Multivariate Extremes at Work for Portfolio Risk Measurement

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Abstract

This paper proposes a methodology to provide risk measures for portfolios during extreme events. The approach is based on splitting the multivariate extreme value distribution of the assets of the portfolio into two parts: the distributions of each asset and their dependence function. The estimation problem is also investigated. Then, stress-testing is applied for market indices portfolios and Monte-Carlo based risk measures – Value-at-Risk and Expected Shortfall – are provided.

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1 Introduction

The behaviour of portfolios during financial crises is an important element of risk management (Basle Committees I and II). The goal of this paper is to construct a methodology to calculate risk measures – as value at risk and expected shortfall – that directly come from the extremal dependence structure between portfolio components. This is achieved by considering their multivariate extreme value (MEV) probability distributions.

Extreme value theory (EVT) is now a well developed tool used to model maxima and minima of financial returns. A seminal paper is Embrechts and Schmidli [1994] in an insurance context. Longin [1996] provides a study of stock market extreme returns. An influential book that provides a "state of the art" of the subject is *Modelling Extremal Events for Insurance and Finance* by Embrechts, Klüppelberg and Mikosch [1997]. However, the extension to the multivariate modelling is not obvious, as pointed out by Embrechts, de Haan and Huang [2000]. However, some examples of applications of MEV theory can be found in the non-financial literature (for example Coles and Tawn [1991], Coles and Tawn [1994], de Hann and de Ronde [1998]). For an overview of the theoretical aspects of the subject, we refer to Resnick [1987].

In the financial literature, some measures for extremal dependence between returns can be found in Straetmans [1999] and Stărică [1999]. Longin [2000] proposed an approach based on EVT for computing value at risk compatible with extreme events. The author provides an *ad hoc* aggregation formula to approximate the value at risk.

As we noted above MEV distributions often become analytically intractable. An interesting way to avoid these difficulties is to use a copula function that allows us to split the univariate extremes from their dependence structure. A general introduction about the application of copulae to finance is Embrechts, McNeil and Straumann [1999] and Bouyé, Durrleman, Nickeghbali, Riboulet and Roncalli [2000]. Concerning the application of copulae to joint extreme events, a bivariate case – two assets – is presented in Longin and Solnik [2001] who use a Gumbel copula to study the conditional correlation structure of international equity returns. However, as we will show in this paper, there are many possible copulae to model the joint extremal dependence. These copulae may exhibit different dependence structures. For example, the Gumbel copula induces clustering in the dependence structure if the dimension is higher than two. Indeed, the higher dimensions are obtained by compound method as we will further see in more details.

In the second section, we introduce univariate EVT and review the link between copulae and

MEV distributions. Then, we present three copulae that can be used in an extreme value context. In the third section, we describe our estimation methodology and provide an application to the joint dependence of german, japanese and US market indices during extreme events. The empirical results are also discussed. In the fourth section, we use the estimated parameters of the MEV distribution to compute risk measures for multi-indices portfolios. Specifically, the risk of the portfolios is studied from two directions: (i) multivariate stress testing and (ii) Monte-Carlo based risk measures. The sixth section concludes.

2 Multivariate Extreme Value Theory

In this section, we first briefly introduce univariate EVT. We then state a theorem that tells us that a MEV distribution can be built from univariate extreme value distributions and a specific family of copulae. We present the three copulae that will be used through this paper. The results for maxima are developed, although equivalent results exist straight forwardly for minima.

2.1 Preliminaries

The general context of univariate extreme value theory is easily explained. A very useful result is the Fisher-Tippett theorem. It tells us that normalised maxima - under particular conditions - follow one of only three (extreme value) distributions. For i.i.d. random variables (X_n), if there are constants $a_n > 0$, $b_n \in R$ and a non degenerate function G with $a_n^{-1}(\chi^+ - b_n) \xrightarrow{d} G$ where $\chi^+ = \max(X_{1,\ldots,}X_n)$, then G corresponds to:

Type I (Gumbel)	$G(x) =$	$\exp(-e^{-x})$	$x \in \mathbb{R}$	
Type II (Frechet)	$G(x) =$	$\begin{cases} 0 \\ \exp(-x^{-\alpha}) \end{cases}$	$x \leq 0$ $x > 0$	$\alpha > 0$
Type III (Weibull)	$G(x) =$	$\begin{cases} \exp(-(-x)^{\alpha}) \\ 1 \end{cases}$	$x \leq 0$ $x > 0$	$\alpha > 0$

In practice the Von-Mises representation encompases this result and provides a unique distribution for all extremes:

$$G(\gamma;\chi^+) = \exp\left\{-\left(1 - \tau\frac{\chi^+ - b}{a}\right)^{1/\tau}\right\} \tag{1}$$

with $\left(1-\tau\frac{\chi^{+}-b}{a}\right)>0$ and $\boldsymbol{\gamma}=(\tau,a,b)$. We recover the three cases as the $\tau=0$ (Gumbel), $\tau=-\alpha^{-1}<0$ (Frechet) and $\tau=\alpha^{-1}>0$ (Weibull). This distribution is called Generalised Extreme Value (GEV) distribution.

The theory of multivariate extremes was introduced by GUMBEL [1960] and an overview can be found in RESNICK [1987]. The main reference given our current objective is DEHEUVELS [1978] which contained a theorem that allows us to split the problem of characterising multivariate extreme value distributions into two distinct problems:

1. the characterisation of the univariate extreme value distributions
2. the existence of a limiting dependence function (or copula) that links univariate extreme value distributions in order to obtain the multivariate extreme value distribution.

This idea is summarized in the following theorem:

Theorem 1 (Deheuvels (1978)) *Let* χ_n^{+} *be such that*

$$\chi_n^{+}=\left(\chi_{1,n}^{+},\ldots,\chi_{d,n}^{+}\right)=\left(\bigvee_{k=1}^{n}X_{1,k},\ldots,\bigvee_{k=1}^{n}X_{d,k}\right) \tag{2}$$

with $(X_{1,n},\ldots,X_{d,n})$ *an i.i.d. sequence of random vectors with distribution function* $\mathbf{F}$*, marginal distributions* $F_1,\ldots,F_d$ *and copula* $\mathbf{C}$*. Then,*

$$\begin{aligned}\lim_{n\to\infty}\Pr\left\{\frac{\chi_{1,n}^{+}-b_{1,n}}{a_{1,n}}\leq x_1,\ldots,\frac{\chi_{d,n}^{+}-b_{d,n}}{a_{d,n}}\leq x_d\right\} &= \mathbf{G}_\infty\left(x_1,\ldots,x_d\right)\\ \forall\left(x_1,\ldots,x_d\right) &\in \mathbb{R}^N\end{aligned} \tag{3}$$

with $a_{j,n}>0, j=1,\ldots,d, n\geq 1$ **iff**

1. $\forall j=1,\ldots,d$, there exist some constants $a_{j,n}$ and $b_{j,n}$ and a non-degenerate limit distribution G_j such that

$$\lim_{n\to\infty}\Pr\left\{\frac{\chi_{1,n}^{+}-b_{j,n}}{a_{j,n}}\leq x_j\right\}=G_j\left(x_j\right)\qquad\forall x\in\mathbb{R} \tag{4}$$

2. there exists a copula $\mathbf{C}_\infty$ such that

$$\mathbf{C}_\infty\left(u_1,\ldots,u_d\right)=\lim_{n\to\infty}\mathbf{C}^n\left(u_1^{1/n},\ldots,u_d^{1/n}\right). \tag{5}$$

If the conditions of the previous theorem are fulfilled, we have

$$\mathbf{G}_{\infty}\left(x_1,\ldots,x_d\right)=\mathbf{C}_{\infty}\left(G_1\left(x_1\right),\ldots,G_d\left(x_d\right)\right) \tag{6}$$

The first condition is not specific to the multivariate case and is already present in univariate EVT. It corresponds to an existence condition. The second condition directly informs us about the dependence structure that allows us to obtain MEV distributions with given margins. The link between the one dimensional extremes is obtained by applying them the function $\mathbf{C}_{\infty}$ that is called copula function. This function is in fact nothing else but a multivariate distribution with uniform margins. The concept of maximum domain of attraction (MDA) is sometimes alternatively used. In the theorem above, each real-valued random variable X_j for $j=1,\ldots,d$ has its own univariate distribution function F_j. And each maximum χ_j (respectively corresponding to X_j) follows an extreme value distribution G_j (amongst the three already presented: Gumbel, Fréchet and Weibull). We say that F_j belongs to the maximum domain of attraction of G_j. This concept can be extended to the multivariate distribution $\mathbf{F}$ that belongs to the MDA of the MEV distribution $\mathbf{G}_{\infty}$. By introducing copula - see Galambos [1978] for more details - the theorem can be restated as follows:

Theorem 2 $\mathbf{F}$ ∈

A corollary of this condition is that extreme value copulae only model positive dependence and this will influence our modelling strategy as we will see further for the empirical estimation of the copulae parameters. For an overview of extreme value copulae, we refer to Joe [1997]. However, many of them are not tractable in high dimensions. For our study, we focus on three copulae: Gumbel, Hüsler and Reiss, and Joe and Hu. Our choice is motivated by the fact that these copulae can be expressed in a recursive form. This property is of special interest from a numerical point of view. Indeed, it means that the copula of dimension d can directly be deduced from the copula of dimension $(d-1)$. In the following subsections, we will provide the functional forms of the chosen copulae, the formulae that allow to extend them to higher dimensions and will discuss the dependence structure they exhibit. Let us denote $\mathbf{u}_d = (u_1, \ldots, u_d) = (G_1(x_1), \ldots, G_d(x_d))$ the d-margins vector and $\boldsymbol{\delta}_d$ the extreme dependence parameters vector whose dimension depends on the copula.

2.2.1 Gumbel

The bivariate Gumbel copula

$$\mathbf{C}(u_1, u_2; \delta) = \exp\left(-\left(\tilde{u}_1^{\delta} + \tilde{u}_2^{\delta}\right)^{\frac{1}{\delta}}\right) \tag{9}$$

with $\delta \in [1, \infty)$. This copula can be extended to higher dimension by compound method:

$$\begin{aligned}
\mathbf{C}(\mathbf{u}_d; \boldsymbol{\delta}_{d-1}) &= \mathbf{C}(\mathbf{C}(\mathbf{u}_{d-1}; \boldsymbol{\delta}_{d-2}), u_d) \\
&= \exp\left\{-\left[\left(\cdots\left(\cdots\left[\left(\tilde{u}_1^{\delta_{d-1}} + \tilde{u}_2^{\delta_{d-1}}\right)^{\frac{\delta_{d-2}}{\delta_{d-1}}} + \tilde{u}_3^{\delta_{d-2}}\right]^{\frac{\delta_{d-3}}{\delta_{d-2}}}\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + \ldots + \tilde{u}_n^{\delta_{d-n+1}}\right)^{\frac{\delta_{d-n}}{\delta_{d-n+1}}} + \ldots + \tilde{u}_{d-1}^{\delta_2}\right)^{\frac{\delta_1}{\delta_2}} + \tilde{u}_d^{\delta_1}\right]^{\frac{1}{\delta_1}}\right\}
\end{aligned} \tag{10}$$

and the dependence structure is given by $\boldsymbol{\delta}_{d-1} = (\delta_1, \ldots, \delta_{d-1})$ as follows:

$$\begin{array}{ccccccc}
\delta_{d-1} & \rightarrow & (\chi_1, \chi_2) & & & & \\
\vdots & & \vdots & \ddots & & & \\
\delta_n & \rightarrow & (\chi_1, \chi_n) & \cdots & (\chi_2, \chi_3) & & \\
\vdots & & \vdots & & \vdots & \ddots & \\
\delta_1 & \rightarrow & (\chi_1, \chi_d) & \cdots & (\chi_n, \chi_d) & \cdots & (\chi_{d-1}, \chi_d)
\end{array} \tag{11}$$

with $\infty > \delta_{d-1} \geq \ldots \geq \delta_3 \geq \delta_2 \geq \delta_1 \geq 1$. The parameter δ_{d-1} characterizes the dependence of one pair and the parameter δ_1 of $(d-1)$ pairs. The Gumbel copula employs a few parameters and then induces clustering.

2.2.2 Hüsler and Reiss

The bivariate Hüsler-Reiss copula is given (HÜSLER and REISS [1989]) by:

$$\mathbf{C}(u_1, u_2; \delta) = \exp\left\{-\tilde{u}_1 \Phi\left(\delta^{-1} + \frac{1}{2}\delta \ln\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right) - \tilde{u}_2 \Phi\left(\delta^{-1} + \frac{1}{2}\delta \ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right)\right\} \tag{12}$$

where $\delta \geq 0$ and $\tilde{u}_i = -\ln u_i = -\ln G_i(x_i)$. Although the Gumbel copula is characterised by $(d-1)$ parameters, the multivariate Hüsler-Reiss copula contains $\frac{d(d-1)}{2}$ parameters ($\delta_{i,j}, 1 \leq i < j \leq d$ and $\delta_{i,j} = \delta_{j,i}$). It can be derived recursively[2]:

$$\mathbf{C}(\mathbf{u}_d; \boldsymbol{\delta}_d) = \mathbf{C}(\mathbf{u}_{d-1}; \boldsymbol{\delta}_{d-1}) \times \exp\left\{-\int_0^{-\ln u_d} \boldsymbol{\Phi}_{d-1}\left(\boldsymbol{\kappa}_{d-1}(\mathbf{u}_{d-1}, q); \boldsymbol{\rho}_{d-1}\right) dq\right\} \tag{16}$$

with

$$\boldsymbol{\rho}_{d-1} = \begin{pmatrix} 1 & & & & \\ \rho_{d,1,2} & 1 & & & \\ \rho_{d,1,3} & \rho_{d,2,3} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \rho_{d,1,d-1} & \rho_{d,2,d-1} & \cdots & \rho_{d,d-2,d-1} & 1 \end{pmatrix}$$

where $\rho_{d-1,i,j} = \frac{\delta_{i,d-1}\delta_{j,d-1}}{2}\left(\delta_{i,d-1}^{-2} + \delta_{j,d-1}^{-2} - \delta_{i,j}^{-2}\right)$ and

$$\begin{cases} \mathbf{u}_d = (u_1, \ldots, u_d) \\ \boldsymbol{\delta}_d = (\delta_{i,j}, 1 \leq i < j \leq d) \\ \boldsymbol{\kappa}_{d-1}(\mathbf{u}_{d-1}, q) = (\kappa_{1,d}(u_1, q), \ldots, \kappa_{d-1,d}(u_{d-1}, q)) \quad \text{with} \quad \kappa_{i,d}(u_i, q) = \delta_{i,d}^{-1} + \frac{1}{2}\delta_{i,d} \ln\left(-\frac{q}{\ln u_i}\right) \\ \text{for} \quad i = 1, \ldots, d-1 \end{cases}$$

and $\boldsymbol{\Phi}_k(.; \boldsymbol{\rho})$ corresponds to the multivariate gaussian cumulative function with correlation $\boldsymbol{\rho}$.

[2] The expression of this copula directly comes from the link between Multivariate Extreme Value (MEV) Distributions and Min-Stable Multivariate Exponential (MSMVE). Indeed, with $\mathbf{C}$ an MEV copula, if one can write:

$$\mathbf{C}(u_1, \ldots, u_n) = \mathbf{D}(\tilde{u}_1, \ldots, \tilde{u}_n) \tag{13}$$

with $\tilde{u}_i = -\ln u_i$ then $\mathbf{D}$ is an MSMVE distribution. Let use the definition of the dependence with $\mathbf{A} = -\ln \mathbf{D}$, as in JOE [1997] (p. 184), the Hüsler-Reiss is defined recursively:

$$\mathbf{A}(\mathbf{y}_n; \boldsymbol{\delta}_n) = \mathbf{A}(\mathbf{y}_{n-1}; \boldsymbol{\delta}_{n-1}) + \int_0^{y_n} \boldsymbol{\Phi}_{n-1}\left(\boldsymbol{\kappa}_{n-1}(\exp(-\mathbf{u}_{n-1}), q); \boldsymbol{\rho}_{n-1}\right) dq \tag{14}$$

and equation (16) follows. In the trivariate case, we have:

$$\mathbf{C}(\mathbf{u}_3; \boldsymbol{\delta}_3) = \mathbf{C}(\mathbf{u}_2; \boldsymbol{\delta}_2) \times \exp\left\{-\int_0^{-\ln u_3} \boldsymbol{\Phi}_2(\boldsymbol{\kappa}_2(\mathbf{u}_2, q); \rho)\, dq\right\} \tag{15}$$

where $\rho = \rho_{3,1,2} = \frac{\delta_{1,3}\delta_{2,3}}{2}\left(\delta_{1,3}^{-2} + \delta_{2,3}^{-2} - \delta_{1,2}^{-2}\right)$.

2.2.3 Joe and Hu

Another interesting copula has been defined by Joe and Hu [1996]:

$$\mathbf{C}\left(\mathbf{u}_d;\boldsymbol{\delta}_d\right)=\exp\left\{-\left[\sum_{i=1}^{d}\sum_{j=i+1}^{d}\left[\left(p_i\tilde{u}_i^{\theta}\right)^{\delta_{i,j}}+\left(p_j\tilde{u}_j^{\theta}\right)^{\delta_{i,j}}\right]^{\frac{1}{\delta_{i,j}}}+\sum_{i=1}^{d}\nu_i p_i\tilde{u}_i^{\theta}\right]^{\frac{1}{\theta}}\right\} \tag{17}$$

with $p_i=(\nu_i+d-1)^{-1}$ and where $\boldsymbol{\delta}_d$ has the following elements: $\delta_{i,j}$ the pairwise coefficients, ν_i the bivariate and multivariate asymmetry coefficients and θ a common parameter. To extend this copula to higher dimensions, one only has to extend the sum components of the formula. The bivariate margins are given by:

$$\mathbf{C}_{ij}\left(u_i,u_j\right)=\exp\left\{-\left[\left[\left(p_i\tilde{u}_i^{\theta}\right)^{\delta_{i,j}}+\left(p_j\tilde{u}_j^{\theta}\right)^{\delta_{i,j}}\right]^{\frac{1}{\delta_{i,j}}}+\left(\nu_i+d-2\right)p_i\tilde{u}_i^{\theta}+\left(\nu_j+d-2\right)p_j\tilde{u}_j^{\theta}\right]^{\frac{1}{\theta}}\right\} \tag{18}$$

3 Empirical estimation of copulae parameters

Estimation is a two-step procedure. First, the parameters of the marginal distributions are estimated, then the original variables are mapped to uniforms using these estimated parameters and the dependence parameters are estimated. A detailed description of this procedure can be found in Joe and Xu [1996]. In practice, *for each margin*, a sample of size nT can be divided into T blocks of n observations. Then, T maxima are available: $\chi_n^{+(t)}=\max(X_{n(t-1)+1,\ldots,}X_{nt})$ with $t=1...T$. The likelihood function is:

$$L(\boldsymbol{\gamma};\chi^{+})=\prod_{t=1}^{T}\mathbf{g}(\boldsymbol{\gamma};\chi_n^{+(t)})\mathbf{1}_{\left\{1-\tau\frac{\chi_n^{+(t)}-b}{a}>0\right\}} \tag{19}$$

with $\mathbf{g}(\boldsymbol{\gamma};\chi^{+})=\frac{1}{a}\left(1-\tau\frac{\chi^{+}-b}{a}\right)^{\frac{1}{\tau}-1}\exp\left\{-\left(1-\tau\frac{\chi^{+}-b}{a}\right)^{1/\tau}\right\}$. The log-likelihood estimator for each margin is:

$$\begin{aligned}\hat{\boldsymbol{\gamma}}&=\arg\max_{\boldsymbol{\gamma}\in\Theta}\ \ln L(\boldsymbol{\gamma};\chi_n^{+(1)},\ldots,\chi_n^{+(T)})\\ \hat{\boldsymbol{\gamma}}&=\arg\max_{\boldsymbol{\gamma}\in\Theta}\left\{-T\ln(a)+(\frac{1}{\tau}-1)\sum_{t=1}^{T}\ln\left(1-\tau\frac{\chi_n^{+(t)}-b}{a}\right)-\sum_{t=1}^{T}\left(1-\tau\frac{\chi_n^{+(t)}-b}{a}\right)^{1/\tau}\right\}\end{aligned} \tag{20}$$

where $\chi_n^{+(t)}$ is the maxima of the t^{th} block. The score vector $\mathbf{s}(\boldsymbol{\gamma})$ is as usual:

$$\mathbf{s}(\boldsymbol{\gamma})=\frac{\partial\log\mathbf{g}(\boldsymbol{\gamma};\chi^{+})}{\partial\boldsymbol{\gamma}} \tag{21}$$

where the derivatives are developed in the Appendix. Finally, to compute the standard errors, an estimator $[\mathbf{Q}(\boldsymbol{\gamma})]^{-1}$ of the asymptotic covariance matrix is used :

$$[\mathbf{Q}(\boldsymbol{\gamma})] = T^{-1}\sum_{t=1}^{T}\mathbf{s}(\boldsymbol{\gamma})\mathbf{s}(\boldsymbol{\gamma})^{\top}$$

We apply this approach to daily returns for MSCI US (MSUS), MSCI Germany (MSGE) and MSCI Japan (MSJP) indices. The dataset starts from 1/1/1981 to 1/1/2001. Estimation by blocks, as described above, has been applied. Different block sizes have been tested to insure the consistency of estimation. For a detailed discussion about the impact on the estimation of the block size and the interpretation of the parameters of univariate asymptotic extreme distributions for financial series, we refer to LONGIN [1996]. The results are presented for a block size equals to 21 that corresponds to one month.

$-\chi^-$	MSGE	MSUS	MSJP
Location parameter $\hat{b}$	0.0263	0.0194	0.0255
	(0.0014)	*(0.0013)*	*(0.0018)*
Scale parameter $\hat{a}$	0.0094	0.0084	0.0109
	(0.0012)	*(0.0011)*	*(0.0015)*
Tail index $\hat{\tau}$	-0.2824	-0.3259	-0.3617
	(0.1023)	*(0.0981)*	*(0.1422)*

Table 1: MLE for the parameters of the univariate GEV for the minima

χ^+	MSGE	MSUS	MSJP
Location parameter $\hat{b}$	0.0281	0.0203	0.0315
	(0.0014)	*(0.0013)*	*(0.0018)*
Scale parameter $\hat{a}$	0.0103	0.0065	0.0118
	(0.0012)	*(0.0011)*	*(0.0015)*
Tail index $\hat{\tau}$	-0.0957	-0.2064	-0.2502
	(0.1023)	*(0.0981)*	*(0.1422)*

Table 2: MLE for the parameters of the univariate GEV for the maxima

From tables 1 and 2, it appears that extreme returns follow a Fréchet distribution (the tail indices are negative for all market indices). The degree of fatness is given by the absolute level of the tail index. As confirmed by the figures 1 and 2, MSGE has the lower degree of fatness for both minima and maxima, and MSJP has the greater degree of fatness for both minima and maxima. The second

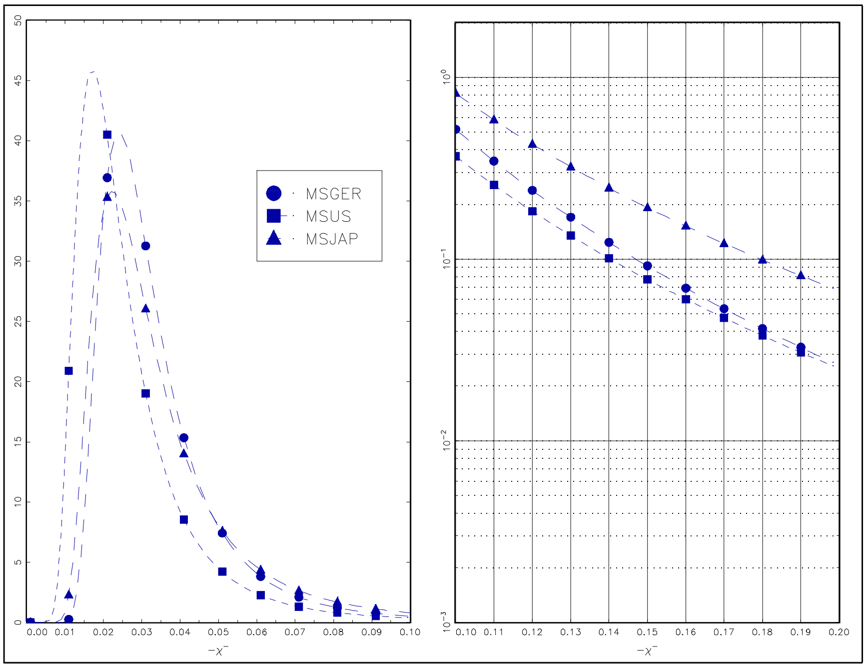


Figure 1: Estimated GEV distributions for minima

step of estimation consists of estimating the parameters for different dependence structures. The log-likelihood ℓ of the multivariate extreme distribution is:

$$
\begin{aligned}
\ell(\boldsymbol{\chi}_1,\ldots,\boldsymbol{\chi}_d;\hat{\gamma},\boldsymbol{\delta}) &= \ln \mathbf{g}\left(\boldsymbol{\chi}_d,\ldots,\boldsymbol{\chi}_d;\hat{\gamma},\boldsymbol{\delta}\right) \\
&= \sum_{t=1}^{T} \ln \mathbf{c}(G(\chi_1^{(t)};\hat{\gamma}_1),\ldots,G(\chi_d^{(t)};\hat{\gamma}_d);\boldsymbol{\delta}) \qquad (22)
\end{aligned}
$$

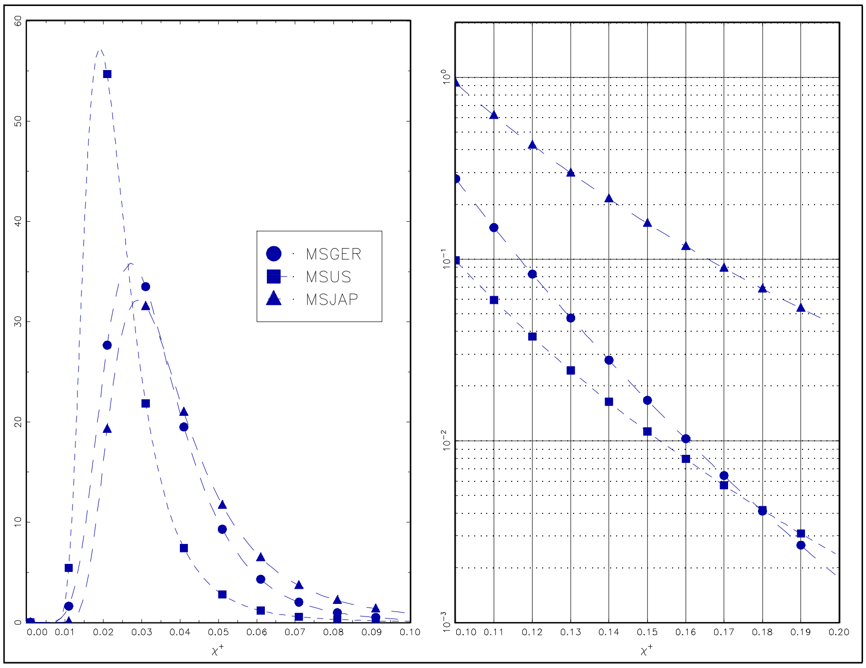


Figure 2: Estimated GEV distributions for maxima

where $\boldsymbol{\chi}_i = \left(\chi_i^{(1)}, \ldots, \chi_i^{(T)}\right)$ for $i = 1, \ldots, d$, $\mathbf{g}$ the asymptotic MEV density and $\mathbf{c}$ the associated copula density[3]. The ML estimator of the dependence parameters[4] is:

$$\hat{\boldsymbol{\delta}} = \arg\max_{\boldsymbol{\delta} \in \boldsymbol{\Delta}} \ln \mathbf{g}(\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_d; \hat{\boldsymbol{\gamma}}, \boldsymbol{\delta}) \tag{24}$$

with $\boldsymbol{\Delta}$ the set of dependence parameters. As seen above, extreme value copulae can only model positive dependence. Consequently, if one wants to model the minima and maxima simultaneously, one needs to split the estimation problem. For example, if we are interested in estimating the bivariate dependence parameters for the extrema of three variables, we will have to estimate twelve dependence

[3]Let $\mathbf{g}$ be the N-dimensional density function of $\mathbf{G}$ defined as follows:

$$\mathbf{g}(x_1, \ldots, x_N) = \frac{\partial\, \mathbf{G}(x_1, \ldots, x_N)}{\partial\, x_1 \cdots \partial\, x_N} \tag{23}$$

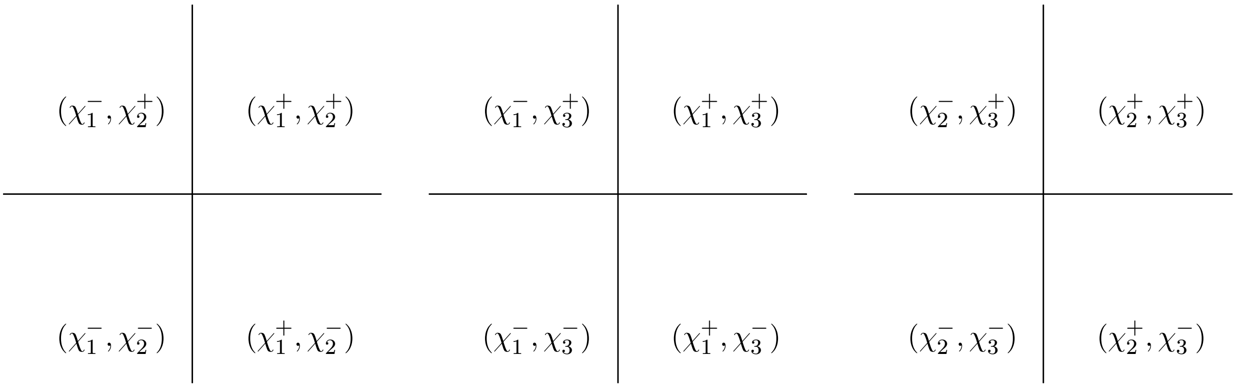
With the notation $u_n = G_n(x_n)$ for $n = 1, \ldots, N$, we have

$$\mathbf{c}(u_1, \ldots, u_N) = \frac{\partial\, \mathbf{C}(u_1, \ldots, u_N)}{\partial\, u_1 \cdots \partial\, u_N}$$

with $\mathbf{c}$ the copula density of $\mathbf{C}$.

[4]A criticism of this estimation methodology might arise from the fact that extrema may not occur simultaneously (same day) in one month. However, we believe that this method is asymptotically valid since the asymptotic MEV distribution is usually found by assuming componentwise extrema. An alternative estimation method called *threshold estimation method* could be used. It would lead us to use a multivariate generalised pareto distribution that is directly linked to a MEV distribution. We refer to LONGIN and SOLNIK [2001] for an application of this technique to financial series. Moreover, the goal of our paper is to focus on risk management implications rather than estimation methods.

structures – four for each pair –, as summarized in the following pictures:



By extending this methodology, it will be necessary to estimate eight different copulae for the trivariate case. The results for the three market indices are reported below. The subscripts 1, 2 and 3 are respectively used for MSGE, MSUS and MSGE.

	G			**HR**			**HJ**			
	$\hat{\delta}_1$	$\hat{\delta}_2$	*ldv*	$\hat{\delta}_{12}$	$\hat{\delta}_{13}$	$\hat{\delta}_{23}$	$\hat{\delta}_{12}$	$\hat{\delta}_{13}$	$\hat{\delta}_{23}$	$\hat{\theta}$
$(\chi_1^+, \chi_2^+, \chi_3^+)$	1.50	1.88	2	1.76	2.66	2.13	1.40	1.95	1.52	fixed
	(0.14)	*(0.23)*		*(0.31)*	*(0.93)*	*(0.39)*	*(0.29)*	*(0.52)*	*(0.31)*	fixed
$(\chi_1^+, \chi_2^+, -\chi_3^-)$	1.32	1.42	2	1.72	1.84	1.67	1.25	1.71	1.20	fixed
	(0.14)	*(0.19)*		*(0.30)*	*(0.41)*	*(0.29)*	*(0.25)*	*(0.39)*	*(0.24)*	fixed
$(\chi_1^+, -\chi_2^-, \chi_3^+)$	1.49	1.91	2	1.69	2.84	2.45	1.23	2.25	1.72	fixed
	(0.15)	*(0.24)*		*(0.33)*	*(1.15)*	*(0.61)*	*(0.24)*	*(0.86)*	*(0.40)*	fixed
$(\chi_1^+, -\chi_2^-, -\chi_3^-)$	1.52	2.03	1	1.62	1.66	2.88	1.35	1.39	2.45	fixed
	(0.15)	*(0.25)*		*(0.30)*	*(0.35)*	*(0.65)*	*(0.27)*	*(0.28)*	*(0.95)*	fixed
$(-\chi_1^-, -\chi_2^-, -\chi_3^-)$	1.51	1.77	2	1.69	2.53	1.34	1.45	2.06	1.29	fixed
	(0.15)	*(0.22)*		*(0.45)*	*(0.81)*	*(0.24)*	*(0.31)*	*(0.72)*	*(0.26)*	fixed
$(-\chi_1^-, -\chi_2^-, \chi_3^+)$	1.53	2.00	2	2.70	3.28	1.39	2.12	2.19	1.10	fixed
	(0.15)	*(0.24)*		*(1.29)*	*(0.93)*	*(0.22)*	*(0.80)*	*(0.84)*	*(0.18)*	fixed
$(-\chi_1^-, \chi_2^+, -\chi_3^-)$	1.26	1.81	2	1.60	2.79	1.51	1.22	2.31	1.12	fixed
	(0.14)	*(0.23)*		*(0.33)*	*(0.91)*	*(0.30)*	*(0.21)*	*(0.89)*	*(0.19)*	fixed
$(-\chi_1^-, \chi_2^+, \chi_3^+)$	1.45	2.05	2	1.53	3.40	2.13	1.28	2.59	1.64	fixed
	(0.14)	*(0.25)*		*(0.28)*	*(0.91)*	*(0.44)*	*(0.26)*	*(1.04)*	*(0.39)*	fixed

Table 3: MLE for the parameter of the trivariate copulae (**G**: Gumbel, **HR**: Hüsler-Reiss, **HJ**: Hu-Joe)

Let us comment the results of Table 3. The abbreviation *ldv* means "less dependent variable". This is motivated by the fact that only two parameters are estimated for the Gumbel copula. One parameter measures the dependence for one pair, the other one – corresponding to a lower dependence – is common for the two remaining pairs. In most cases, the dependence is higher between the extrema of MSGE and MSJP, except for the minima of MSUS and MSJP. The highest extremal dependences appear for: (i) a long position in the MSJP and a short position in the MSGE, and (ii) a short position in the MSUS and a short position in the MSJP. We note that these dependence measures are *conditional* to the dependence with the extremes (maxima or minima) of the remaining indice. Not surprisingly, the dependence hierarchy is the same for the three copulae. The Hüsler-Reiss copula is asymmetric, three parametric dependences are possible. Indeed, the choice of the Hüsler-Reiss MEV distribution depends on the two market indices that are firstly selected. The dependence with the higher likelihood has been selected and reported in the table. Some numerical difficulties erose in finding the maximum likelihood for the Hu-Joe copula. This led us to constrain the common dependence parameter θ to 1.

4 Application to risk management

The results above can be applied to risk management in two ways. First, it is possible to compute stress test values that would correspond to the evolution of the portfolio under extremal scenarii. Secondly, the parametric estimates of the MEV distributions can be used to simulate the joint extrema of the portfolio components.

4.1 Stress testing scenarii design

Draisma, de Haan and Peng [1997] define a failure area as the set of extrema with a given probability that one of them is exceeded. We will adopt a different definition by considering the set that corresponds to a simultaneous exceedence. Formally, this set $\mathcal{A}_p$ is:

$$\mathcal{A}_p^{s_1 \ldots s_n} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_n}, \Pr\left(\chi_1^{s_1} > x_1, \ldots, \chi_n^{s_n} > x_n\right) = p \right\} \tag{25}$$

with for $i = 1, \ldots, n$, $s_i = +$ for maxima or $-$ for minima. In the bivariate case, four sets need to be defined: $\mathcal{A}_p^{++}, \mathcal{A}_p^{+-}, \mathcal{A}_p^{-+}, \mathcal{A}_p^{--}$. In the trivariate case, eight sets are necessary: $\mathcal{A}_p^{+++}$, $\mathcal{A}_p^{++-}$, $\mathcal{A}_p^{-++}$, $\mathcal{A}_p^{-+-}$, $\mathcal{A}_p^{+-+}$, $\mathcal{A}_p^{+--}$, $\mathcal{A}_p^{--+}$, $\mathcal{A}_p^{---}$. More generally, for an n-dimensional problem, the number of sets equals 2^n. The probability involved for the characterisation of the failure area is nothing else but the survival distribution function that can be expressed with copulae (Joe [1997]) as:

$$\Pr\left(\chi_1^{s_1} > x_1, \ldots, \chi_n^{s_n} > x_n\right) = 1 + \sum_{M \in \mathcal{M}} (-1)^{|M|} \mathbf{C}_M \left(\mathbf{G}\left(x_j; \boldsymbol{\gamma}_j\right), j \in M; \boldsymbol{\delta}_M\right) \tag{26}$$

where $|M|$ denotes the cardinality of M, an element of $\mathcal{M}$ the set of marginal distributions of $\mathbf{C}$. For $n=3$, we have[5]:

$$\begin{aligned}\Pr\left(\chi_1^{s_1}>x_1,\chi_2^{s_2}>x_2,\chi_3^{s_3}>x_3\right) &= 1-\mathbf{C}_{12}\left(\mathbf{G}\left(x_1;\boldsymbol{\gamma}_1^{s_1}\right),\mathbf{G}\left(x_2;\boldsymbol{\gamma}_2^{s_2}\right);\boldsymbol{\delta}_{12}^{s_1s_2}\right)\\ &\quad-\mathbf{C}_{13}\left(\mathbf{G}\left(x_1;\boldsymbol{\gamma}_1^{s_1}\right),\mathbf{G}\left(x_3;\boldsymbol{\gamma}_3^{s_3}\right);\boldsymbol{\delta}_{13}^{s_1s_3}\right)\\ &\quad-\mathbf{C}_{23}\left(\mathbf{G}\left(x_2;\boldsymbol{\gamma}_2^{s_2}\right),\mathbf{G}\left(x_3;\boldsymbol{\gamma}_3^{s_3}\right);\boldsymbol{\delta}_{23}^{s_2s_3}\right)\\ &\quad+\mathbf{C}\left(\mathbf{G}\left(x_1;\boldsymbol{\gamma}_1^{s_1}\right),\mathbf{G}\left(x_2;\boldsymbol{\gamma}_2^{s_2}\right),\mathbf{G}\left(x_3;\boldsymbol{\gamma}_3^{s_3}\right);\boldsymbol{\delta}^{s_1s_2s_3}\right) \qquad (28)\end{aligned}$$

with $\mathbf{C}_{ij}$ the marginal copulae.

From this definition, a natural question arises: which probability level should be chosen ? An elegant answer – often used in the statistical literature and introduced by Gumbel [1958] for extreme value distributions – is to associate a waiting period t to the probability level p such that $t=\frac{1}{p}$. The univariate daily stress test scenarii for different waiting periods are reported in table 4.

	Minima			Maxima		
Waiting period	MSGE	MSUS	MSJP	MSGE	MSUS	MSJP
5 years	−6.1%	−5.2%	−7.0%	5.8%	4.2%	7.3%
10 years	−7.6%	−6.8%	−9.3%	6.7%	5.0%	9.0%
50 years	−12.4%	−11.9%	−17.0%	9.2%	7.5%	14.3%

Table 4: Univariate daily stress testing scenarii

To illustrate the concept of failure area with two variables, we provide an example for the maxima of two virtual indices with the same univariate stress testing scenarii (+9%) but with different degrees of dependence (Figure 3). The higher the dependence, the lower the distance between the failure area and univariate stress testing scenarii. We then build the trivariate failure areas under the hypothesis of a MEV distribution obtained from the Hűsler and Reiss copula. The associated probability level

[5] This formula comes directly as

$$\begin{aligned}\Pr\left(\chi_1^{s_1}>x_1,\chi_2^{s_2}>x_2,\chi_3^{s_3}>x_3\right) &= 1-\Pr\left(\chi_1^{s_1}\le x_1,\chi_2^{s_2}\le x_2\right)\\ &\quad-\Pr\left(\chi_1^{s_1}\le x_1,\chi_3^{s_3}\le x_3\right)\\ &\quad-\Pr\left(\chi_2^{s_2}\le x_2,\chi_3^{s_3}\le x_3\right)\\ &\quad+\Pr\left(\chi_1^{s_1}\le x_1,\chi_2^{s_2}\le x_2,\chi_3^{s_3}\le x_3\right) \qquad (27)\end{aligned}$$

corresponds to a 50 years waiting period. The failure areas have been obtained by numerically solving equation (25). The three dimensional space – each axis corresponds to one indice – is splitted into two parts: a short position in the MSUS (Figure 4) and a long position in the MSUS (Figure 5). For both figures, each point of the discretized surface is a three dimensional stress testing scenario that corresponds to values of the triplet (MSGE,MSUS,MSJP). The univariate stress testing scenarii (Table 4) are also represented. By definition, the trivariate failure areas are included in the parallelepiped arising from these univariate scenarii.

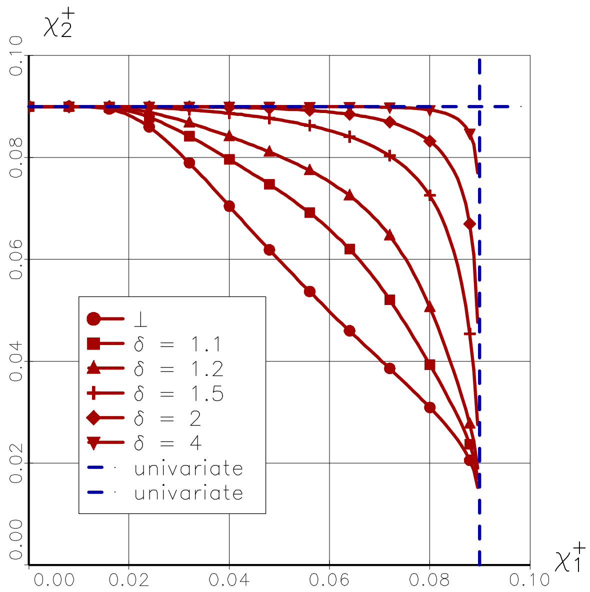


Figure 3: Bivariate failure area $\mathcal{A}_p^{++}$ with different parameter values for the Gumbel copula

4.2 Monte-Carlo based risk measures

Stress testing becomes intractable in higher dimensions because the number of points of the failure areas increases very quickly. Moreover, the failure areas have to be re-built if one wants portfolio values for different probability levels. Then, an alternative is to simulate the variables that follow a MEV distribution. To illustrate the Monte-Carlo applications, we will consider the following portfolios:

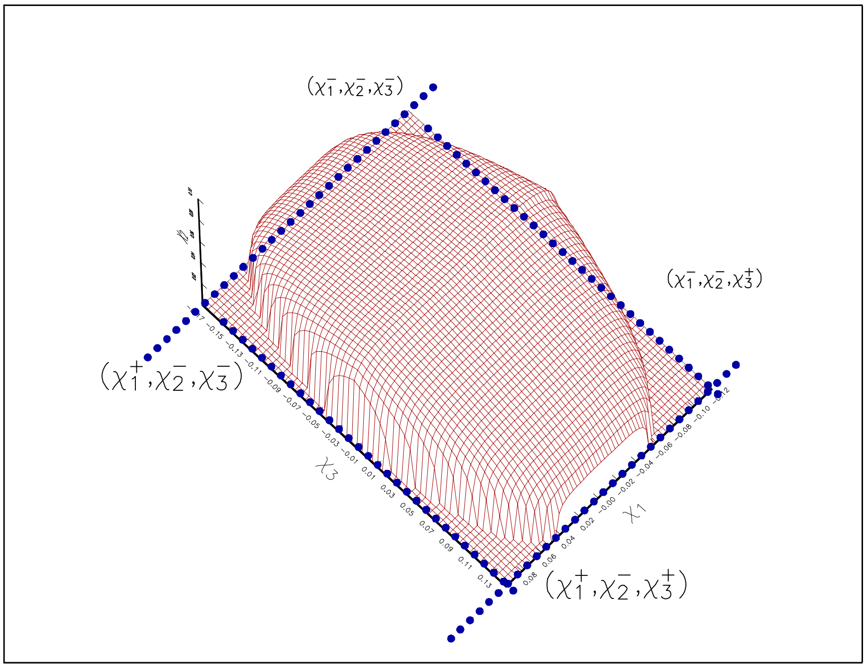


Figure 4: Trivariate failure areas $\mathcal{A}_p^{+-+}, \mathcal{A}_p^{+--}, \mathcal{A}_p^{--+}, \mathcal{A}_p^{---}$ for MSGE (χ_1), MSUS (χ_2) and MSJP (χ_3) with a 50 years waiting period (surface). Univariate stress test scenarios are also represented (dotted line)

Portfolio positions	MSGE	MSUS	MSJP
P_1	0	1	1
P_2	1	0	1
P_3	1	1	1
P_4	1	0	-1

The following algorithm, based on the conditional distributions, can be used to simulate extrema with a given n-variate copula $\mathbf{C}$:

1. Generate n independent uniform variates $(t_1, \ldots, t_n)$;

2. The n uniform variates are given recursively for $j = n, \ldots, 1$:

$$u_j = \mathbf{C}^{-1}\left(t_j \mid u_1, \ldots, u_{j-1}\right) \tag{29}$$

where

$$\begin{aligned} \mathbf{C}\left(u_j \mid u_1, \ldots, u_{j-1}\right) &= \Pr\left\{U_j \leq u_j \mid (U_1, \ldots, U_{j-1}) = (u_1, \ldots, u_{j-1})\right\} \\ &= \frac{\partial^{j-1}\mathbf{C}\left(u_1, \ldots, u_j, 1, \ldots, 1\right)/\partial u_1 \ldots \partial u_{j-1}}{\partial^{j-1}\mathbf{C}\left(u_1, \ldots, u_{j-1}, 1, \ldots, 1\right)/\partial u_1 \ldots \partial u_{j-1}} \end{aligned} \tag{30}$$

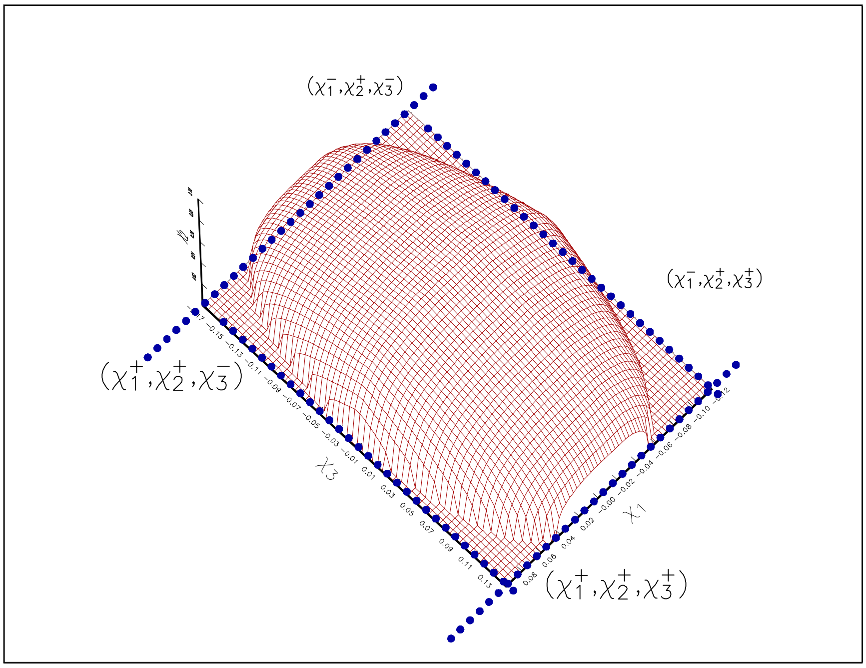


Figure 5: Trivariate failure areas $\mathcal{A}_p^{+++}, \mathcal{A}_p^{++-}, \mathcal{A}_p^{-++}, \mathcal{A}_p^{-+-}$ for MSGE (χ_1), MSUS (χ_2) and MSJP (χ_3) with a 50 years waiting period (surface). Univariate stress test scenarios are also represented (dotted line)

3. The extrema are obtained by inverting the estimated GEV distribution: $\chi_j = G_j^{-1}\left(u_j; \boldsymbol{\gamma}_j\right)$ for $j = 1, \ldots, n$.

Fortunately, this algorithm might be simplified for specific copula families. A detailed description for archimedean copulae can be found in Lindskog [2000]. From simulated data, two risk measures are computed: value at risk (VaR) and expected shortfall (ES). Following Artzner, Delbaen, Eber and Heath [1999], these measures are defined as follows:

Definition 1 *For a given probability space* $(\ ,\mathcal{F},\mathbb{P})$*, the value-at-risk* VaR_p *of the net worth* X *with distribution* $\mathbb{P}$ *is such that*

$$VaR_p(X) = -\inf\left(x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq p\right) \tag{31}$$

with $p \in (0,1)$.

Definition 2 *The expected shortfall* ES_p *is directly defined from* VaR_p *as follows:*

$$ES_p = -\mathbb{E}\left(X \mid X \leq -VaR_p(X)\right) \tag{32}$$

ES is coherent – see the definition of a coherent measure of risk in ARTZNER, DELBAEN, EBER and HEATH [1999] –, but VaR is generally not. Let us define the empirical estimation of these quantities. Let $\{X_i\}_{i=1\ldots n}$ be a vector of n realizations of the random variable X. The order statistics are such that $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Then the empirical estimators of these measures are:

$$\mathrm{VaR}_p^{(n)}(X) = X_{\lfloor np \rfloor :n}$$

$$\mathrm{ES}_p^{(n)}(X) = -\frac{\sum_{i=1}^{\lfloor np \rfloor} X_{i:n}}{\lfloor np \rfloor}$$

with $\lfloor np \rfloor = \max\{j \mid j \leq np,\, j \in \mathbb{N}\}$.

The estimated values of VaR and ES are reported in the Tables 5 (50000 simulations) and 6 (100000 simulations). Two probability levels are considered, respectively corresponding to 10 years and 50 years waiting periods.

Copula	**G**				**HR**				**HJ**			
	VaR		ES		VaR		ES		VaR		ES	
Portfolio	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y
P_1	-13.4%	-26.3%	-17.1%	-29.6%	-13.4%	-26.4%	-17.3%	-30.1%	-13.0%	-26.7%	-17.2%	-29.4%
P_2	-15.4%	-25.1%	-18.4%	-31.5%	-14.9%	-25.0%	-18.7%	-30.7%	-15.2%	-17.9%	-18.4%	-30.6%
P_3	-20.6%	-36.0%	-25.1%	-40.7%	-20.4%	-36.4%	-25.5%	-40.5%	-20.8%	-36.2%	-25.4%	-41.0%
P_4	-15.1%	-23.0%	-17.9%	-27.2%	-15.5%	-23.0%	-17.6%	-27.1%	-15.4%	-23.5%	-18.4%	-26.9%

Table 5: VaR and ES with 50000 Monte-Carlo simulations

Copula	**G**				**HR**				**HJ**			
	VaR		ES		VaR		ES		VaR		ES	
Portfolio	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y	10y	50y
P_1	-15.2%	-27.3%	-18.3%	-33.8%	-15.3%	-27.2%	-18.1%	-33.3%	-15.3%	-27.5%	-18.2%	-33.6%
P_2	-15.9%	-28.5%	-19.2%	-33.9%	-15.8%	-28.2%	-19.0%	-33.6%	-15.5%	-28.1%	-19.0%	-33.8%
P_3	-22.1%	-38.0%	-26.4%	-46.1%	-22.2%	-38.2%	-26.6%	-46.5%	-22.0%	-37.6%	-26.3%	-45.5%
P_4	-15.8%	-25.4%	-18.4%	-29.4%	-15.9%	-25.9%	-18.6%	-29.5%	-15.5%	-25.4%	-18.2%	-29.1%

Table 6: VaR and ES with 100000 Monte-Carlo simulations

From the tables above, it appears that the three models provide similar results both for VaR and ES. In other words, the clustering induced by the Gumbel copula does not seem to affect the risk measures dramatically. Moreover, the convergence looks quite acceptable for 50000 simulations. The values are consistent with the univariate stress testing scenarii. Indeed, VaRs are lower than the sum of the univariate values for all portfolios.

5 Conclusion

A methodology based on the MEV parametric distributions has been investigated in two directions: stress scenario design and Monte-Carlo based risk measures. It appears that the results are similar for the three copulae examined. Two extensions might be interesting. First, it would be useful to know which copula is the right one after the maximum likelihood estimation step by constructing specific tests to discriminate the more robust copula. Indeed, since the models are non-nested, their likelihoods can not be compared directly. Second, analytical formula might be developed by providing bounds for the risk measures. This has been done in a non-extremal context in Durrleman, Nikeghbali and Roncalli [2000] and Embrechts, Hoeing and Juri [2001].

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6 Appendix

SCORE VECTOR FOR GEV DISTRIBUTION

The score vector described in equation (21) is detailed:

$$
\begin{aligned}
\frac{\partial \log \mathbf{g}(\boldsymbol{\gamma}; x)}{\partial a} &= -\frac{a + (b-x)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} - 1\right)}{a(a+\tau(x-b))} \\
\frac{\partial \log \mathbf{g}(\boldsymbol{\gamma}; x)}{\partial b} &= \frac{\tau - 1 + \left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}}}{(a+\tau(x-b))} \\
\frac{\partial \log \mathbf{g}(\boldsymbol{\gamma}; x)}{\partial \tau} &= \frac{\tau(b-x)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} + \tau - 1\right) + (a+\tau(x-b))\log\left(\frac{a+\tau(x-b)}{a}\right)\left(\left(\frac{a+\tau(x-b)}{a}\right)^{\frac{1}{\tau}} - 1\right)}{\tau^2(a+\tau(x-b))}
\end{aligned}
$$

TRIVARIATE DENSITIES

The density for the trivariate Gumbel copula is developed. The copula function can be obtained by compound method:

$$
\begin{aligned}
\mathbf{C}(u_1, u_2, u_3; \delta_1, \delta_2) &= \mathbf{C}(u_3, \mathbf{C}(u_1, u_2; \delta_2); \delta_1) \\
&= \exp\left(-\left[\tilde{u}_3^{\delta_1} + \left(\tilde{u}_1^{\delta_2} + \tilde{u}_2^{\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right]^{\frac{1}{\delta_1}}\right)
\end{aligned} \tag{33}
$$

By computing the three iterative derivatives, it follows that:

$$
\begin{aligned}
\mathbf{c}(u_1, u_2, u_3; \delta_1, \delta_2) = {} & \mathbf{c}(u_1, u_2; \delta_2) \times \mathbf{c}(u_3, \mathbf{C}(u_1, u_2; \delta_2); \delta_1) \\
& + (\partial_1 \mathbf{C})(u_1, u_2; \delta_2) \times (\partial_2 \mathbf{C})(u_1, u_2; \delta_2) \times (\partial_{221} \mathbf{C})(u_3, \mathbf{C}(u_1, u_2; \delta_2); \delta_1)
\end{aligned} \tag{34}
$$

where

$$\begin{cases} \mathbf{c}\left(u_1,u_2;\delta\right) & = \mathbf{C}(u_1,u_2;\delta)(u_1u_2)^{-1}\frac{(\ln u_1 \ln u_2)^{\delta-1}}{\boldsymbol{\vartheta}(u_1,u_2;\delta)^{2-1/\delta}}\left[\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{1/\delta}+\delta-1\right] \\ \left(\partial_1\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = u_1^{-1}\mathbf{C}(u_1,u_2;\delta)\left(1+(\frac{\tilde{u}_2}{\tilde{u}_1})^{\delta}\right)^{1/\delta-1} \\ \left(\partial_2\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = \left(\partial_1\mathbf{C}\right)\left(u_2,u_1;\delta\right) \\ \left(\partial_{221}\mathbf{C}\right)\left(u_1,u_2;\delta\right) & = u_1^{-1}u_2{}^{-2}\mathbf{C}(u_1,u_2;\delta)\times\tilde{u}_1^{-1+\delta}\,\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{-3+\frac{1}{\delta}}\tilde{u}_2^{-2+\delta} \\ & \left\{\left(-1+\delta+\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right)\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)\ln(u_2)\right. \\ & -\left(\tilde{u}_1^{\delta}+\delta^2\,\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)-\tilde{u}_1^{\delta}\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right. \\ & +\delta\left[-2\,\tilde{u}_1^{\delta}+\tilde{u}_1^{\delta}\,\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\right. \\ & \left.+\,\tilde{u}_2^{\delta}-2\,\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\tilde{u}_2^{\delta}\right] \\ & \left.\left.+\;2\,\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{1}{\delta}}\tilde{u}_2^{\delta}-\boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)^{\frac{2}{\delta}}\tilde{u}_2^{\delta}\right)\right\} \\ \text{with} & \boldsymbol{\vartheta}\left(u_1,u_2;\delta\right)=\left(\tilde{u}_1^{\delta}+\tilde{u}_2^{\delta}\right) \end{cases}$$

We develop the Hüsler-Reiss copula density for the trivariate case.

$$\frac{\partial\mathbf{C}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_3}=\frac{1}{u_3}\mathbf{C}\left(\mathbf{u}_2;\boldsymbol{\delta}_2\right)\times\boldsymbol{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\exp\left\{-\int_0^{-\ln u_3}\boldsymbol{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,q\right);\rho\right)dq\right\} \tag{35}$$

Then,

$$\frac{\partial^2\mathbf{C}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_2\partial u_3}=\frac{1}{u_3}\left(\boldsymbol{\alpha}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)+\boldsymbol{\beta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)+\boldsymbol{\gamma}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)\right) \tag{36}$$

where

$$\begin{cases} \boldsymbol{\alpha}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) & = \partial_2\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\boldsymbol{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ \boldsymbol{\beta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) & = \mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\boldsymbol{\Phi}\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ \boldsymbol{\gamma}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) & = \mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\boldsymbol{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\varsigma_{21}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \end{cases}$$

with

$$\begin{cases} \partial_j\mathbf{C}\left(u_i,u_j;\delta\right) & = \frac{\partial\mathbf{C}(u_i,u_j,\delta)}{\partial u_j}=\mathbf{C}\left(u_i,u_j;\delta\right)\times u_j^{-1}\times\Phi\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\ln u_j}{\ln u_i}\right)\right)\ \text{for}\ \left(i,j\right)=\left(1,2\right),\left(2,1\right) \\ \frac{\partial\kappa_{i3}(u_i,u_3)}{\partial u_i} & = -\frac{\delta_{i3}}{2u_i\ln u_i}\ \text{for}\ i=1,2 \\ \boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) & = \exp\left\{-\int_0^{-\ln u_3}\boldsymbol{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,q\right);\rho\right)dq\right\} \\ \varsigma_{ij}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) & = \exp\left\{-\int_0^{-\ln u_3}\frac{\partial\kappa_{i3}(u_i,q)}{\partial u_i}\boldsymbol{\Phi}\left(\frac{\kappa_{j3}(u_j,q)-\rho\kappa_{i3}(u_i,q)}{\sqrt{1-\rho^2}}\right)dq\right\}. \end{cases}$$

Note that this comes partly from $\frac{\partial\boldsymbol{\Phi}_2}{\partial x}\left(x,y;\rho\right)=\boldsymbol{\Phi}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$. The copula density can be deduced:

$$\begin{aligned} \mathbf{c}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) &= \frac{\partial^3\mathbf{C}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_1\partial u_2\partial u_3} \\ &= \frac{1}{u_3}\left(\frac{\partial\boldsymbol{\alpha}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_1}+\frac{\partial\boldsymbol{\beta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_1}+\frac{\partial\gamma\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right)}{\partial u_1}\right) \end{aligned} \tag{37}$$

where

$$\begin{cases} \frac{\partial\alpha(\mathbf{u}_3;\boldsymbol{\delta}_3)}{\partial u_1} & = c\left(u_1,u_2;\delta_{12}\right)\times\mathbf{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & +\partial_2\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{13}(u_1,u_3)}{\partial u_1}\times\mathbf{\Phi}\left(\frac{\kappa_{23}(u_2,u_3)-\rho\kappa_{13}(u_1,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & +\partial_2\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\mathbf{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\boldsymbol{\varsigma}_{12}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ \frac{\partial\beta(\mathbf{u}_3;\boldsymbol{\delta}_3)}{\partial u_1} & = \partial_1\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\mathbf{\Phi}\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & +\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\frac{\partial\kappa_{13}(u1,u_3)}{\partial u_1}\times\frac{1}{\sqrt{1-\rho^2}}\times\boldsymbol{\phi}\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\theta}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & +\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{23}(u_2,u_3)}{\partial u_2}\times\mathbf{\Phi}\left(\frac{\kappa_{13}(u_1,u_3)-\rho\kappa_{23}(u_2,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\varsigma}_{12}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ \frac{\partial\gamma(\mathbf{u}_3;\boldsymbol{\delta}_3)}{\partial u_1} & = \partial_1\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\mathbf{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right)\times\boldsymbol{\varsigma}_{21}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & +\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\frac{\partial\kappa_{13}(u_1,u_3)}{\partial u_1}\times\mathbf{\Phi}\left(\frac{\kappa_{23}(u_2,u_3)-\rho\kappa_{13}(u_1,u_3)}{\sqrt{1-\rho^2}}\right)\times\boldsymbol{\varsigma}_{21}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \\ & -\mathbf{C}\left(u_1,u_2;\delta_{12}\right)\times\mathbf{\Phi}_2\left(\boldsymbol{\kappa}_2\left(\mathbf{u}_2,u_3\right);\rho\right) \\ & \times\int_0^{-\ln u_3}\left(\frac{\partial\kappa_{23}(u_2,q)}{\partial u_2}\frac{\partial\kappa_{13}(u_1,q)}{\partial u_1}\mathbf{\Phi}\left(\frac{\kappa_{13}(u_1,q)-\rho\kappa_{23}(u_2,q)}{\sqrt{1-\rho^2}}\right)\right)dq\times\boldsymbol{\varsigma}_{21}\left(\mathbf{u}_3;\boldsymbol{\delta}_3\right) \end{cases}$$

with the bivariate copula density such that:

$$\begin{aligned} c\left(u_1,u_2;\delta\right) &= \frac{\mathbf{C}\left(u_1,u_2;\delta\right)}{u_1u_2}\left[\mathbf{\Phi}\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right)\times\mathbf{\Phi}\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_2}{\tilde{u}_1}\right)\right)\right. \\ & \left.-\frac{\delta}{2\ln u_2}\boldsymbol{\phi}\left(\delta^{-1}+\frac{1}{2}\delta\ln\left(\frac{\tilde{u}_1}{\tilde{u}_2}\right)\right)\right] \end{aligned}$$

The likelihood can then be numerically computed.

For the trivariate Joe-Hu copula, the analytical expression is simpler. For indication, the bivariate copula density is

$$\mathbf{c}\left(\mathbf{u}_2;\boldsymbol{\delta}_2\right)=\mathbf{C}\left(\mathbf{u}_2;\boldsymbol{\delta}_2\right)\tilde{u}_2^{-1}u_1u_2^{-1}\xi\left(\mathbf{u}_2;\boldsymbol{\delta}_2\right)^{-1+\frac{1}{\theta}}\left(\left(\left(p_1\tilde{u}_1^{\theta}\right)^{\delta}+\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}\right)^{-1+\frac{1}{\delta}}\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}+p_2q_2\tilde{u}_2^{\theta}\right)$$

with $\xi\left(\mathbf{u}_2;\boldsymbol{\delta}_2\right)=\left(\left(\left(p_1\tilde{u}_1^{\theta}\right)^{\delta}+\left(p_2\tilde{u}_2^{\theta}\right)^{\delta}\right)^{\frac{1}{\delta}}+p_1q_1\tilde{u}_1^{\theta}+p_2q_2\tilde{u}_2^{\theta}\right)$, $p_i=(\nu_i+n-1)^{-1}$ and $q_i=(\nu_i+n-2)$.