$ is the measure under which both $\hat{B}_t \equiv B_t - (\mu(1-R)/\sigma R)t$ and $\hat{W}_t \equiv W_t$ are Brownian motions.

In conclusion

$$\limsup_{K\uparrow\infty} \lim_{\lambda\downarrow 0} \frac{1}{\lambda^2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda\mathbb{E}[U'(X_T^0)h(S_T)])$$

is greater than the expression (18). Further manipulations yield that

$$\sup_{X_T} \mathbb{E}U(X_T + \lambda h(S_T)) \geq \mathbb{E}U(Z_T^\lambda) \geq V_2(0, X_0, S_0; \lambda) + o(\lambda^2),$$

where V_2 is given in Theorem 4.1, (12). Hence V_2 is a lower bound to order λ^2. Note that we can extend this result to prove that the correction is $O(\lambda^3)$ rather than just $o(\lambda^2)$ by considering higher order Taylor expansions of the utility function.

The Upper Bound

An upper bound on the value function will be found by considering the dual problem as in Section 2. We refine our choice of measure $\mathbb{Q}$ to obtain a higher order bound. For each $\epsilon > 0$ we show $V_2 + \epsilon\lambda^2$ is an upper bound.

Let $M_u = \eta\sqrt{1-\rho^2}\int_0^u (S_t C_t^S / X_t^0) dW_t$ and for any $K > 0$ define

$$T_K = \inf\{u : |M_u| + [M]_u = K\}.$$

Now choose K large enough so that

$$\hat{\mathbb{E}}[[M]_T - [M]_{T_K}] < \epsilon.$$

Let $\mathbb{Q}_K$ be given by

$$\frac{d\mathbb{Q}_K}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T\right)\exp\left(-R\lambda M_{T_K} - \tfrac{1}{2}R^2\lambda^2[M]_{T_K}\right).$$

Then

$$\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right)\right) = \frac{R}{1-R}\alpha^{(R-1)/R}A_K,$$

where

$$\begin{aligned} A_K &= \mathbb{E}\left[\exp\left(\frac{\mu(1-R)}{\sigma R}B_T + \frac{\mu^2(1-R)}{2\sigma^2 R}T\right)\exp\left((1-R)\lambda M_{T_K} + \tfrac{1}{2}R(1-R)\lambda^2[M]_{T_K}\right)\right] \\ &= e^{\{m(1-R)/R\}T}\hat{\mathbb{E}}\left[\exp\left(\tfrac{1}{2}(1-R)\lambda^2[M]_{T_K}\right)\right]. \end{aligned}$$

Note that the measure $\hat{\mathbb{P}}$ is the measure which arose in the calculation of the lower bound. Here $[M]_{T_K}$ is bounded so A_K can be written as an expansion in λ

$$\begin{aligned} A_K &= e^{\{m(1-R)/R\}T}\left[1 + \tfrac{1}{2}(1-R)\lambda^2\hat{\mathbb{E}}[M]_{T_K} + O(\lambda^4)\right]. \\ &\leq e^{\{m(1-R)/R\}T}\left[1 + \tfrac{1}{2}(1-R)\lambda^2(\hat{\mathbb{E}}[M]_T - \epsilon I_{(R>1)}) + O(\lambda^4)\right]. \end{aligned} \tag{19}$$

Now

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_K} h(S_T) &= \mathbb{E}h(S_T)\frac{d\mathbb{P}^0}{d\mathbb{P}}\left[e^{-\lambda R M_{T_K}-\frac{1}{2}\lambda^2R^2[M]_{T_K}}\right] \\ &= \mathbb{E}^0 h(S_T)\left[1-\lambda R M_{T_K}\right] = \mathbb{E}^0 h(S_T) - \lambda R \mathbb{E}^0 M_{T_K} h(S_T) + O(\lambda^2)\end{aligned}$$

If we now show

$$\mathbb{E}^0 M_{T_K} h(S_T) = x\hat{\mathbb{E}}M^2_{T_K} = x\hat{\mathbb{E}}[M]_{T_K} \tag{20}$$

then

$$\mathbb{E}^{\mathbb{Q}_K} h(S_T) = \mathbb{E}^0 h(S_T) - \lambda R x\hat{\mathbb{E}}[M]_{T_K} + O(\lambda^2) \le \mathbb{E}^0 h(S_T) - \lambda R x(\hat{\mathbb{E}}[M]_T - \epsilon) + O(\lambda^2). \tag{21}$$

Using Itô on M_tC_t and since M_t, C_t are $\mathbb{P}^0$-martingales,

$$\mathbb{E}^0 M_{T_K} h(S_T) = \eta^2(1-\rho^2)\mathbb{E}^0\int_0^{T_K}\frac{S_t^2\left(C_t^S\right)^2}{X_t^0}dt. \tag{22}$$

Now using $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{\frac{\mu(1-R)}{\sigma R}B_T - \frac{1}{2}(\frac{\mu(1-R)}{\sigma R})^2 T}$ and $\frac{d\mathbb{P}^0}{d\mathbb{P}}$, we derive the relationships

$$\frac{\left(X^0\right)^{1-R}}{x^{1-R}} = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}e^{m(1-R)T}$$

and

$$\frac{d\mathbb{P}^0}{d\mathbb{P}} = \frac{\left(X^0\right)^{-R}}{x^{-R}}e^{-m(1-R)T}.$$

Thus

$$\begin{aligned}\hat{\mathbb{E}}M^2_{T_K} &= \mathbb{E}\left[\frac{M^2_{T_K}}{e^{m(1-R)T}}\frac{\left(X^0\right)^{1-R}}{x^{1-R}}\right] = \frac{1}{x}\mathbb{E}^0 M^2_{T_K}X^0_T \\ &= \frac{1}{x}\mathbb{E}^0\int_0^{T_K}X^0(dM)^2 = \frac{\eta^2(1-\rho^2)}{x}\mathbb{E}^0\int_0^{T_K}\frac{S_t^2\left(C^S\right)_t^2}{X_t^0}dt\end{aligned} \tag{23}$$

using Itô on $M_t^2X_t^0$. Thus (20) holds using (22) and (23).

Now

$$\begin{aligned}\sup_{X_T}\mathbb{E}(U(X_T+\lambda S_T)) &\le \inf_{\alpha}\inf_{\mathbb{Q}}\left\{\frac{R}{1-R}\alpha^{\frac{1-R}{R}}A_K + \alpha x + \lambda\alpha\mathbb{E}^{\mathbb{Q}_K}h(S_T)\right\} \\ &= \frac{1}{1-R}(A_K)^R(x+\lambda\mathbb{E}^{\mathbb{Q}_K}h(S_T))^R\end{aligned} \tag{24}$$

and using (19) and (21) we see for some constant c_0

$$\inf_{\alpha}\mathbb{E}\left(V\left(\alpha\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right) + \alpha(x+\mathbb{E}^{\mathbb{Q}_K}h(S_T))\right) \le V_2(0,x,S_0;\lambda) + c_0\epsilon\lambda^2 + O(\lambda^3).$$

Higher Order Expansions

By combining the upper and lower bounds we conclude that, for λ and h satisfying Assumption A, the expansion given to order λ^2 given in Theorem 4.1 is valid. In order to extend this result, and to prove that the expansion can be continued to higher orders, it is necessary to refine the strategy θ used in calculating the lower bound, and the martingale measure $\mathbb{Q}$ for the upper bound. There are no obvious problems with this approach, although the calculations would become very involved.

□

We can also calculate the expansion for the reservation price, p the agent would be willing to pay for λ units of the claim $h(S_T)$. As discussed earlier, this involves solving

$$X_t^{1-R} = (X_t - p)^{1-R}\left[1 + \frac{\lambda \mathbb{E}^0 h(S_T)}{X_t - p} - \frac{\lambda^2}{2} R\eta^2(1-\rho^2)\hat{\mathbb{E}}\int_0^T \frac{S_u^2 (C^S)_u^2}{X^0{}_u^2} du\right]^{1-R} \tag{25}$$

where $X_0^0 = x - p$.

Theorem 4.2 *For h and λ satisfying Assumption A, the time t price p for λ units of $h(S_T)$ delivered at time T, given a current wealth X_t is:*

$$\begin{aligned} p(t, X_t, S_t; \lambda) = p &= \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2} x \hat{\mathbb{E}}[M]_T + O(\lambda^3) \\ &= \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2}\frac{\eta^2}{x}(1-\rho^2)\hat{\mathbb{E}}\int_t^T \frac{S_u^2 (C^S)_u^2}{(X_u^0/x)^2} du \\ &= \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2}\eta^2(1-\rho^2)\mathbb{E}^0\int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} du \end{aligned} \tag{26}$$

Note that when $\rho = 1$, $\delta = 0$ we recover the price in the complete market case, (8).

Example 1 continued:

We wish to calculate the price of the call with $h(S_T) = (S_T - \bar{K})^+$. From (26)

$$p = \lambda \mathbb{E}_t^0 (S_T - \bar{K})^+ - \lambda^2 \frac{R}{2}\eta^2(1-\rho^2)\mathbb{E}^0\int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} du \tag{27}$$

The first term is simple, we have a closed form expression in (14). The second term is more involved and we simulate this. The simulation results gave standard errors of less than 0.2%.

Note if we fix δ the second order term is increasing in ρ. The minimal martingale measure term is unchanged with ρ, so the price in (27) is increasing in ρ. This is consistent with the idea that as the correlation approaches 1, the traded asset gives a better hedge, and the position is less risky. The agent is thus willing to pay more for the claim.

A plot of the second order term in (27) is given in Figure 3. Parameters used are: $\lambda = 0.01, S_0 = 100, T = 1, \rho = 0.8, R = 0.5, \mu = 0.04, \eta = 0.30, \sigma = 0.35$, $\delta = 0$, $x = 500$ and $0 \le \bar{K} \le 200$. In both Figures 3 and 4, we plot the second order term divided by λ^2:

$$\frac{R}{2}\eta^2(1-\rho^2)\mathbb{E}^0\int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} du \tag{28}$$

or equivalently the second order term with $\lambda = 1$. In Figure 3, this is about 0.07, for $\bar{K} = 100$. For comparison, the first order term (with $\lambda = 1$ and for $S = 100$, $\bar{K} = 100$ say) is 11.924. Thus the second term is about 0.6 % of the first. Of course, if we use a larger value for R, we would get a

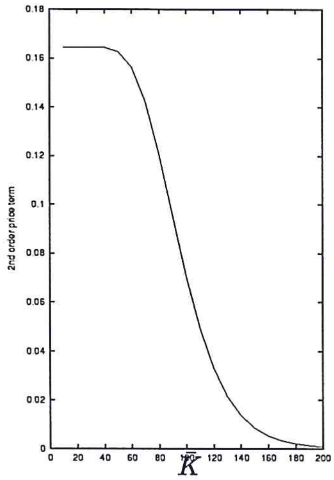

Figure 3: *The 2nd order term of the reservation price of the claim $(S_T - \bar{K})^+$ for $0 \leq \bar{K} \leq 200$, as given in (28). Parameter values are* $\lambda = 0.01, S = 100, T = 1, R = 0.5, \mu = 0.04, \rho = 0.8, \eta = 0.30, \sigma = 0.35$, $\delta = 0$ *and* $x = 500$.

larger ratio here. To obtain the comparison for different λ, we can simply multiply this percentage by λ.

Returning to Figure 3, as $\bar{K} \to 0$, the payoff approaches S_T and we recover the linear case in Henderson and Hobson [19]. In this case we have a simple second order term and it has greatest effect on the price. As $\bar{K} \to 200$ the option will not pay out and hence the second order term tends to zero.

For a strike of $\bar{K} = 100$ the second order term in (27) is graphed in Figure 4 for varying values of the asset price S. Note this appears to look like a call option payoff. With the parameter $\delta = 0$ note that the first term is simply the complete (risk neutral) price.

□

Example 2 continued: For $h(S_T) = S_T^2$ the price can be calculated explicitly.

$$p = \lambda S_t^2 e^{(2\delta+\eta^2)(T-t)} - 2\lambda^2 R\eta^2(1-\rho^2)\frac{S_t^4}{x}\left(\frac{e^{(3\eta^2+6\delta+\frac{\mu^2}{\sigma^2 R^2})(T-t)} - 1}{3\eta^2 + 6\delta + \frac{\mu^2}{\sigma^2 R^2}}\right) + O(\lambda^3)$$

□

Returning to some more general remarks on Theorem 4.2, if we consider the reservation price for the random payment of $\lambda h(S_T)$, and convert it into a unit price, we find

$$\frac{p}{\lambda} = \mathbb{E}_t^0 h(S_T) - \lambda \frac{R}{2}\frac{\eta^2}{x}(1-\rho^2)\hat{\mathbb{E}}\int_t^T \frac{S_u^2 (C^S)_u^2}{(X_u^0/x)^2} du + O(\lambda^2). \tag{29}$$

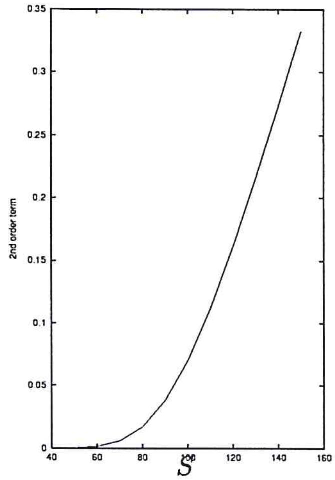

Figure 4: *The 2nd order term of the reservation price of the claim* $(S_T - \bar{K})^+$ *for* $50 \leq \bar{S} \leq 150$, *as given in (28). Parameter values are* $\lambda = 0.01, T = 1, \bar{K} = 100, R = 0.5, \mu = 0.04, \rho = 0.8, \eta = 0.30, \sigma = 0.35$, $\delta = 0$ *and* $x = 500$.

The 'marginal' price of a derivative is the price at which diverting a little money into the derivative at time zero, has a neutral effect on the achievable utility. This is given by

$$\lim_{\lambda \downarrow 0} \frac{p}{\lambda} = \mathbb{E}_t^0 h(S_T). \tag{30}$$

Of note is that the marginal price is independent of the risk-aversion parameter R. This is an example of a general result which states that the marginal price is independent of the utility function, see Davis [5], Hobson [20, Theorem 1] or Karatzas and Kou [22]. Further the marginal price is the expected payoff under the minimal martingale measure $\mathbb{P}^0$. Importantly, and unlike in the complete market scenario of Section 3, the marginal price the agent is prepared to pay for $h(S_T)$ depends on the drift μ of the traded asset.

As we remarked above, conclusions about the marginal price the agent is prepared to pay for the asset are independent of the agent's utility. However the reservation price for a non-negligible quantity of non-traded asset does depend on the utility as expressed in the λ^2 term in the expansion (26). Note that the correction term to order λ^2 is negative, since $[M]_T \geq 0$, using (26). This is because utilities are concave, so that the agent is prepared to pay a lower (unit) price for larger quantities.

We now examine the unit price in (29). Putting $\lambda > 0$ gives a 'buy price' of

$$\mathbb{E}_t^0 h(S_T) - \phi$$

and likewise $\lambda < 0$ gives 'sell price'

$$\mathbb{E}_t^0 h(S_T) + \phi$$

where $\phi = |\lambda|\frac{R}{2}x\hat{\mathbb{E}}[M]_T > 0$ using the equivalent formulation in (26). Perhaps this difference 2ϕ can be seen as a proxy for the bid ask spread and on the option and might be useful for comparing two claims on non-traded assets.

We can see also from (29) that if initial wealth increases, with fixed S and λ, then the holding in derivatives is diluted, and the price larger. For the Merton utility, the absolute risk aversion $-\frac{U''(x)}{U'(x)} = \frac{R}{x}$ is a decreasing function of wealth and thus the higher wealth, the higher price the agent is willing to pay. In contrast, for the exponential utility which has constant absolute risk aversion, the price would be independent of wealth. We examine this utility function briefly in the next section.

5 A Comparison with Exponential Utility

Another popular utility function widely used in the literature is the exponential utility, $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, see for example, Hodges and Neuberger [21], Svensson and Werner [28], Duffie and Jackson [12], Davis [6], Cvitanic et al [3], Delbaen et al [8] and Rouge and El Karoui [27]. This utility has constant absolute risk aversion, and its popularity is derived in part from its separability properties.

Again, let $V(t, X_t, S_t; \lambda)$ be the value function for the agent who at time t has wealth X_t and who will receive $\lambda h(S_T)$ at time T. Here we take $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$. Then

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= \sup_{\theta} \mathbb{E}_t U(X_T + \lambda h(S_T)) \\ &= -\frac{1}{\gamma}e^{-\gamma X_t} \inf_{\theta} \mathbb{E}_t (e^{-\gamma \int_t^T \theta_u (dP_u/P_u) - \gamma\lambda h(S_T)}) \\ &= -\frac{1}{\gamma}e^{-\gamma X_t} g(T-t, \log S_t) \end{aligned}$$

where $g(0,z) = e^{-\lambda\gamma h(e^z)}$. Using the fact that V is a supermartingale for any strategy, and a martingale for the optimal strategy, we find that g solves the pde:

$$\dot{g} - \nu g_z + \tfrac{1}{2}\eta^2 g_z - \tfrac{1}{2}\eta^2 g_{zz} + \tfrac{1}{2}\frac{(\sigma\eta\rho g_z + \mu g)^2}{\sigma^2 g} = 0.$$

We follow Hobson [20] and the example in Henderson and Hobson [19] to solve this equation. If we set $g(\tau, y) = e^{\alpha\tau} G(\tau, y + \beta\tau)^b$ then we find that G solves

$$\begin{aligned} &b\dot{G} - \tfrac{1}{2}\eta^2 b G_{yy} - \tfrac{1}{2}\eta^2(b(b-1) - \rho^2 b^2)\frac{G_y^2}{G} \\ &\quad + \left[b(\beta + \tfrac{1}{2}\eta^2 - \nu + \frac{\eta\rho\mu}{\sigma})\right] G_y + \left[\alpha + \frac{\mu^2}{2\sigma^2}\right] G = 0. \end{aligned}$$

Choosing

$$b = \frac{1}{(1-\rho^2)}, \qquad \alpha = -\frac{\mu^2}{2\sigma^2}, \qquad \beta = \nu - \frac{\eta\rho\mu}{\sigma} - \tfrac{1}{2}\eta^2 = \delta - \tfrac{1}{2}\eta^2$$

we find that G solves

$$\dot{G} = \tfrac{1}{2}\eta^2 G_{yy}.$$

This is the heat equation, with solution

$$G(\tau, y) = \int_{-\infty}^{\infty} G(0, y+z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz$$

so

$$\begin{aligned} g(\tau, y) &= e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\int_{-\infty}^{\infty} G(0, y + (\delta - \tfrac{1}{2}\eta^2)\tau + z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz \right]^{\frac{1}{1-\rho^2}} \\ &= e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\mathbb{E}(G(0, y + (\delta - \tfrac{1}{2}\eta^2)\tau + \eta\sqrt{\tau}N)) \right]^{\frac{1}{1-\rho^2}} \end{aligned}$$

where N is a standard normal random variable. Using the boundary condition $G(0,y) = e^{-(\lambda\gamma/b)h(e^y)} = e^{-\lambda\gamma(1-\rho^2)h(e^y)}$:

$$\begin{aligned} V(t, X_t, S_t; \lambda) &= -\frac{1}{\gamma} e^{-\gamma X_t - \frac{\mu^2}{2\sigma^2}(T-t)} \\ &\times \left[\mathbb{E}\left(\exp(-\lambda\gamma(1-\rho^2) h(S_t e^{\delta(T-t)} e^{\eta\sqrt{T-t}N - \frac{1}{2}\eta^2(T-t)})) \right) \right]^{\frac{1}{1-\rho^2}}. \end{aligned}$$

It follows that the reservation price p^e for receiving a random payoff $\lambda h(S_T)$, given as the solution to $V(0, X_0 - p^e, S_0; \lambda) = V(0, X_0, S_0, 0)$, is

$$p^e = -\frac{1}{\gamma(1-\rho^2)} \log \mathbb{E}\left[\exp\left\{ -\lambda\gamma(1-\rho^2) h(S_0 e^{\delta T} e^{\eta\sqrt{T}N - \frac{1}{2}\eta^2 T}) \right\} \right].$$

We want to find an expansion in terms of small λ which we can compare to our results using the Merton utility. The expansion is

$$p^e = \lambda \mathbb{E} h(S_0 e^{\delta T - \frac{1}{2}\eta^2 T + \eta\sqrt{T}N}) - \frac{\gamma}{2}\lambda^2(1-\rho^2) Var(h(S_0 e^{\delta T - \frac{1}{2}\eta^2 T + \eta\sqrt{T}N})) + O(\lambda^3)$$

which is equivalent to the price found in Davis [6]. We find that to leading order the price is precisely the expected value of the claim under the minimal martingale measure. Hence we concentrate on the correction term. Note that the second order correction is linear in the risk aversion parameter γ. We equate the local absolute risk aversion in the Merton and exponential utility models to compare the results. This involves identifying the parameter γ with R/x. The price becomes

$$p^e = \lambda \mathbb{E} h(S_0 e^{\delta T - \frac{1}{2}\eta^2 T + \eta\sqrt{T}N}) - \frac{R}{2x}\lambda^2(1-\rho^2) Var(h(S_0 e^{\delta T - \frac{1}{2}\eta^2 T + \eta\sqrt{T}N})) + O(\lambda^3). \tag{31}$$

Example 1 continued: We can evaluate (31) for the call option with $\lambda > 0$. Calculations give

$$p^e = \lambda \mathbb{E}^0(S_T - \bar{K})^+ - \frac{R}{2}\frac{\lambda^2}{x}(1-\rho^2)[\mathbb{E}^0(S_T - \bar{K})^2 I_{(S_T > \bar{K})} - (\mathbb{E}^0(S_T - \bar{K})^+)^2] \tag{32}$$

where $\mathbb{E}_t^0(S_T - \bar{K})^+$ is given in (14) and

$$\begin{aligned} \mathbb{E}(S_T - \bar{K})^2 I_{(S_T > \bar{K})} &= S_t^2 e^{(2\delta + \eta^2)(T-t)} N\left(\frac{\ln(\frac{S_t}{\bar{K}}) + (\delta + \frac{3}{2}\eta^2)(T-t)}{\eta\sqrt{(T-t)}} \right) \\ &+ \bar{K}^2 N\left(\frac{\ln(\frac{S_t}{\bar{K}}) + (\delta - \frac{1}{2}\eta^2)(T-t)}{\eta\sqrt{(T-t)}} \right) - 2\bar{K} e^{\delta(T-t)} S_t N\left(\frac{\ln(\frac{S_t}{\bar{K}}) + (\delta + \frac{1}{2}\eta^2)(T-t)}{\eta\sqrt{(T-t)}} \right). \end{aligned} \tag{33}$$

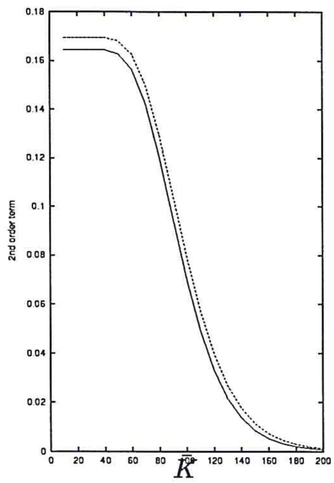

Figure 5: *The 2nd order term of the reservation price of the claim with payoff* $(S_T - \bar{K})^+$ *for* $0 \leq \bar{K} \leq 200$. *The lower line uses the power law utility whilst the higher line uses the exponential utility. Parameter values are* $\lambda = 0.01, S_0 = 100, T = 1, R = 0.5, \mu = 0.04, \rho = 0.8, \eta = 0.30, \sigma = 0.35,\ \delta = 0$ *and* $x = 500$.

Figure 5 graphs the second order term in (32) and the Merton price (27) over values of the strike $\bar{K}$. The numbers are extremely close (since we equated local absolute risk aversion) however the exponential utility gives a larger correction over the whole range. This is worthy of further investigation.

Now we consider the dependence of the reservation price on the risk aversion parameter with surprising results. Comparing the forms of the exponential and power utilities in the limit as the risk aversion parameter tends to zero, we see

$$\lim_{\gamma \downarrow 0} \frac{1 - e^{-\gamma x}}{\gamma} = x = \lim_{R \downarrow 0} \frac{x^{1-R}}{1-R},$$

but in the former case the domain of definition is $\mathbb{R}$ whereas in the latter it is $\mathbb{R}^+$. Hence there is no reason to expect identical behaviour in the limit as risk aversion decreases to zero.

In Figure 6 we graph the second order price term as a function of R. Note we plot $-(p^e - \lambda \mathbb{E}^0(S_T - \bar{K})^+)/\lambda^2$ and the equivalent price for the Merton case. The solid line uses the exponential utility whilst the broken line uses the power utility. We see that over most of the parameter range, as risk aversion increases so the reservation price falls. The agent is willing to pay less for the non-traded stock as she becomes less tolerant of risk. However, surprisingly this relationship reverses for the power utility as R gets very small. For the parameter choices in Figure 5 this happens for R below approximately 0.1. As R decreases below this value the agent is prepared to pay less for the risky non-traded asset even though she is becoming more tolerant of risk.

Recall the optimal strategy given in Theorem 4.1:

$$\theta^*(t, X_t, S_t; \lambda) = \frac{\mu}{\sigma^2 R}(X_t + \lambda C_t) - \frac{\eta\rho}{\sigma}\lambda S_t C_t^S + O(\lambda^2).$$

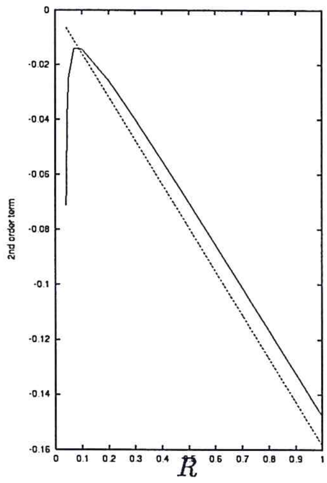

Figure 6: *The 2nd order term of the reservation price of the claim with payoff* $(S_T - \bar{K})^+$ *for* $0.03 \leq R \leq 1$. *Parameter values are* $\lambda S_0 = 1.0, T = 1, \bar{K} = 100, \mu = 0.04, \rho = 0.8, \eta = 0.30, \sigma = 0.35$, $\delta = 0$ *and* $x = 500$. *The broken line uses the power utility whilst the solid line uses the exponential utility.*

As $R \downarrow 0$, both the first term, the Merton proportion, and the first order correction term become large. Fluctuations in the value of P and S are magnified into large fluctuations in the final wealth. The price an agent is prepared to pay for a random payoff depends on two factors. The first is her level of risk aversion, but the second is the magnitude of the unhedgeable component of the random payoff. Thus even though the agent is only mildly risk averse, the large fluctuations in final wealth have a non-negligible effect on expected utility, and hence price. □

Example 2 continued: For $h(S_T) = S_T^2$

$$p^e = \lambda \mathbb{E}^0 S_T^2 - \frac{R}{2}\frac{\lambda^2}{x}(1-\rho^2)[\mathbb{E}^0 S_T^4 - (\mathbb{E}^0 S_T^2)^2]$$

where $\mathbb{E}^0 S_T^4 = S_0^4 e^{4(\delta + \frac{3}{2}\eta^2)(T-t)}$. □

6 Conclusion

This paper has studied the utility maximisation pricing of claims on non-traded assets, using a close asset to hedge. Under the assumptions of CRRA and exponential Brownian motion, the techniques of duality were used to approximate the option hedge and obtain a reservation price. The results hold under the assumption that the money value in the non-traded asset is small in comparison to wealth. As expected, the reservation price depends on the drifts of the assets and the level of risk aversion. The marginal price however, is independent of the utility function.

The examples of a call and 'power' option were analysed, concentrating on the call. We show the reservation price is increasing in correlation, hence the agent is willing to pay more when he is more likely to have a reasonable hedge. Under our choice of parameters, the second order correction term was about 0.6% of the first order (minimal martingale measure) term when $\lambda = 1$. A comparison of the reservation price with the exponential utility price showed, for the call, the corrections were very close. However, they behave very differently as a function of risk aversion as risk aversion tends to zero. This might be explained by the fact that the power utility is defined for positive wealth whereas the exponential utility also allows wealth to go negative.

A shortcoming of this analysis and others in the literature is that it cannot be used to price a short call position due to the unbounded nature of the payoff. Recall, the payoff used in this paper must satisfy Assumption A. The utility approach is not suitable for short calls and further work could be done in this area.

Finally, an area where these results can be applied is that of executive stock options. These are options on the stock of the company, and are given to executives as part of their compensation package. However, frequently executives are not permitted to trade away the risk using the stock or derivatives on the stock, so that they are essentially receiving options on an non-traded asset. Work is being done on this topic by the author.

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