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**Mark C W Wong**
**and**
**Stewart D Hodges**


April 2001


*Financial Options Research Centre*
*Warwick Business School*
*University of Warwick*
*Coventry*
*CV4 7AL*
*Phone: (0)24 76 524118*



**FORC Preprint: 2001/118**

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Mark C. W. Wong* Stewart D. Hodges†

This version: 20 April 2001

*Wong (corresponding author) is a doctoral researcher at the Warwick Business School, University of Warwick, CV4 7AL, UK. Phone: +44 (0)24 7652 4691, Fax: +44 (0)24 7652 4650, Email: Mark.Wong@warwick.ac.uk.

†Hodges is Professor of Financial Management and director of the Financial Options Research Centre (FORC) at the Warwick Business School, University of Warwick, CV4 7AL, UK. Phone: +44 (0)24 7652 3606, Fax: +44 (0)24 7652 4167, Email: forcsh@razor.wbs.warwick.ac.uk.

# A Reduced-form Model incorporating Fundamental Variables


**Abstract**

This paper proposes a reduced-form model of corporate debt, by taking into account stochastic interest rates, a firm's equity values, and hazard rates of default. We innovatively introduce structural characteristics of the firm into the model. Distinguishing features of the model are fourfold. Firstly, the model is able to capture the effects of economic fundamentals on properties of credit spreads. Secondly, it preserves a high degree of flexibility in generating credit spreads. Thirdly, the analytical and tractable form of the model enhances its empirical applicability. Finally, the model can easily be generalized to deal with counterparty default risk.



**JEL Classification:** G12, G13

**Keywords:** Interest rates, Equity values, Hazard rates, Structural characteristics, Moving average, Counterparty default risk

The valuation of corporate debt is central to theoretical and empirical work in corporate finance. The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. Pioneered by Merton (1974), the structural approach has taken the dynamics of the assets of the issuing firm as given, and priced corporate bonds as contingent claims on the assets. Assuming a constant interest rate economy, Merton's model provides an important insight into the determinants of the risk structure, and shows how the default risk premium is affected by changes in the firm's business risk, debt maturity, and the prevailing interest rate.

Black and Cox (1976) and Geske (1977) provide generalizations that take into account the effects of coupon and bond indenture provisions. The application of stochastic interest rates in the valuation of corporate debt using Merton's (1974) framework is discussed in Shimko, Tjima, and Van Deventer (1993). Longstaff and Schwartz (1995a) adapt Black and Cox's (1976) model to a more realistic setting. Firstly, they allow interest rates to be stochastic, with dynamics proposed by Vasicek (1977). Secondly, they do not require the recovery rate to be equal to the boundary value upon first passage, but assume an exogenously given rate of face value $K$. Thirdly, the default boundary is assumed to be $K$. This approach explicitly allows for deviations from absolute priority.

By introducing bankruptcy costs and tax effects, the framework has been extended to a richer extent, taking into account issues in corporate finance. Examples that consider endogenous capital structure, liquidation policy, re-capitalization, and re-organization of debt include Brennan and Schwartz (1984), Leland (1994, 1998), Leland and Toft (1996), Anderson and Sundaresan (1996), and Mella-Baral and Perruadin (1997). These models allow for endogenous default, optimally determined by equity holders when asset levels fall to a sufficiently low level.

A recent paper by Collin-Dufresne and Goldstein (2001) employs a structural approach to investigate the effects of a firm's capital structure on debt pricing. They propose a structural model of default with stochastic interest rates and the firm's asset values that captures mean-reverting feature of leverage ratios. Effectively, their model allows for an amount of debt to be issued in the future when the firm's asset values increase. They derive the value of a risky discount bond in the form of an infinite series in line with Longstaff and Schwartz's (1995) model. The levels of credit spread generated appear to be more consistent with empirical findings.

The structural approach to the valuation of risky debt has been criticized for not being able to generate sufficient credit spreads for small maturities of debt. Although these structural models can answer questions about the implications for debt pricing in changes of firm-specific variables such as debt restructuring, this important feature is compromised by their inability to generate realistic credit spreads for short maturity bonds. In practice, even for small maturities, the market does not neglect the possibility that some disaster may happen. As noted in Kim, Ramaswamy, and Sundaresan (1993), realistic values of leverage and the volatility of the value of firm asset seem incapable of producing the credit spreads that are actually observed in the market.

In contrast to the structural models, the literature has adopted an alternative approach that offers a high degree of tractability for credit risky bonds. This reduced-form approach bypasses the complications of handling a firm's economic fundamentals, and deals directly with market prices and spreads. The method involves relating default time to the stopping time of an exogenously given hazard rate process. Models in this area include those of Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1994), Lando (1995), Madan and Unal (1998), Duffee (1999), and Duffie and Singleton (1999).

There have been many applications of Duffie and Singleton's (1999) framework in the literature. Lando (1998) illustrates how doubly stochastic Poisson processes, also known as Cox processes, can be applied to model prices of financial instruments in which credit risk is a significant factor. The idea is based on Duffie and Singleton's (1999) model with the specification of a hazard rate process as a Cox process. Because of the general nature of Cox processes, Lando's (1998) approach allows default characteristics of firms, such as rating transitions to be captured into his model.

Duffie and Huang (1996) apply Duffie and Singleton's model to price swaps with counterparties of different default risks. A switching-type, default-adjusted short rate process is used depending on whether the swap value is positive or negative. Asymmetric default risk of the counterparties and non-linearity of promised cash flows are then explored in their paper. Another application of Duffie and Singleton's (1999) model is a recent paper by Jarrow and Yu (2001). The paper studies the impact of counterparty default risk on the pricing of defaultable securities, where correlated defaults due to an exposure of common risk factors and firm-specific risks are considered. As with Duffie and Huang's (1996) model, Jarrow and Yu (2001) specify in their models switching-type hazard rate processes depending on which counterparties have gone bankrupt. In principle, a framework with multiple layers of counterparty relationship can potentially be applied to pricing defaultable securities.

A major advantage of reduced-form models is that they provide us with a model that is close to the data; it is always possible to fit some version of the model. However, the fitted model may not perform well on "out of sample" analysis. Another potential drawback in the construction of an underlying hazard rate process is that these models lack a connection of a firm's economic fundamentals to default events. As a consequence, these models provide no guidance of structural interpretation in the changes of firm-specific variables. Firm-specific risk and financial fundamentals are not evaluated and may even be ignored.

In addition to the basic incompatibility in the default mechanisms of the two approaches discussed by Wong and Hodges (2001), there is another key theoretical difference between them. A structural model completely rules out the use of a hazard rate process that is common in the reduced-form approach, and such a structural model implies a hazard rate that would be zero before default and infinite at default. Madan and Unal (1998) have come up with a reduced-form model whose hazard rate process concurs with the diffusion-based structural approach in this respect. However, the model still lacks an interpretation of a firm's structural characteristics.

To capture the effects of capital structure, a hybrid-type model has been suggested in the literature. Madan and Unal (2000) propose a two-factor hazard rate model to price risky debt. Consistent with the hazard rate literature, the probability of sudden default is governed by the hazard rate. They derive the hazard rate function endogenously in terms of the firm's non-interest sensitive asset values and default-free interest rates. Assuming that default follows a Poisson arrival rate and loss in the case of default has a cumulative distribution function, they come up with a structural definition of the hazard rate process as a product of the two quantities. Though the structural approach is appealing, they fail to obtain an exact analytical solution for the bond price. Instead, an analytical approximation is derived after they express the hazard rate function as a first-order approximation of its Taylor expansion. Other attempts to introduce structural properties into the reduced-form framework include Cathcart and El-Jahel (1998), Jarrow (2000), Jarrow and Turnbull (2000), and Hübner (2001). In this paper, we extend their results by incorporating current and lagged effects of individual stocks into the pricing of corporate bonds.

This paper proposes a reduced-form model of corporate debt by taking into account stochastic interest rates, a firm's asset values, and hazard rates of default. Consistent with the literature of the reduced-form models, we assume that default can only happen unexpectedly. As in Duffie and Singleton (1999) and Duffee's (1999) work, we take a hazard rate process as exogenously given. Unlike those models, there is a crucial distinction in the specification of the process in our model. We introduce structural characteristics of the firm into the hazard rate process, through a factor providing a measure of a firm's performance in equity. The use of such a measure has two important features. Firstly, instead of solely using a firm's current value as conventional Merton-type models do, we take the past performance of the firm into account. Unlike Madan and Unal (2000), we employ relative values of *observable* equity prices to measure a firm's performance as well as leverage effect. Secondly, having high equity values alone may not necessarily be a good indicator of a firm's creditworthiness. In our model, we take a broader view of the financial health of a firm by considering the current asset level relative to its past positions. The debt becomes more risky when the relative levels are lower, whereas when the relative levels are higher, the debt becomes safer. As a consequence, a peculiar feature of financial markets that news on corporate earnings is normally reflected in equity prices first, and then bond prices, can be captured in our model.

The objectives of this paper are as follows. We seek to propose a flexible model of corporate debt in analytical form. The structural characteristics of a firm and stochastic interest rates are taken into account. Our crucial assumption is that the default hazard of the firm, unlike the structural approach, depends on the current relative price of equity to its recent past levels. We consider a moving average of logarithm of recent stock prices. The use of this measure is an innovative idea that allows economic fundamentals of the firm to be captured in the hazard rate process, and hence bond prices. Three features are noteworthy. Firstly, as with other structural models, we show the structural impact of interest rate movements and their correlation with equity returns on the pricing of risky debt. For example, we demonstrate that the levels of spread increase with interest rate volatility, equity return volatility, and the correlation. Secondly, as a reduced-form model, the model preserves a high degree of flexibility in generating credit spreads. Numerical computations show that the model is flexible enough to generate many different term structures of credit spreads by using appropriately chosen parameters. We investigate analytically how parameter values affect the shape of credit spread curve in terms of its intercept, slope at zero maturity, and spread level for long maturity. Finally, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhance its empirical applicability.

The paper is divided into seven sections. In the next section, we state in advance a main result of this paper. We postpone detailed discussion of economic implications and construction of underlying processes to Section II and III. Section IV shows the short- and long-term behaviour of credit spreads, and their relationships to the structural characteristics of the firm. Emphasis is placed on the flexibility of the model in generating credit spreads in relation to model parameters. Section V shows how we can extend the model to deal with counterparty default risk, whose impact on credit spreads is also presented. Finally, we conclude in Section VI with a summary and a discussion of further research.

# I The Model

For ease of exposition, we state in advance the solution of our model in this section, and postpone detailed discussion of economic implications and construction of underlying processes to Section II and III. We consider a risky zero-coupon bond of unit face value and maturity date $T$. The default-free interest rate process is $r_t$. Consistent with Duffie and Singleton (1999), we suppose that default occurs randomly, and that the risky debt has a risk-neutral short spread process $s_t$. We also allow economic fundamentals of the firm to be captured in the short spread, through a process $Y_t$. For construction and interpretations of processes $s_t$ and $Y_t$, please refer to Section III.

Given that these processes $s_t$, $Y_t$, and $r_t$ have the following affine representations (*):

$$(*)\begin{cases} dr_t = k_r(\theta_r - r_t)\,dt + \sigma_r\,dB_t^r, \\ dY_t = (r_t - \alpha Y_t - \sigma_s^2/2 - a)\,dt + \sigma_s\rho\,dB_t^r + \sigma_s\sqrt{1-\rho^2}\,dB_t^s, \text{and} \\ ds_t = (\delta\theta_h + k_h s_t + \delta k_{hy}Y_t + \delta k_{hr}r_t)dt + \delta\sigma_{hr}dB_t^r + \delta\sigma_{hs}dB_t^s + \sigma_h\sqrt{\delta}\sqrt{s_t}dB_t^h, \end{cases}$$

where $B_t^r$, $B_t^s$, and $B_t^h$ are independent standard Brownian motions, then as in Duffie and Kan (1996), we assume that the bond price can be expressed in exponential-affine form in terms of the three factors. The time-$t_0$ price, $D(t_0,T)$, of the risky bond is of the form:

$$D(t_0,T) = \exp\Big(A(t_0,T) + B_1(t_0,T)s_{t_0} + B_2(t_0,T)Y_{t_0} + B_3(t_0,T)r_{t_0}\Big). \tag{1}$$

Now we state a main result in this paper as follows:

**Proposition 1** *Suppose that the bond price satisfies equation (1). Then*

$$\begin{aligned} B_1(t_0,T) &= -\frac{2[1-e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t_0)}]}{(\sqrt{k_h^2+2\delta\sigma_h^2}-k_h)+(\sqrt{k_h^2+2\delta\sigma_h^2}+k_h)e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t_0)}}, \\ B_2(t_0,T) &= \delta k_{hy}\int_{t_0}^T e^{-\alpha(u-t_0)}B_1(u,T)du, \\ B_3(t_0,T) &= -\int_{t_0}^T e^{-k_r(u-t_0)}\Big[1-\delta k_{hr}B_1(u,T)-B_2(u,T)\Big]du, \text{ and} \\ A(t_0,T) &= \int_{t_0}^T \Big[\delta\theta_h B_1(u,T) - (\sigma_s^2/2+a)B_2(u,T) + k_r\theta_r B_3(u,T) + \frac{1}{2}\sigma_s^2 B_2(u,T)^2 \\ &\quad + \frac{1}{2}\sigma_r^2 B_3(u,T)^2 + \frac{\delta^2}{2}(\sigma_{hr}^2+\sigma_{hs}^2)B_1(u,T)^2 + \sigma_r\delta\sigma_{hr}B_1(u,T)B_3(u,T) \\ &\quad + \sigma_r\sigma_s\rho B_2(u,T)B_3(u,T) + \delta(\sigma_s\sigma_{hr}\rho + \sigma_s\sigma_{hs}\sqrt{1-\rho^2})B_1(u,T) \\ &\quad \times B_2(u,T)\Big]du, \end{aligned}$$

*where $B_2(t_0,T)$, $B_3(t_0,T)$ are expressible in terms of a hypergeometric function and its integrals.*

**Proof.** See Appendix A. ■

Proposition 1 shows that the bond price has an analytical form which can be expressed in terms of hypergeometric functions of the type ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$. In addition to tractability, the analytical form of the model also enables us to undertake comparative statics analysis and empirical research. Furthermore, the influences of interest rates, a firm's economic fundamentals, as well as the probabilities of default are synthesized into the price of the risky debt in equation (1).

# II The Framework

In this section, we set out an overall structure of the model. We assume a frictionless economy with a prevailing default-free interest rate $r_t$. Under a risk-neutral measure, the evolution of the short rate follows a Vasicek process (1977):

$$dr_t = k_r(\theta_r - r_t)\ dt + \sigma_r\ dB_t^r, \tag{2}$$

where $B_t^r$ is a standard Brownian motion under the risk-neutral measure, $k_r$ is the speed at which the interest rate $r_t$ tends to its long term mean $\theta_r$, and $\sigma_r$ is the volatility of changes in the instantaneous interest rate.

We consider a firm whose equity has value $S_t$, which follows a diffusion process with constant volatility of rate of return:

$$dS_t/S_t = (r_t - a)\ dt + \sigma_s\rho\ dB_t^r + \sigma_s\sqrt{1-\rho^2}\ dB_t^s, \tag{3}$$

where $B_t^s$ is another standard Brownian motion under the same measure, and is independent of $B_t^r$, $a$ is the total dividend rate to shareholders, $\sigma_s$ is the volatility of equity returns, and $\rho$ is the correlation between the increments of $r_t$ and the instantaneous returns of equity.

We consider a risky zero-coupon bond, issued by the firm, of unit face value and maturity date $T$. Consistent with Duffie and Singleton (1999), we suppose that default occurs at a random time $\tau$, $\tau < T$, and that the corresponding risk-neutral hazard rate process is $h_t$. Assuming that in the event of default, the mean fractional loss of the market value of the claim is a constant $\delta$, where $0 < \delta < 1$, then the short spread can be expressed as:[1] [2]

$$s_t = \delta h_t. \tag{4}$$

According to Duffie and Singleton (1999), the time-$t_0$ price, $D(t_0, T)$, of the risky bond is of the following form:

$$D(t_0, T) = E_{t_0}\Big[\exp(-\int_{t_0}^{T} R_u \, du) \; X\Big], \tag{5}$$

where $R_t = r_t + s_t$. The intuition behind the model is as follows. By discounting at the adjusted short rate $R_t$, the model accounts for both the probability and timing of default, as well as for the loss effects on default. Furthermore, the bond corresponds to having a thinned default intensity $\delta h_t$, and a recovery rate of *zero* in the event of default. Hence the bond can be valued alternatively as in the Cox process case in Lando (1998).

Before we prove Proposition 1, we need to specify other processes through which the firm's economic fundamentals are incorporated into the model. The next section achieves this by relating the equity prices to the instantaneous hazard rate.

[1] There are four different formulations of the loss function suggested in the literature: the default pay-off is either a fraction of (i) par (Madan and Unal (1998)), (ii) of par plus accrued interest (J.P. Morgan (1999) and Jarrow and Turnbull (2000)), (iii) of a risk-free bond with the same structure of cash flows (Jarrow and Turnbull (1995)), and (iv) of the market value of the security just prior to default (Duffie and Singleton (1999)). In this paper, we adopt Duffie and Singleton's (1999) approach and assume that the loss rate is a constant fraction of the bond price immediately before default.

[2] The mean loss rate is heavily dependent on the seniority of the bond.

# III Modeling of the Hazard Rate Process

Having high equity values alone may not necessarily provide a good indicator of firm's creditworthiness. Directors of a firm may consider issuing more debt after realizing an increase in the firm's asset values, but the increase in debt levels would bring extra risk into the firm's capital structure. On the contrary, they may consider reducing debt levels by issuing new equity when realising a continual decline in the equity prices, the firm's debt-equity ratio would subsequently be lowered. This observation is consistent with Malitz's (1994) findings that bond covenants typically allow directors to have a degree of flexibility in changing the debt levels in the future.

In our model, we take a broader view of the financial health of the firm to allow for such a dynamic restructuring of capital structure, by considering the current equity level relative to its past positions. The debt becomes more risky when the relative levels are lower; when the relative levels are higher, the debt becomes safer. With this motivation, we consider a continuous moving average $M_t$ of $\log(S_t)$:

$$dM_t = \alpha\Big( \log(S_t) - M_t\Big) \; dt, \tag{6}$$

where $\alpha > 0$ is a smoothing parameter. Note that instead of taking averages of the equity prices, we define $M_t$ as the continuous moving average of $\log(S_t)$ for the sake of tractability. To solve equation (6) for $t \geq s \geq 0$, we have the following expression:

$$M_t = e^{-\alpha(t-s)} M_s + \int_s^t \alpha e^{-\alpha(t-u)} \log(S_u) \; du. \tag{7}$$

This variable has been employed in the literature of bond pricing and stochastic volatility models. Instead of using equity prices, Collin-Dufresne and Goldstein (2001) structure a log-default threshold in a similar way by considering a firm's asset values. Tompkins (2000) has shown that the exponentially-weighted return series of futures prices on a stock index is significantly related to the volatility of the futures prices, and hence leverage effect.[3] Equation (7) is a straightforward generalization of exponential moving average models in discrete case. This expression shows that the moving average $M_t$ depends on the equity in two manners: ($i$) the equity prices before time $s$, and ($ii$) those entering the system from time $s$ to $t$. More precisely, $M_t$ is a continuous exponentially-weighted mean of its value at time $s$ and all values of $\log(S_u)$ between time $s$ and $t$, for which the weights are $e^{-\alpha(t-s)}$ and $1-e^{-\alpha(t-s)}$ respectively. It is evident that the higher the value of $\alpha$, the more the moving average is dependent on the recent values of equity price. The value of $\alpha$ must be chosen to ensure that the current value of $M_t$ does not depend overwhelmingly on those in the past. We will show later in this paper how the choice of $\alpha$ affects the term structure of credit spreads.

We define a measure of the relative levels of equity price as:

$$Y_t = \log(S_t) - M_t.$$

This variable measures how far $\log(S_t)$ is from its recent mean level, and provides an attribute to indicate the firm's business outlook. Since, as documented in Kwan (1996), both current and lagged values of equity return have been shown to have impact on changes in bond yields, we incorporate these empirical properties into our model and postulate that the structural characteristics of the firm enter into the prices of risky bond through the process $Y_t$ in the following way:

$$dh_t = (\theta_h + k_h h_t + k_{hy} Y_t + k_{hr} r_t)dt + \sigma_{hr} dB_t^r + \sigma_{hs} dB_t^s + \sigma_h \sqrt{h_t} dB_t^h, \tag{8}$$

where $B_t^h$ is a standard Brownian motion independent of $B_t^r$ and $B_t^s$. Using equation (4), the short spread $s_t$ follows the following process:

$$ds_t = (\delta\theta_h + k_h s_t + \delta k_{hy} Y_t + \delta k_{hr} r_t)dt + \delta\sigma_{hr} dB_t^r + \delta\sigma_{hs} dB_t^s + \sigma_h \sqrt{\delta}\sqrt{s_t} dB_t^h. \tag{9}$$

[3] Tompkins (2000) uses this variable as an attribute to measure leverage effect. He found that recent relative prices are negatively correlated to the series of 20-day unconditional volatility of stock index futures. This result is consistent with the negative leverage effects that Christie (1982) has pointed out for individual stocks.

It is interesting to note the role of $Y_t$ in the processes (8) and (9). The presence of the process $Y_t$ is a structural difference between the short spread process (9) and many others that have been suggested in the literature. For example, Duffee (1999) applies Duffie and Singleton's (1999) idea to fit yields on bonds issued by individual investment-grade firms to a reduced-form model, in which no factors of firm's economic fundamentals are taken into account.[4] Four important features are captured in our setting:

(i) Both the hazard rate and the short spread are modeled as square-root processes.[5] We know from the work of Longstaff and Schwartz (1995b) that credit spread displays a significant amount of stability. To be consistent with this property, a mean reverting feature of credit spreads is incorporated into the model by specifying $k_h < 0$;

(ii) As documented in Duffee (1998), yield spreads for high-quality firms are positive,

[4]Duffee (1999) considers a three-factor model in which the instantaneous, default-free short rate process $r_t$ is assumed to be a linear combination of two square-root diffusion processes. Short spread $s_t$ is modeled as another linear combination of three square-root diffusions, where two of them are the same as those in the short rate process. No factors of firm's economic fundamentals are taken into account. An analytical form of solution for bond prices is obtained. Although empirical results appear to be encouraging as the average error in fitting corporate bond yields is less than 10 basis points, Duffie and Singleton (1999) argue that the models used by Duffee (1999) are theoretically incapable of capturing the negative correlation between credit spreads and U.S. Treasury yields while maintaining non-negative default hazard rates. They succeed in coming up with an alternative model with more flexible correlation structures for $(r_t, s_t)$, but the system cannot be solved analytically for bond prices. We discuss a method of solution for our model in Appendix A.

[5]In this formulation, the risk-neutral hazard-rate and short spread processes can become negative. However, it can be shown by Monte-Carlo simulations that when $\theta_h$ is sufficiently large, it is unlikely for the processes to hit 0. In particular, for the numerical examples in this paper, Monte-Carlo simulations show that if we assume that the true hazard rate process is of the form: $h_t^* = max\{h_t, 0\}$, then our model tends to underestimate the true levels of credit spreads by no more than 10 basis points. Therefore, given the tractability of the subsequent expressions, this is an acceptable approximation.

even at the short end of spread curve. This suggests that there is a positive spread at zero maturity, which is $s_{t_0} = \delta h_{t_0}$ in the model;

(iii) The short spreads are stochastic, fluctuating with the firm's structural characteristics, captured by $Y_t$. It is interesting to note that any latent variable with the same structural and mathematical properties can be employed in the place of $Y_t$; and

(iv) The short spreads can be structured to be systematically related to variations in the default-free term structure, as documented in empirical literature.[6]

It is important to investigate the properties of processes (8) and (9) in relation to the process $Y_t$, $\alpha$, and other parameters. Recall that $M_t$ is defined as a mean of $M_s$ and $\log(S_u)$ from time $s$ to $t$, weighted by $e^{-\alpha(t-s)}$ and $1 - e^{-\alpha(t-s)}$ respectively. A noticeable feature is that the contribution of $M_s$ becomes negligible when $\alpha(t-s)$ is large. The larger the value of $\alpha$, the more significant contribution of the second term to the overall average $M_t$. As a consequence, the averaging is mainly performed on $\log(S_u)$ from time s to t. An intuition is that when equity prices are continually rising, $Y_t$ tends to be positive. However, when equity prices are continually declining, $Y_t$ tends to be negative. Such a property of $Y_t$ provides us with a clue as to the appropriate signs of $k_{hy}$ and $k_{hr}$. To capture the property of negative correlation between the interest rate $r_t$ and the short spread $s_t$, we specify that $k_{hy}$, $k_{hr} \leq 0$. On the other hand, a positive value of $\sigma_{hr}$ induces positive correlation between the increments of $r_t$ and $s_t$. By construction, this model also has a fairly high degree of flexibility in correlation structures. We will discuss the flexibility of the model further in Section IV.

Before we finish this section, we state in the following proposition that the specification of $Y_t$ is affine. Hence, together with previous results, we have structured the

[6]See Duffee (1998), and Longstaff and Schwartz (1995a) for empirical justifications.

framework in terms of the three main processes, $r_t$, $s_t$, and $Y_t$, in affine representations. We are now ready to present a proof of Proposition 1 (see Appendix A).

**Proposition 2** *The process $Y_t$ satisfies the following stochastic differential equation:*

$$dY_t = (r_t - \alpha Y_t - \sigma_s^2/2 - a)\ dt + \sigma_s \rho\ dB_t^r + \sigma_s \sqrt{1-\rho^2}\ dB_t^s. \tag{10}$$

**Proof.** Rewrite process (3) as

$$d\log(S_t) = (r_t - \sigma_s^2/2 - a)\ dt + \sigma_s \rho\ dB_t^r + \sigma_s \sqrt{1-\rho^2}\ dB_t^s.$$

By definitions of $Y_t$ and $M_t$, the result follows. ■

# IV Properties of Credit Spreads

To better understand the impacts of the underlying processes on risky debt, we conduct an analysis of credit spreads as follows. Let the credit spread $s(t_0, T)$ be the difference in yields between the risky bond $D(t_0, T)$ and default free bond $B(t_0, T)$. Then

$$s(t_0, T) = \frac{\log\Big(B(t_0, T)/D(t_0, T)\Big)}{T - t_0},$$

where

$$\begin{aligned} B(t_0, T) &= \exp\Big( - B_0(t_0, T) r_{t_0} + \frac{(B_0(t_0, T) - T + t_0)(k_r^2 \theta_r - \sigma_r^2/2)}{k_r^2} - \frac{\sigma_r^2 B_0(t_0, T)^2}{4k_r}\Big), \\ B_0(t_0, T) &= \frac{1 - e^{-k_r(T - t_0)}}{k_r}. \end{aligned}$$

By proposition 1, credit spread $s(t_0, T)$ is of the following form:

$$\begin{aligned} s(t_0, T) &= -\frac{A(t_0, T)}{T - t_0} - \frac{B_1(t_0, T)}{T - t_0} s_{t_0} - \frac{B_2(t_0, T)}{T - t_0} Y_{t_0} - \frac{B_3(t_0, T)}{T - t_0} r_{t_0} \\ &\quad - \frac{B_0(t_0, T)}{T - t_0} r_{t_0} + \frac{(B_0(t_0, T)/(T - t_0) - 1)(k_r^2 \theta_r - \sigma_r^2/2)}{k_r^2} \\ &\quad - \frac{\sigma_r^2 B_0(t_0, T)^2}{4k_r(T - t_0)}. \end{aligned} \tag{11}$$

It is evident that the spread $s(t_0, T)$ is a linear function depending on the current states of economy, $r_{t_0}$, $s_{t_0}$, and $Y_{t_0}$. The role of $Y_{t_0}$ in the spread function appears to concur with a finding in the work of Kwan (1996) that current bond yield changes are negatively correlated with the issuing firm's current and lagged stock returns, and so firm-specific information tends to be embedded first into individual stock prices and then reflected in individual bond prices. Furthermore, the linear relationship of our model can potentially capture both the specific and systematic risks of the firm. This is an important point, as shown in Elton, Gruber, Agrawal, and Mann (2000), that the most significant components of credit spreads result from expected default risk, taxes, and systematic risk in the stock market.[7]

To evaluate the effects of the three factors on the yield spreads, we state the short and long-term properties of the spread function $s(t_0, T)$ in the following proposition:

**Proposition 3** *The spread function $s(t_0, T)$ has the following properties:*

*(i) Short-term level of spreads* $s(t_0, t_0) = \delta h_{t_0}$,

*(ii) Short-term slope of the spread curve* $= \delta\theta_h + k_h s_{t_0} + \delta k_{hy} Y_{t_0} + \delta k_{hr} r_{t_0}$,

*(iii) Long-term properties of $s(t_0, T)$ are stationary, and*

$$\begin{aligned}\lim_{T\to\infty} s(t_0, T) &= -\delta\theta_h l_1 + (\sigma_s^2/2 + a)l_2 - \frac{1}{2}\sigma_s^2 l_2^2 - \frac{1}{2}\sigma_r^2 l_3^2 - \frac{\delta^2}{2}(\sigma_{hr}^2 + \sigma_{hs}^2)l_1^2 \\ &\quad -\sigma_r\delta\sigma_{hr}l_1 l_3 - \sigma_r\sigma_s\rho l_2 l_3 - \delta(\sigma_s\sigma_{hr}\rho + \sigma_s\sigma_{hs}\sqrt{1-\rho^2})l_1 l_2 \\ &\quad -k_r\theta_r l_3 - \frac{k_r^2\theta_r - \sigma_r^2/2}{k_r^2},\end{aligned}$$

*where*

$$l_1 = \frac{-2}{\sqrt{k_h^2 + 2\delta\sigma_h^2} - k_h},$$

[7]Elton, et al. (2000) show that almost all of the differences between government and corporate credit yields are explained by expected default risk, taxes, and systematic risk. We neglect tax effects in our discount bond model.

$l_2 = \frac{\delta k_{hy} l_1}{\alpha}$, *and*

$l_3 = -\frac{1}{k_r} + \frac{\delta k_{hr} l_1}{k_r} + \frac{l_2}{k_r}$.

**Proof.** See Appendix B. ■

As discussed in Section III, our model preserves the property that the short-term spreads are positive. This is the case if the firm has a non-zero loss rate and positive default hazard for short maturity debt. More interestingly, the slope of short-term spread is a linear combination of *initial* values of zero maturity spread, interest rate, and the relative position of the equity price to its average of past equity levels. The dependence of the *current* state of economy allows us to generate richer term structures of credit spread. Assuming that $\theta_h > 0$, $k_h$, $k_{hy}$, and $k_{hr} \leq 0$, the spread curve tends to be upward (or download) sloping when $Y_{t_0} < 0$ (or $Y_{t_0} > 0$). This concurs with our intuition that there is a tendency for the spreads to rise when a firm's equity is continually declining. On the contrary, when the firm's equity is continually rising, the spreads tend to be sloping downward. On the other hand, the current interest rates have similar effects on the slope of short term spreads. Such empirical properties have been documented in Duffee (1998), who demonstrates that non-callable bond yield spreads fall when the levels of Treasury term structure rises. Furthermore, the extent of the decline depends on the initial credit quality of the bond. Duffee (1998) shows that the decline is small for high-grade bonds and large for low-grade bonds. By appropriate choices of $k_h$, $k_{hy}$, $k_{hr}$, and $Y_{t_0}$, the model appears to have a high degree of flexibility in reconciling these empirical results. For long-term debt, the levels of spread are stationary and independent of the current states of economy. The spread levels depend only on the present estimates of parameters. For example, it is trivial to observe that the levels of long-term spread increase with the value of $\theta_h$.

In the following, we illustrate the model by numerical results. In order to study the properties of credit spreads, we consider a particular case of the model with a base case environment in which the parameters take the following values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$. In this case, we are assuming $Y_{t_0} = 0$, that there are no particular substantial upward or downward movements in recent equity prices. Also we specify the model in such a way that while it becomes simpler as $k_{hr} = 0$, $\sigma_{hr} = 0$, and $\sigma_{hs} = 0$, it is rich enough to capture a negative correlation between the interest rate and the spread movements.

[Please Insert Figure 1 Here.]

[Please Insert Figure 2 Here.]

[Please Insert Figure 3 Here.]

Figures 1, 2, and 3 illustrate structural properties of the model. The plots show that the movements of spread are similar to those of Merton-type frameworks, as documented, for example, in Shimko, Tjima, and Van Deventer (1993). Figure 1 shows that the levels of spread tend to increase with the correlation. Figure 3 shows that equity return volatility has a significant impact on the levels of credit spread. The effects of equity volatility tend to increase the spreads through the $Y_{t_0}$ term. Furthermore, interest rate volatility has a similar effect on spread levels.

The effects of $\alpha$ are demonstrated in Figures 4 and 5. Figure 4 shows that in the case where there is a recent decline in equity prices and $\alpha$ is increasing, the spreads tend to move toward to the level of spread (dashed line with dots) generated in the case where $k_{hy} = 0$. Similar results are shown in Figure 5, where equity prices are assumed to be continually rising. The rationale behind this is as follows. Recall that the higher the value of $\alpha$, the greater the dependence of the moving average on the recent values of equity price. When $\alpha$ is increasing, the value of $Y_t$ tends to move to zero, and hence the effects of $Y_t$ in the hazard rate process and short rate process vanish.

Figures 4 and 5 also illustrate that the spread curves tend to be upward (or downward) sloping when the equity prices are continually declining (or rising). The result is trivial as, by Proposition 3, the slope of a spread curve at short maturity is negatively related to $Y_{t_0}$. Furthermore, it appears that the terms $Y_t$ in processes (8) and (9) have another importance in generating spread curves with a humped shape. Numerical computations show that this is likely to be the case, as the slope at short maturity tends to be positive when equity prices are recently declining.

[Please Insert Figure 4 Here.]

[Please Insert Figure 5 Here.]

# V An Extension: A Model with Counterparty Default Risk

It is interesting to see how we can extend our model to deal with the default risk of firm's counterparty. In this section, we introduce this element of risk into the model by employing the ideas in Jarrow and Yu (2001). We consider a simple primary-secondary framework of two firms, A and B. Firm A is a primary firm whose default process depends only on macro-variables. Firm B is a secondary firm having default process dependent on the macro-variables and the default probability of firm A. This assumption can be taken to mean that firm B is holding a significant amount of long (or short) positions of assets issued by firm A, and firm A is not holding any firm B's equity or debt. In principle, default processes of the two firms should be correlated. However, we intend to weaken this assumption, as we are only concerned with the impact of firm A's default risk on the credit spread of a bond issued by firm B. For the sake of technical simplicity, we assume that firm A has a constant default rate process,

$$h_t^A = h^A \; > 0,$$

and the default rate process $h_t^B$ of firm B consists of two parts relating to: (i) its own economic fundamentals, and (ii) a default hazard induced by the default hazard of firm A. We define $h_t^B$ as

$$h_t^B = h_t + p\mathbf{1}_{\{\tau^A \le t\}},$$

where $\tau^A$ is the default time of firm A, and $h_t$ is defined by equation (8) and $p > 0$ (or $p < 0$) is a constant. The interpretation of this equation is that firm B is holding some assets issued by firm A, and the default of firm A increases (or decreases) the instantaneous hazard rate of firm B. Furthermore, we can relate the value of $p$ to the nature of underlying assets, since a portfolio of holding a significant amount of long (or short) positions of firm A's assets is normally associated with a large positive (or negative) value of $p$. Assuming that debt issued by firm B has a loss rate $\delta$ of its market value in the event of default,[8] then the short spread process of firm B's debt is of the following form:

$$s_t^B = s_t + \delta p\mathbf{1}_{\{\tau^A \le t\}},$$

where $s_t$ is defined by equation (9). This equation means that the short spread increases (or decreases) by an amount of $\delta p$ after firm A have gone bankrupt. The price of a discount bond with a unit face value issued by firm B is then:

$$D^B(t_0, T) = D(t_0, T)E_{t_0}\Big[\exp(-\int_{t_0}^{T} \delta p\mathbf{1}_{\{\tau^A \le u\}}\, du)\Big], \tag{12}$$

if firm B has not defaulted by time $t_0$. By construction, as $h^A$ is assumed to be constant, the bond price can be separated into a product of two parts. The first part is exactly the same as the solution given in equation (1) and Proposition 1. The second part of the price is entirely due to the risk of holding the portfolio of firm A's assets. The following proposition shows how the default risks of firm A affect the credit spread of firm B's debt.

[8]This assumption on the loss of market value is different from Jarrow and Yu's (2001) approach . They assume that the loss rate is a fraction of face value and final payoff is always made at maturity.

**Proposition 4** *Assuming that the above conditions hold, and that both firm A and B have not defaulted by time $t_0$, then*

*(i) the bond price $D^B(t_0,T)$ of firm B is given by equation (12), where*

$$E_{t_0}\left[\exp(-\int_{t_0}^{T}\delta p\mathbf{1}_{\{\tau^A\leq u\}}\,du)\right]=\begin{cases}\frac{\delta p e^{-h^A(T-t_0)}-h^A e^{-\delta p(T-t_0)}}{\delta p-h^A} & \text{if } h^A\neq\delta p\\ e^{-h^A(T-t_0)}[h^A(T-t_0)+1] & \text{if } h^A=\delta p.\end{cases}$$

*(ii) The credit spread $s^B(t_0,T)$ of the risky bond is of the form:*

$$s^B(t_0,T)=s(t_0,T)+CS^A(t_0,T),$$

*where $s(t_0,T)$ is given by equation (11), $CS^A(t_0,T)$ is a component of the total spread, due to the default risk of firm A only, and*

$$CS^A(t_0,T)=\begin{cases}\frac{-1}{T-t_0}\log\left(\frac{\delta p e^{-h^A(T-t_0)}-h^A e^{-\delta p(T-t_0)}}{\delta p-h^A}\right) & \text{if } h^A\neq\delta p\\ \frac{-1}{T-t_0}\log\left(e^{-h^A(T-t_0)}[h^A(T-t_0)+1]\right) & \text{if } h^A=\delta p.\end{cases}$$

*Furthermore, $CS^A(t_0,T)$ has the following properties:*

$$\begin{aligned}\lim_{T\to t_0^+}CS^A(t_0,T) &= 0, \text{ and}\\ \lim_{T\to\infty}CS^A(t_0,T) &= Min\{h^A,\delta p\}.\end{aligned}$$

**Proof.** See Appendix C. ■

Proposition 4 shows that the default risk of counterparty A has an impact on the credit spread of firm B's debt. Interestingly, the effect is small when the maturity is short; when the maturity is long, an additional amount of $Min\{h^A,\delta p\}$ adds to the level $s(t_0,T)$ of the spreads. The short-end property of the credit spreads is due to the assumption that firm A has not yet defaulted at the issue time $t_0$. This implies that the zero maturity level of spread curve remains the same as $s_{t_0}=\delta h_{t_0}$. The effect of default risk of the counterparty becomes prominent only for debt of longer maturities. At the long end of the spread curve, the spread level $CS^A(t_0,T)$ is positive if $p>0$ when firm B is holding the portfolio of long positions of firm A's assets. The spread level $CS^A(t_0, T)$ is negative if $p < 0$ when the holding is largely short positions of firm A's assets.

The long-term properties of the counterparty default risk have an economic implication that can be visualised in the following situations. When the counterparty A has very high hazard of default, firm B would prefer to choose a portfolio in such a way that the value of $p$ is minimized, for example, by maintaining an optimal composition of long and short positions of firm A's assets in the portfolio. On the contrary, when the counterparty A has a very small hazard rate, it would be safe for firm B to hold a large portfolio of firm A's assets. Figure 6 shows the term structures of the spread level $CS^A$ for different values of $p$. Note that the spread curves $CS^A(t_0, T)$ increase more steeply for larger values of $p$.

[Please Insert Figure 6 Here.]

# VI Conclusion

The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. The two approaches have pros and cons. Although appealing, structural models have been criticized for not being able to generate sufficient credit spreads for small maturities of debt. The models' reliance on economic fundamentals and the value of a firm's asset make them hard to estimate in practice. On the contrary, the reduced-form approach has a major advantage in that it provides us with a model of very high tractability and ease of calibration. However, most reduced-form models have a structural drawback that lacks a connection between a firm's economic fundamentals and default events, although some suggestions for improvement have been put forward in the literature. This motivates us to propose in this paper the flexible analytical model which provides a compromise between the two approaches.

Our model of corporate debt has taken into account stochastic interest rates, a firm's asset values, and hazard rates of default. Consistent with the literature of the reduced-form models, we have assumed that default can only happen unexpectedly. As in Duffie and Singleton (1999) and Duffee (1999), we take a hazard rate process as exogenously given. Unlike those models, there is a crucial and innovative distinction in the specification of the process in our model. We have introduced structural characteristics of the firm into the hazard rate process, through a moving average providing a measure of the firm's performance in equity as well as its leverage effect. Furthermore, our model also has a fairly high degree of flexibility in correlation structures.

The model has another four important features. Firstly, instead of solely using a firm's current value as the conventional Merton-type and recent Madan and Unal's (2000) models do, we take a broader view of the financial health of the firm by considering the current asset level relative to its past positions. The implication of this is that the debt becomes more risky when the relative levels are lower. When the relative levels are higher, the debt becomes safer. As with other structural models, we have shown that our model is able to capture the effects of economic fundamentals on properties of credit spreads. For example, in a simplified version of the model, we have demonstrated that the levels of spread increase with interest rate volatility, equity return volatility, and the correlation.

Secondly, our model preserves a high degree of flexibility in generating credit spreads. Numerical computations have shown that the model is flexible enough to generate many different term structures of credit spreads by using appropriately chosen parameters. We have investigated analytically how parameter values affect the shape of the credit spread curve, in terms of its intercept, slope at zero maturity, and spread level for long maturity.

Thirdly, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhance its empirical applicability.

Fourthly, as an interesting extension, we have demonstrated how we can generalize our model to deal with the default risk of a counterparty. Although in a simple setting, we believe that the extended model has captured essential features of counterparty default risk, whose properties have been shown to have an economic implication in holding a portfolio of assets issued by the counterparty.

The analytical tractability of the model has another advantage. Given the value of the market loss rate $\delta$, we are able to estimate other model parameters, and hence compute analytically survival probabilities of firms. However, as shown in Duffie and Singleton (1999), if $\delta$ is unknown and has to be estimated from data, we also have the same identification problem of the market loss rate $\delta$ and the hazard rate process $h_t$ from our model. Jarrow (2001) develops a procedure for segregating the two variables by using equity and bond prices. While it remains to be seen whether Jarrow's idea can be justified by empirical work, the modeling of survival probabilities of firms in terms of hazard rate processes is an important area of research in credit risk analysis. More importantly, by linking the hazard rate processes to underlying fundamental variables, the result may provide us with deeper insights into understanding the default mechanism.

Finally, the flexibility of our model also paves the way for further generalizations. For example, the modeling of a framework with several counterparties and the pricing of credit default swaps are attractive avenues for further work.

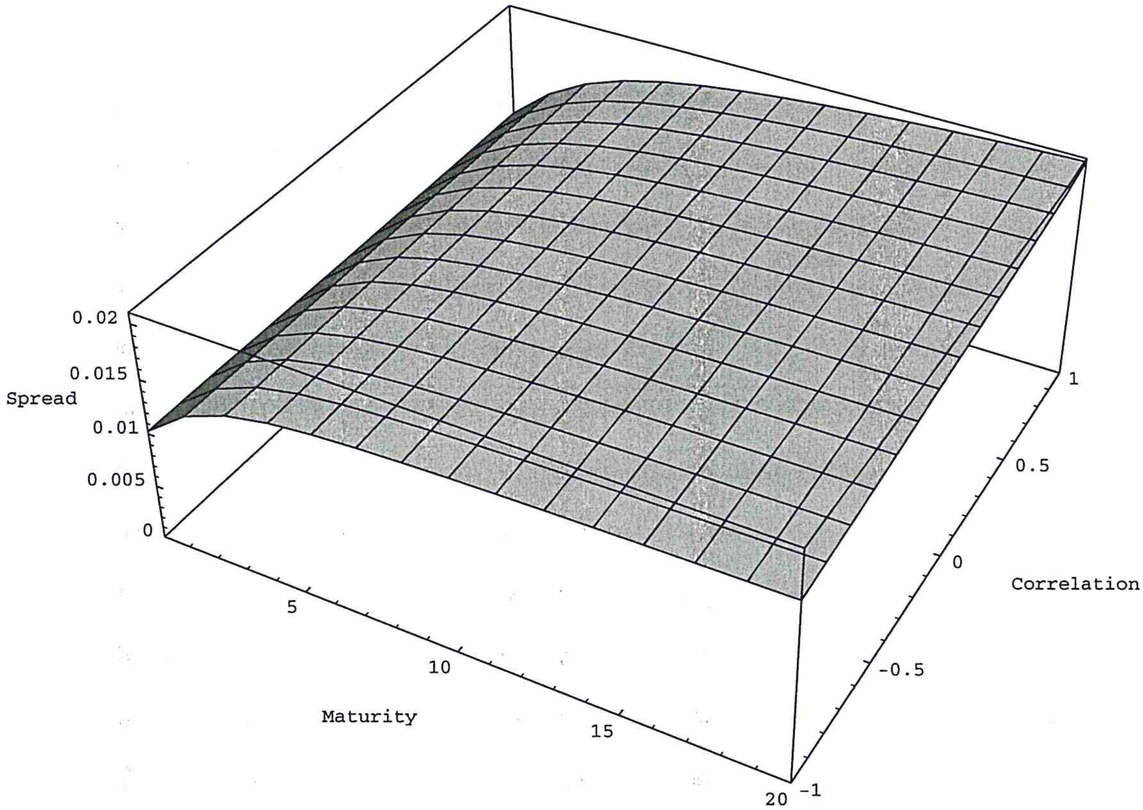


Figure 1: **Credit spread surface as a function of maturity $T$ and correlation $\rho$.** This figure shows the term structures of credit spreads of a risky discount bond with different values of correlation $\rho$: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

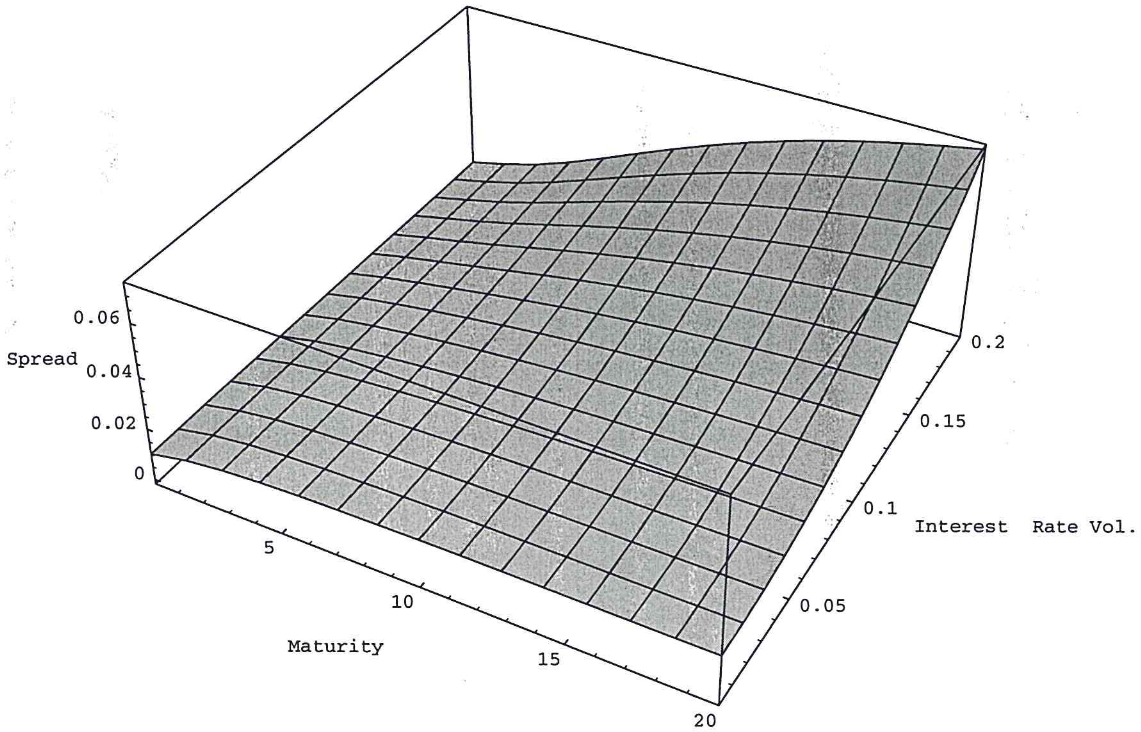


Figure 2: **Credit spread surface as a function of maturity $T$ and interest rate volatility $\sigma_r$.** This figure shows the term structures of credit spreads of a risky discount bond with different values of interest rate volatility $\sigma_r$: $k_r = 0.2$, $\theta_r = 0.06$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

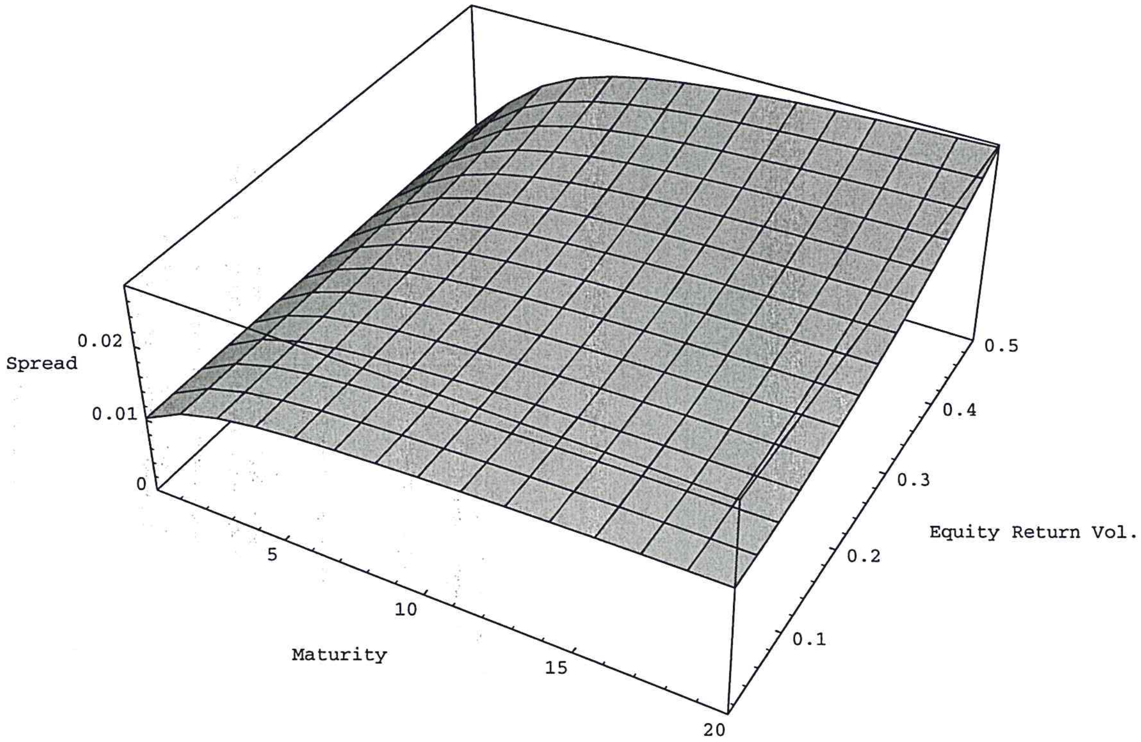


Figure 3: **Credit spread surface as a function of maturity $T$ and equity return volatility $\sigma_s$.** This figure shows the term structures of credit spreads of a risky discount bond with different values of equity return volatility $\sigma_s$: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

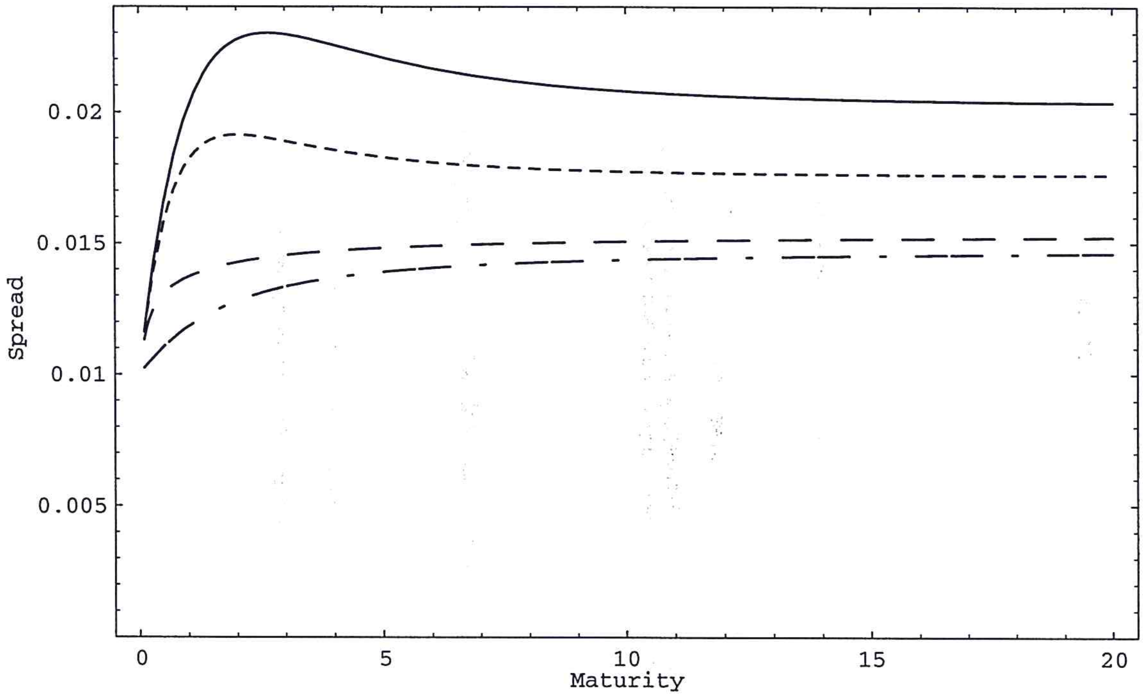


Figure 4: **Term structures of credit spread for different values of $\alpha$ when $Y_{t_0} < 0$.** This figure shows the term structures of credit spreads of a risky discount bond with different values of $\alpha$: (i) $\alpha = 1$ (solid line), (ii) $\alpha = 2$ (short dashed line), (iii) $\alpha = 10$ (long dashed line), (iv) $k_{hy} = 0$ (dashed line with dots). Parameter values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = -0.3$, and $r_{t_0} = 0.05$, unless stated otherwise.

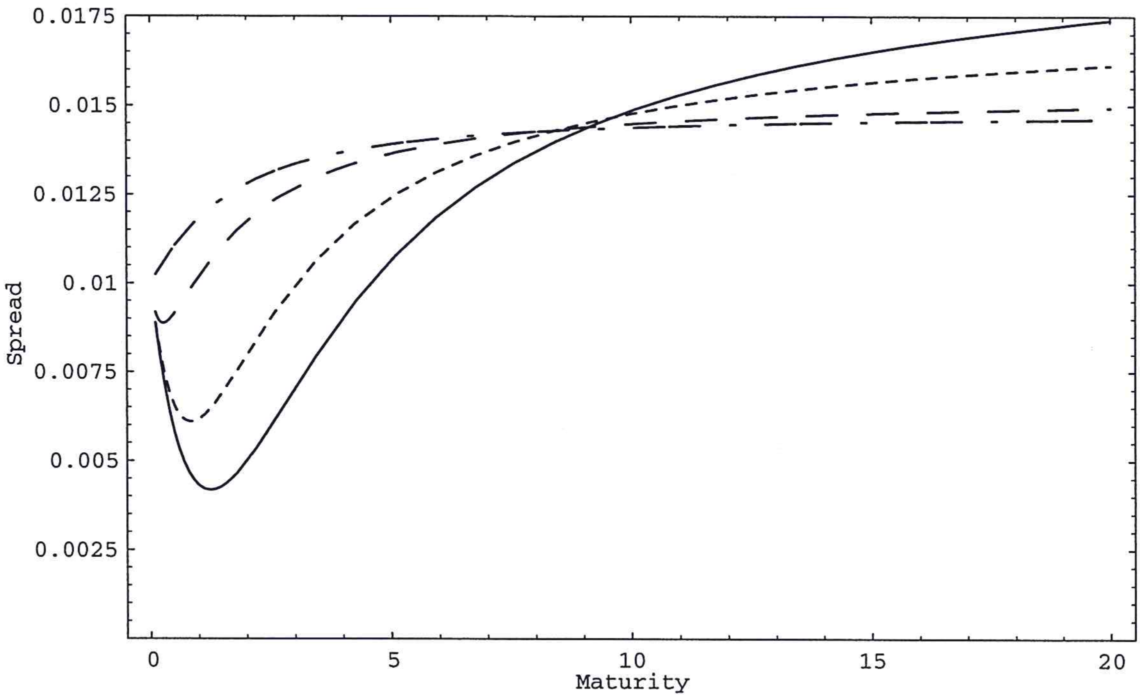


Figure 5: **Term structures of credit spread for different values of $\alpha$ when $Y_{t_0} > 0$.** This figure shows the term structures of credit spreads of a risky discount bond with different values of $\alpha$: (i) $\alpha = 1$ (solid line), (ii) $\alpha = 2$ (short dashed line), (iii) $\alpha = 10$ (long dashed line), (iv) $k_{hy} = 0$ (dashed line with dots). Parameter values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = +0.3$, and $r_{t_0} = 0.05$, unless stated otherwise.

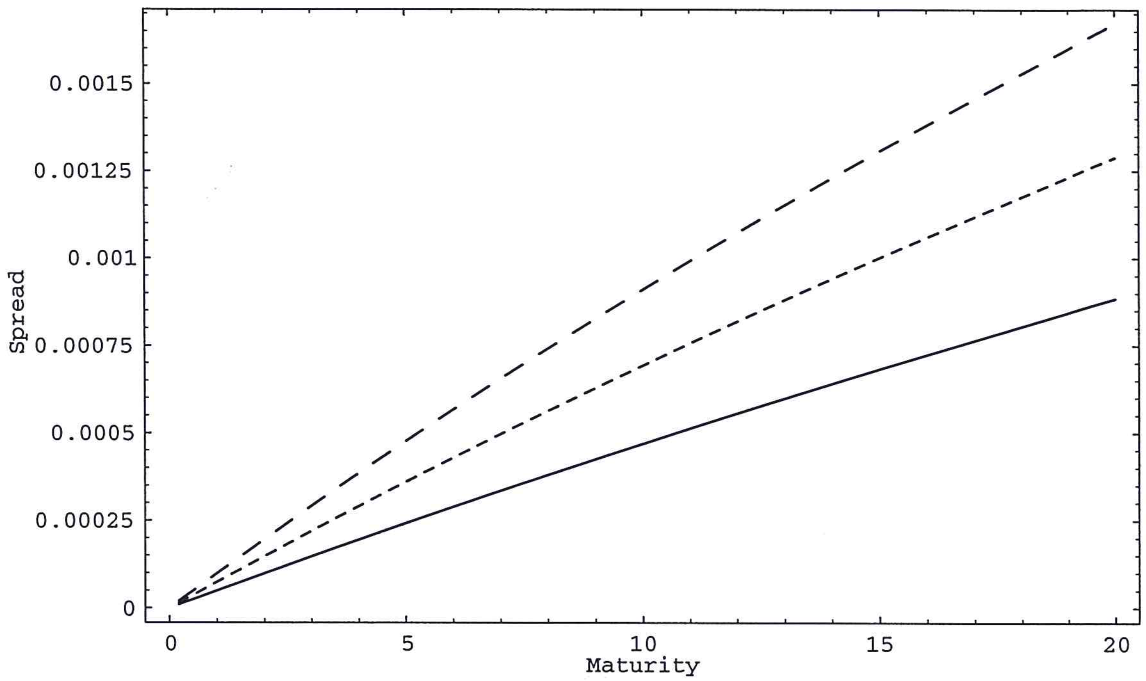


Figure 6: **Term structures of credit spread $CS^A(t_0, T)$ for different values of $p$.** This figure shows the term structures of credit spreads due to the default risk of firm A for different values of $p$: (i) $p = 0.02$ (solid line), (ii) $p = 0.03$ (short dashed line), and (iii) $p = 0.04$ (long dashed line). Parameter values: $h^A = 0.01$, $\delta = 0.5$, and $t_0 = 0$.

# Appendices

## A Proof of Proposition 1

**Proof.** Given the three affine processes $s_t$, $Y_t$, and $r_t$ as in $(*)$, by Duffie and Kan [1996], we can express the solution in the form of the equation (1). For equation (5) to satisfy the corresponding backward Kolmogorov partial differential equation,[9] we have the following set of differential equations:[10]

$$
\begin{aligned}
0 &= -1 + B_1'(t,T) + k_h B_1(t,T) + \frac{\delta}{2}\sigma_h^2 B_1(t,T)^2, && (13)\\
0 &= B_2'(t,T) + \delta k_{hy} B_1(t,T) - \alpha B_2(t,T), && (14)\\
0 &= -1 + B_3'(t,T) - k_r B_3(t,T) + \delta k_{hr} B_1(t,T) + B_2(t,T), \mathit{and} && (15)\\
0 &= A'(t,T) + \delta\theta_h B_1(t,T) - (\sigma_s^2/2 + a) B_2(t,T) + k_r\theta_r B_3(t,T) + \frac{1}{2}\sigma_s^2 B_2(t,T)^2 \\
&\quad + \frac{1}{2}\sigma_r^2 B_3(t,T)^2 + \frac{\delta^2}{2}(\sigma_{hr}^2 + \sigma_{hs}^2) B_1(t,T)^2 + \sigma_r \delta \sigma_{hr} B_1(t,T) B_3(t,T) && (16)\\
&\quad + \sigma_r\sigma_s\rho B_2(t,T)B_3(t,T) + \delta(\sigma_s\sigma_{hr}\rho + \sigma_s\sigma_{hs}\sqrt{1-\rho^2}) B_1(t,T) B_2(t,T),
\end{aligned}
$$

with the boundary conditions $A(T,T) = 0$, $B_1(T,T) = 0$, $B_2(T,T) = 0$, and $B_3(T,T) = 0$. Assuming that $k_h$, $\delta$, $\sigma_h$, $\alpha$, and $k_r$ are positive, results follow by solving the above system iteratively. Note that $B_2(t_0,T)$ can be expressed in terms of hypergeometric functions of the form ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$, and so are $B_3(t_0,T)$ and $A(t_0,T)$. ■

## B Proof of Proposition 3

**Proof.** Part (i) and (ii) are trivial.

[9] We know from the Feynman-Kac formula that, under some technical conditions, equation (5) solves the backward Kolmogorov partial differential equation of the problem.

[10] All derivatives are computed with respect to time $t$.

For part (iii), we prove the result by using Proposition 1 and equation (11). Note that $B_1(t_0, T) \to l_1$ as $T \to \infty$, where $l_1 = \frac{-2}{\sqrt{k_h^2 + 2\delta\sigma_h^2} - k_h}$.

Let $B_2(t_0, T) \to l_2$ as $T \to \infty$, where $l_2$ is independent of time $t_0$. Then for $t > t_0 \geq 0$,

$$\begin{aligned} & \delta k_{hy} \int_{t_0}^{T} e^{-\alpha(s-t_0)} B_1(s,T)\, ds \\ = \; & \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} B_1(s,T)\, ds + \delta k_{hy} \int_{t}^{T} e^{-\alpha(s-t_0)} B_1(s,T)\, ds \\ = \; & \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} B_1(s,T)\, ds + \delta k_{hy} e^{-\alpha(t-t_0)} \int_{t}^{T} e^{-\alpha(s-t)} B_1(s,T)\, ds. \end{aligned}$$

Taking limits on both sides, as $T \to \infty$,

$$l_2 = \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} l_1 \, ds + e^{-\alpha(t-t_0)} l_2,$$

this implies that $l_2 = \frac{\delta k_{hy} l_1}{\alpha}$.

Similarly, we can prove that $l_3 = -\frac{1}{k_r} + \frac{\delta k_{hr} l_1}{k_r} + \frac{l_2}{k_r}$. Hence,

$$\lim_{T\to\infty} \frac{B_i(t_0, T)}{T - t_0} = 0,$$

for i = 0,1,2, and 3.

To compute $\lim_{T\to\infty} \frac{A(t_0,T)}{T-t_0}$, we consider

$$\lim_{T\to\infty} \frac{\int_{t_0}^{T} B_1(s,T)\, ds}{T - t_0}.$$

Note that $B_1(s,T)$ depends on $s$ and $T$ through their difference $T - s$, and that $\lim_{T\to\infty} \int_{t_0}^{T} B_1(s,T)\, ds \to -\infty$ as $T \to \infty$. Therefore,

$$\begin{aligned} \lim_{T\to\infty} \frac{\int_{t_0}^{T} B_1(s,T)\, ds}{T - t_0} &= \lim_{T\to\infty} \frac{\partial}{\partial T} \int_{t_0}^{T} B_1(s,T)\, ds \\ &= \lim_{T\to\infty} -\frac{\partial}{\partial t_0} \int_{t_0}^{T} B_1(s,T)\, ds \\ &= \lim_{T\to\infty} B_1(t_0, T) \\ &= l_1 \end{aligned}$$

All remaining terms in $\lim_{T\to\infty} \frac{A(t_0,T)}{T-t_0}$ can be computed similarly. The result follows.

■

# C Proof of Proposition 4

**Proof.** Part (i): Note that for $t_0 \leq u \leq T$, $\mathbf{1}_{\{\tau^A \leq u\}} = \mathbf{1}_{\{\tau^A \leq u\}}\mathbf{1}_{\{\tau^A \leq T\}}$.

$$
\begin{aligned}
& E_{t_0}\Big[\exp(-\int_{t_0}^{T} \delta p \mathbf{1}_{\{\tau^A \leq u\}}\, du)\Big] \\
= \; & E_{t_0}\Big[\exp(-\delta p(T-\tau^A)\mathbf{1}_{\{\tau^A \leq T\}})\Big] \\
= \; & \int_{t_0}^{\infty} e^{-\delta p(T-s)\mathbf{1}_{\{s \leq T\}}} h^A e^{-h^A(s-t_0)} ds \\
= \; & \int_{t_0}^{T} e^{-\delta p(T-s)} h^A e^{-h^A(s-t_0)} ds + \int_{T}^{\infty} h^A e^{-h^A(s-t_0)} ds \\
= \; & \int_{t_0}^{T} e^{-\delta p T + h^A t_0} h^A e^{-(h^A - \delta p)s} ds + e^{-h^A(T-t_0)}.
\end{aligned}
$$

If $\delta p \neq h^A$, then the above integral becomes $\frac{h^A\left[e^{-h^A(T-t_0)} - e^{-\delta p(T-t_0)}\right]}{\delta p - h^A}$. Otherwise, it becomes $e^{-\delta p T + h^A t_0} h^A(T - t_0)$. The results follow.

Part (ii) is trivial. ■

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