**On Empirical Risk Measurement With Asymmetric Returns Data**

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Abstract. By formulating a nested test of the asymmetric response model of Bawa, Brown, and Klein (1981), the mean-lower partial moment CAPM (LPM-CAPM) of Bawa and Lindenberg (1977) and the mean-variance CAPM of Sharpe (1963, 1964), Lintner (1965) and Mossin (1969), this paper investigates the relative merits of symmetric and asymmetric risk measures using UK equity data for differently sized companies and at different frequencies. Our analysis shows that, when equity returns are not normal - which is the case for most daily and weekly returns, and for a large portion of smaller firms - the CAPM is rejected in 30%-50% of cases, and the optimal choice of alternative model is LPM-CAPM in over two thirds of these. These, and our further results, have strong consequences for the accurate measurement of equity risk, performance and prices, as downside and/or asymmetric risk measures often outperform the traditional CAPM framework, thus rendering it's related and widely-used current approaches sub-optimal for some company sizes/data frequency combinations.

JEL Classifications: C10, G12

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# 1. INTRODUCTION

Correctly measuring the risk of an asset, or a portfolio of assets, is of fundamental importance for asset pricing and performance measurement in Finance. The most popular equilibrium model which yields a measure of systemic market risk, or "beta", is the

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mean-variance CAPM of Sharpe (1963 and 1964), Lintner (1965) and Mossin (1969). Empirically, this imposes a parameter restriction on the joint distribution of market and portfolio returns, which can be tested using a linear data-generating function.

Given the continuing criticism of the variance as risk measure and the well-documented poor empirical performance of CAPM (see the excellent survey of Jagannathan and McGratten (1995) for an extensive bibliography on this), several asset-pricing models which serve as alternatives to CAPM have been proposed. Motivated by the belief that risk is ultimately related to shortfall rather than volatility, such models typically imply downside or asymmetric betas (see Bawa and Lindenberg (1977), Harlow and Rao (1989), Satchell (1996) or Pedersen (1999b)). These models demand a more general data-generating function for asset returns, do not depend on mean-variance rules, and imply the use of alternative risk and performance measures, which complement or generalise the traditional trio associated with CAPM (introduced in Treynor (1965), Sharpe (1966) and Jensen (1972)). Such non-traditional measures have already appeared in different guises in Finance literature and have the advantage of being able to capture asymmetries and fat tails in equity returns (see Henriksson (1984), Henriksson and Merton (1981), Kim and Zumwalt (1979), Fabozzi and Francis (1977 and 1979), Chen (1982), Bawa and Lindenberg (1977), Harlow and Rao (1989), Sortino and Price (1996) or Pedersen and Satchell (1999)).

This paper empirically investigates which underlying data generating model best captures the essential features of some types of returns, thus defining risk and performance measures based on the best empirical fit to the data, whilst remaining within the boundaries of reasonable theoretical justification. This is a topic which has already attracted some interest. For instance, Price, Price, and Nantell (1982) and Homaifar and Graddy (1990) examine non-nested tests, which focus on average value differences - or the number of positive differences - in the "betas" derived from two alternative models over a large class of assets. However, both these studies assume betas are uncorrelated across assets and that volatility is constant across both assets and markets, which is clearly violated for any set of real financial asset classes. Roll (1973) and Grauer (1981) also address this problem, but use a simple visual comparison of the implied security markets lines to infer their conclusions. Most recently, the problem of modelling asymmetric risks has been attacked from the angle of modelling conditional skewness (see Harvey and Siddique (1999) for references) and estimation of more general non-linear pricing kernels (an excellent summary of these is given in Dittmar (2000)).

Our analysis begins with the work of Harlow and Rao (1989) and Eftekhari and Satchell (1996), who present a nested test of differences between CAPM and the mean-lower partial moment CAPM (LPM-CAPM) of Bawa and Lindenberg (1977). The LPM-CAPM is the only model which both explicitly derives a systemic risk measure directly comparable with the CAPM beta, whilst assuming individuals have sound axiomatic downside risk preferences. By illustrating how this test depends on a maintained hypothesis in the form of a non-linear parameter restriction, we derive a test for LPM-CAPM against a more general asymmetric response model. Thus we effectively construct a nested test for the differences in three risk measures, which are capable of capturing increasingly "non-normal" features of asset returns; the traditional mean-variance CAPM beta, the LPM-CAPM downside risk beta, and a general asymmetric measure of risk introduced in Bawa, Brown and Klein (1981), which was extended to performance measurement in Pedersen and Satchell (1999).

We use ten years daily, weekly, and monthly returns for FTSE100, FTSE250, and FTSE SmallCap constituents in order to test the three models, since both company size and data frequency have been well-documented as affecting skewness and kurtosis in returns. We find that high frequency returns reject the mean-variance CAPM model more frequently than monthly returns. This is consistent with lower frequency data (which are temporal aggregates of higher frequency data) approaching normality via the Central Limit Theorem. We also find that the rejection rate of the mean-variance CAPM model increases for smaller companies. This reflects the fact that small companies often have more skewed and leptokurtotic returns due to lower trading volume, greater takeover and acquisition speculation, and bankruptcy risk.

Conditional on returns data not rejecting a Bera-Jarque test for normality at the 5% level, we confirm that the common perception that *"when equity returns are conditionally normal, use CAPM"* is more or less accurate (in 88% of cases, this held true). Alarmingly, though, for smaller companies and all companies with daily or weekly data - which were non-normal in all of our cases - about 30-40% of cases typically required an alternative model than CAPM. That is, whilst the CAPM is applicable for some small companies and high frequency returns, one should conduct a more detailed analysis in these cases in order to determine whether symmetric or asymmetric risk are the dominant explanatory factors. Given the widespread and arbitrary use of CAPM in the industry today, this could have severe consequences for risk and performance measurement, and asset pricing.

The paper is organised as follows: in the next section, we review the testing procedure introduced by Harlow and Rao (1989) and later used by Eftekhari and Satchell (1996), whilst Section 3 extends this and nests the three models we are testing. Section 4 considers potential models of excess market returns and derives the precise forms of likelihoods needed for our analysis (with details in the Appendix). The data used for our main empirical analysis is presented in Section 5 and results are discussed in Section 6. Section 7 is reserved for our conclusions.

## 2. THE HARLOW & RAO TEST FOR DOWNSIDE BETA

As alluded to in the Intnroduction, the two main theoretical models we shall test are the Capital Asset Pricing Model (CAPM) of Sharpe (1963 and 1964), Lintner (1965) and Mossin (1969), and the mean-lower partial moment CAPM (LPM-CAPM) of Bawa and Lindenberg (1977). We refer the reader to the original texts for details of the theoretical derivations of these models; for our purposes, we focus on the fact that the only difference between them is that, rather than variance, the investors in the latter model minimise the lower partial moment

$$\left[E(\min\left[0, R_p(t) - R_f(t)\right]^2)\right]^{\frac{1}{2}} \tag{1}$$

where $R_p(t)$ are portfolio returns at time $t$ and $R_f(t)$ the riskfree rate at time $t$, rather than. The use of downside risk measures such as (1) has been advocated by a large number of theorists and practitioners (see, for instance, Markowitz (1952), Fishburn (1980), Bawa (1975), Menezes, Geiss, and Tressler (1980), Sortino and Van der Meer (1991), Balzer (1994), Kijima and Ohnishi (1993), and Pedersen (1999a) where an extensive recent bibliography can be found). For the models under consideration, this change of risk measure has only one main implication for the resulting pricing equation, namely that the equilibrium measure of risk, the "beta", is different in the two models. However, the CAPM is always implied when returns are spherically symmetric or quadratic utility is assumed (see Bawa and Lindenberg (1997), Chamberlain (1983), Ingersoll (1987), Chow and Denning (1994) or Satchell (1996)). Hence, CAPM is nested in LPM-CAPM, which suggests a role for standard statistical techniques to test for differences in the models. The next section reviews existing applications of such tests and Section 3, by examining an implicit restriction these rely upon, extends the analysis to derive a new test of the LPM-CAPM against a more general asymmetric response model; this test then forms the basis of our empirical investigations in Section 5.

**2.1. The General Data Generating Function.** In order to describe the econometric relationship between the models, we appeal to the sample estimates of the two key equilibrium risk measures discussed above. In sample terms, CAPM beta is

$$\widehat{\beta}_{CAPM} = \frac{cov\left(R_p, R_m\right)}{var\left(R_m\right)} = \frac{\sum_{t=1}^{T}\left(R_p(t) - \overline{R}_p\right)\left(R_m(t) - \overline{R}_m\right)}{\sum_{t=1}^{T}\left(R_m(t) - \overline{R}_m\right)^2} \tag{2}$$

and LPM-CAPM beta is

$$\widehat{\beta}_{LPM} = \frac{CLPM_{R_f}\left(R_p, R_m\right)}{LPM_{R_f}\left(R_m\right)} = \frac{\sum_{t=1}^{T}\left(R_p(t) - \overline{R}_f(t)\right)\min\left[0, R_m(t) - R_f(t)\right]}{\sum_{t=1}^{T}\left(\min\left[0, R_m(t) - R_f(t)\right]\right)^2} \tag{3}$$

The underlying model which can capture both these is the asymmetric response model

$$R_p(t) - R_f(t) = \beta_{1p}R_m^-(t) + \beta_{2p}R_m^+(t) + \pi\delta(t) + \varepsilon_p(t) \tag{4}$$

where $R_m^-(t) = R_m(t) - R_f(t)$ when $R_m(t) < R_f(t)$ and zero otherwise, $R_m^+(t) = R_m(t) - R_f(t)$ when $R_m(t) > R_f(t)$ and zero otherwise, and $\delta(t)$ is an index function which is one when $R_m(t) > R_f(t)$ and zero otherwise[1]. This model was first introduced by Bawa, Brown, and Klein (1981), but has since been adapted by Harlow and Rao (1989) and Eftekhari and Satchell (1996). Note that the market portfolio is split, which allows us to capture asymmetric responses of portfolio returns to changes in market conditions. The disturbances, $\varepsilon_p(t)$, are serially uncorrelated, independent of all other variables, and have mean zero.

To place this firmly within popular finance literature, it is worth briefly establishing further links to previous works. When $\pi = 0$, (4) is the equation used by Kim and Zumwalt (1979) to test for differences in CAPM-beta in Bull and Bear markets. Their work built on the analysis of Fabozzi and Francis (1977 and 1979) and itself was extended to include time-varying betas in Chen (1982). An identical empirical framework also formed the basis for the tests of Henriksson (1984) and Henriksson and Merton (1981), who studied the performance of market timers in Bull and Bear market conditions.

Both Harlow and Rao (1989) and Eftekhari and Satchell (1996) advance by assuming that $\pi = \phi\left(\beta_{1p} - \beta_{2p}\right)$ in (4), where $\phi$ is the conditional expectation of $R_m(t)$ given that $R_m(t) > R_f(t)$, i.e.

$$\phi = E\left[R_m(t) - R_f(t)\left|R_m(t) > R_f(t)\right.\right] = \frac{E\left[R_m^+(t)\right]}{\Pr(R_m(t) > R_f(t))} \tag{5}$$

[1]For convenience, this assumes both mean-variance CAPM and LPM-CAPM are valid (i.e. that all alphas of the assets are zero). However, one could insert a constant in the regression - the following analysis would still be valid.

This restriction is employed since it allows on to derive the desired test for distinguishing between the equilibrium models. In it's present format, (4) without (5) does not allow such a derivation (see Harlow and Rao (1989) for details). In particular, it can be shown that, under the restriction imposed by (5), $\widehat{\beta}_{1p} = \widehat{\beta}_{LPM}$, whilst $\widehat{\beta}_{2p}$ measures the response of the portfolio to upside market returns. Also, when $\pi = 0$ and $\beta_{1p} = \beta_{2p}$ in (4), $\widehat{\beta}_{1p} = \widehat{\beta}_{CAPM}$. Thus, by testing the hypothesis

$$H_{CAPM} : \beta_{1p} = \beta_{2p} \text{ and } \pi = 0 \tag{6}$$

against the alternative

$$H_{LPM} : \beta_{1p} \neq \beta_{2p} \text{ or } \pi \neq 0 \tag{7}$$

**given that** $\pi = \phi(\beta_{1p} - \beta_{2p})$, one may establish a statistical difference between the two models. The next section further elaborates on this and derives a new framework in which the LPM-CAPM itself is tested against the asymmetric response model (4).

## 3. NESTING THREE ALTERNATIVE RISK MEASURES

We have thus highlighted how the critical assumption

$$\pi = \phi\left(\beta_{1p} - \beta_{2p}\right) \tag{8}$$

is needed to prove that the alternative model (7) in the original framework yields the equilibrium LPM-CAPM beta[2]. However, it is also clear that (8) is a test of whether LPM-CAPM is rejected against (4). Indeed, if (8) is rejected, both the LPM-CAPM and the CAPM itself are rejected, and tests for their difference thus made redundant. Thus, we note that by deriving our new nested testing procedure for LPM-CAPM, we also improve the efficiency of the original test for differences between CAPM and LPM-CAPM.

To derive these tests, we shall start with the general case - i.e. (4) without assuming (8) - and then derive a test for

$$H_1 : \pi = \phi\left(\beta_{1p} - \beta_{2p}\right) \tag{9}$$

against

$$H_{1A} : \pi \neq \phi\left(\beta_{1p} - \beta_{2p}\right) \tag{10}$$

[2]Note that when $\pi = 0$ and $\beta_{1p} = \beta_{2p}$, (8) is automatically satisfied for all $\phi$. Hence, (8) is redundant under the null hypothesis $H_{CAPM}$.

A rejection of $H_1$ implies that the data is not well-described by either LPM-CAPM or CAPM. This would speak in favour of the general asymmetric model (4) and its implications for risk and performance measurement, which were examined in Pedersen and Satchell (1999). If we do not reject $H_1$, we test

$$H_2 : \beta_{1p} = \beta_{2p} \tag{11}$$

against

$$H_{2A} : \beta_{1p} \neq \beta_{2p} \tag{12}$$

which then allows us to distinguish between CAPM and LPM-CAPM[3].

We now present the main general steps in our procedure for testing these hypotheses. The next section then discusses key distributions for the excess returns on the market. Firstly, by adopting a suggestion by Eftekhari and Satchell (1996), we note that (4) defines $R_p(t)$ as the conditional distribution of portfolio returns. If we assume that the error $\varepsilon_p(t)$ is distributed as a standard normal variable, the conditional likelihood of $R_p(t)$ given $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ is given by

$$pdf\left(R_p(t)|R_m^-(t),\ R_m^+(t),\delta(t)\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}\left(R_p(t)-\beta_{1p}R_m^-(t)-\beta_{2p}R_m^+(t)-\pi\ \delta(t)\right)^2\right] \tag{13}$$

Given the joint marginal distribution of $R_m^-(t)$, $R_m^+(t)$ and $\delta(t)$, one can thus calculate a full joint probability density function of $\{R_p(t), R_m^-(t),\ R_m^+(t),\delta(t)\}$;

$$pdf\left(R_p(t),R_m^-(t),\ R_m^+(t),\delta(t)\right) = pdf\left(R_p(t)|R_m^-(t),\ R_m^+(t),\delta(t)\right) pdf\left(R_m^-(t),\ R_m^+(t),\delta(t)\right) \tag{14}$$

where the first terms is given by (13) and the second determined by choosing an appropriate assumption for market returns (which we discuss in the next section). Given this joint distribution, full likelihood functions can then be derived under all hypothesis. Note that even if excess returns on the market were normal, this joint distribution (14) is not necessarily multivariate normal, since we explicitly condition on $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ in (13) and weigh them by the coefficients $\beta_{1p}$, $\beta_{2p}$ and $\pi$ respectively[4]. Hence, even in such conditions, the CAPM will not be implied within our framework unless certain parameter restrictions are satisfied..

[3]Note that $H_2$ implies $H_{CAPM}$ - see (6) - since if $\pi = \phi(\beta_{1p} - \beta_{2p})$ and $\beta_{1p} = \beta_{2p}$, then clearly $\pi = 0$. Also, $H_{2A}$ immediately implies $H_{LPM}$ - see (7). Hence, the stated hypothesis are sufficient.

[4]In fact, we can not in general obtain the functional form for the density function of $R_p(t)$. However, the moment-generating function can be obtained and consequently expressions for the central moments of $R_p(t)$ can be deduced (see Pedersen (1998) for details).

Thus, once marginal distributions are determined for the excess market returns and the full unrestricted likelihood function of $\{R_p(t), R_m^-(t),\ R_m^+(t), \delta(t)\}$ written down, Likelihood Ratio Tests can easily be constructed by explicitly imposing (in turn) the restrictions (9) and (11), and comparing the resulting likelihood estimates; this forms the overall approach we take to test for the appropriate model to use for different size companies and with different data frequencies. We next address the question of which assumptions should be chosen for the probability density function of excess market returns, $pdf\,(R_m^-(t),\ R_m^+(t), \delta(t))$, in the context of our chosen empirical analysis based on UK returns data, enabling us to complete the last bit of the theoretical derivations.

## 4. EXCESS RETURNS ON THE MARKET

The problem of specifying a marginal distribution, $pdf\,(R_m^-(t),\ R_m^+(t), \delta(t))$, has recently been addressed by Knight, Satchell, and Tran (1995), whose general approach is to split the contributions of upside and downside excess returns using the identity

$$X(t) = \mu + \delta(t)X_1(t) - (1 - \delta(t))\,X_2(t) \tag{15}$$

where both $X_1(t)$ and $X_2(t)$ are non-negative variables, $\delta(t)$ is a switching variable, and $\mu$ is a constant. To be consistent with (4), we set $\mu = 0$, $X(t) = R_m(t) - R_f(t)$, $X_1(t) = R_m^+(t)$, and $X_2(t) = -R_m^-(t)$. The negative sign in front of $R_m^-(t)$ follows from the fact that $R_m^-(t) \leq 0$, whilst we require $X_2(t) \geq 0$ in (15). Hence, (15) becomes

$$R_m(t) - R_f(t) = \delta(t)R_m^+(t) - (1 - \delta(t))\left[-R_m^-(t)\right] \tag{16}$$

so we sample from $R_m^+(t)$ when $\delta(t) = 0$ and from $-R_m^-(t)$ when $\delta(t) = 1$. The joint marginal density function of $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ is thus given by

$$\left[pdf(-R_m^-(t))\right]^{\delta(t)} \times \left[pdf(R_m^+(t))\right]^{1-\delta(t)} \tag{17}$$

Depending on the data, different assumptions can be made about $R_m^-(t), R_m^+(t)$ and $\delta(t)$ and the explicit unconditional likelihood of $\{R_p(t), R_m^-(t),\ R_m^+(t), \delta(t)\}$ thus derived using (14). We advocate the use of two such "split" distributions given the data to be analysed.

For our empirical analysis, we employ daily, weekly and monthly UK equity returns over the period from 1 August 1991 to 31 July 2001; the FTSE All-share index and a three-month UK Treasury Bill are used to calculate market returns and the risk-free rate. The summary statistics of the returns on the All-share are given in Table 1. The results indicate that the normal distribution may be suitable for monthly All-share returns, whilst

- not unexpectedly - the Bera-Jarque test comfortably rejects normality for the weekly and daily returns. Further investigation shows the significance of fat tails is the key source of the non-normality in the daily market returns whilst weekly returns have both significant skewness and excess kurtosis.

As a benchmark, the normal distribution has been fitted to excess returns on the FTSE All-share and the results are summarised in Table 2. Note that whilst all means $\mu$ are insignificant (t-ratios below 1.65), and thus the market risk premium is not statistically different from zero, the volatilities are all significant. However, because of the non-normality of the daily and weekly market returns shown in Table 1, the estimates in Table 2 may not adequately reflect the excess market returns in these cases.

Distributions such as those modelled by (16) allow alternative separate modelling of negative and positive excess returns, which is appropriate given the above observations, the Bera-Jarque tests and our objectives. In particular, we choose two different distributions; a mixture of truncated normal distributions which are continuous (CMTN) and a structure mixing Gamma distributions (MG). Although they have very different tails characteristics and behaviour at zero, we shall show that they are both well-suited for our purposes; excess market returns are better specified with the two split distributions than with the normal distribution in high frequency data such as daily and weekly returns. What is more, we shall see that all the results of our main tests are virtually identical using these different distributions, suggesting that our procedure is quite robust. We next describe them in greater detail and derive the resulting likelihoods for testing (9) and (11).

**4.1. Continuous Mixed Truncated Normal (CMTN) Distributions.** Our first candidate distribution is a continuous mixed truncated normal distribution. (CMTN), which nests the normal distribution but allows for asymmetry and excess kurtosis (fat tails). To derive this, initially suppose that $R_m^+(t)$ and $-R_m^-(t)$ are given by normal distributions truncated below, i.e.

$$pdf(R_m^+(t)) = \frac{1}{1-\Phi_{R_m^+(t)}(0)}\left[\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2}\right] \tag{18}$$

for $R_m^+(t) > 0$ and zero elsewhere, where $\mu_1$ and $\sigma_1$ are the mean and standard deviation of a normal distribution with c.d.f. $\Phi_{R_m^+(t)}$. Similarly

$$pdf(-R_m^-(t)) = \frac{1}{1-\Phi_{-R_m^-(t)}(0)}\left[\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-R_m^-(t)-\mu_2}{\sigma_2}\right)^2}\right] \tag{19}$$

for $-R_m^-(t) > 0$ and zero elsewhere, where $\mu_2$ and $\sigma_2$ are the mean and standard deviation of a normal distribution with c.d.f. $\Phi_{-R_m^-(t)}$; thus, $R_m^-(t)$ is modelled as a normal distribution truncated above.

Expressing $\Phi_{R_m^+(t)}$ and $\Phi_{-R_m^-(t)}$ in terms of $\Phi$, the c.d.f. of the standard normal distribution and using (17), the joint marginal likelihood of $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ can be given by

$$\left[\frac{p}{\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2}\right]^{\delta(t)} \times \left[\frac{(1-p)}{\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-R_m^-(t)-\mu_2}{\sigma_2}\right)^2}\right]^{1-\delta(t)} \tag{20}$$

where $\delta(t)$ is an independent Bernoulli switching variable, which is one when $R_m(t) > R_f(t)$ and zero otherwise. In its present form (20) does not describe a distribution which is continuous at zero; continuity is, however, a most desirable feature of distributions used in statistical analysis. In addition, most financial return data - including the FTSE All-share we use in our empirical section - has large clusters around zero which would support such an assumption. Consider therefore the following Lemma, whose proof is in the Appendix.

**Lemma 1.** *The restriction*

$$p = \frac{\frac{e^{-\frac{\mu_2^2}{2\sigma_2^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)}}{\frac{e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)}+\frac{e^{-\frac{\mu_2^2}{2\sigma_2^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)}} \tag{21}$$

*is sufficient for continuity of the density whose likelihood is given by (20).*

We label this combined distributional assumption - i.e. (20) where $p$ is given by (21) - as a continuous mixture of truncated normal distributions (CMTN). In Pedersen (1998), the assumption (21) was tested using monthly FTSE returns data, and did not reject continuity even at the 25% significance level[5].

Table 3 reports the estimates of the CMTN for daily, weekly and monthly excess market returns. Note that for both daily and weekly data, $\mu_1$ and $\mu_2$ are negative, suggesting that the estimated CMTN is significantly affected by kurtosis and thus the probability density function is not bell-shaped. By imposing $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$, we used a Likelihood Ratio Test (LRT) to test the assumption of normality in the data. The (LRT) statistic is

[5]We note that a further assumption of differentiability - which restricts the asymmetry allowed - was rejected at the 10% level (see Pedersen (1998) for details).

computed by

$$2(LL_U - LL_R) \sim \chi^2(k) \tag{22}$$

where $LL_U$ is log-likelihood of the CMTN distribution, $LL_R$ is the log-likelihood of the normal distribution, and $\chi^2(k)$ the chi-squared distribution with $k$ degrees of freedom, where $k$ is the number of restrictions. Using the log-likelihood values of the normal and CMTN distributions in Table 5, we calculated the LRT statistics for daily, weekly and monthly returns and find that the LRT statistics are all significant and thus the returns are better specified with the CMTN distribution.

As part of the choice of distribution, we shall also consider how well candidate distributions model $\phi$. The parameter is key for our analysis and also helps us compute how well the sample estimate of the market risk premium is replicated. For the normal distribution, we need the following Lemma, which is also proved in the Appendix.

**Lemma 2.** *If $z \sim N\left(\mu, \sigma^2\right)$, then*

$$\phi = E\left[z\,|z > 0\right] = \mu + \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\left[1 - \Phi\left(-\frac{\mu}{\sigma}\right)\right]} \tag{23}$$

*and*

$$E\left[z\,|z < 0\right] = \mu - \frac{\sigma e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)} \tag{24}$$

*where $\Phi\left(.\right)$ is the cumulative density of a standard normal variable.*

The estimates of $\phi$ and $E\left[R_m(t) - R_f(t)\,|R_m(t) < R_f(t)\right]$ can be obtained using Lemma 2, whilst $p$ can be assessed from (21). For the CMTN distribution, the estimates of daily $\phi$ and $E\left[R_m(t) - R_f(t)\,|R_m(t) < R_f(t)\right]$ are 0.63% and -0.63% respectively. Since $\sigma = 0.0085$ and $\mu = 0.0003$, the probability of negative excess returns is $\Phi\left(-\frac{\mu}{\sigma}\right) = \Phi\left(-0.035\right) = 0.48$, so the daily market risk premium is $E\left[R_m(t) - R_f(t)\right] = (0.52)(0.63\%) + (0.48)(-0.63\%) = 0.03\%$. The similar numbers for weekly and monthly returns indicate that, for weekly returns, $\phi = 1.50\%$, and the market risk premium is 0.09%, whilst, for monthly data, $\phi = 2.96\%$, and the market risk premium is 0.42%. Note that these confirm the estimate of $\mu$ recorded in Table 2.

As these results are positive, and we can explicitly model "upside" and "downside" returns, CMTN forms the first of our alternative distributions of excess market returns to be used in our tests of the main hypothesis (9) and (11). We hence derive the log-likelihood function of $\{R_p(t), R_m^-(t),\ R_m^+(t), \delta(t)\}$ using the probability density function of (14), when the conditional density of $R_p(t)$ given $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ is (13) and the marginal distribution as specified by (20). This, and the log-likelihood functions under the main hypothesis (9) and (11), are all given in the Appendix.

**4.2. Mixed Gamma Distributions.** Our second candidate distribution for excess market returns (i.e. potential model for $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$) is the Mixed Gamma (MG) proposed by Knight, Satchell, and Tran (1995). The distribution assumes that both $R_m^+(t)$ and $-R_m^-(t)$ are described by Gamma distributions

$$pdf(x) = \left\{ \begin{array}{cc} \frac{\lambda^{\alpha} x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{array} \right\} \tag{25}$$

in which $\Gamma$ denotes the Gamma function, $\alpha > 0$ and $\lambda > 0$, and $\delta(t)$ is an independent Bernoulli switching variable which is one with probability $p$ and zero with probability $1-p$. Under these assumptions, Knight, Satchell, Tran (1995) show that the joint likelihood of $R_m^+(t)$, $R_m^-(t)$ and $\delta(t)$ can be given as

$$\left[\frac{p\lambda_1^{\alpha_1}\left[R_m^+(t)\right]^{\alpha_1-1}\exp\left[-\lambda_1 R_m^+(t)\right]}{\Gamma(\alpha_1)}\right]^{\delta(t)} \times \left[\frac{(1-p)\,\lambda_2^{\alpha_2}\left[-R_m^-(t)\right]^{\alpha_2-1}\exp\left[-\lambda_2(-R_m^-(t))\right]}{\Gamma\left(\alpha_2\right)}\right]^{1-\delta(t)} \tag{26}$$

where the parameters $(\alpha_1, \lambda_1)$ are from the Gamma distribution for $R_m^+(t)$ and $(\alpha_2, \lambda_2)$ from the Gamma distribution modelling $-R_m^-(t)$. This has very different tail characteristics than the CMTN distributions in the previous section (see Knight, Satchell, and Tran (1995) for further details). In addition, whilst the truncated normal distribution is unrestricted, the MG distribution (25) must have density of zero at zero. Consequently, we shall allow very different features of excess market returns to be picked up depending upon the choice of distribution.

The result of fitting the MG distribution to the market excess returns is given in Table 4. All estimates are significant and for monthly returns the results are similar to those reported in Hwang and Satchell (2001) and Knight, Satchell, and Tran (1995); large values of $\lambda_1$ and $\lambda_2$, and $\alpha_1 > 1$ and $\alpha_2 > 1$. The density has maximum value (i.e. mode) at $(\alpha_i - 1)/\lambda_i$ when $\alpha_i > 1$; for example, for the monthly returns, the conditional densities for positive and negative excess returns have maximum value at $(\widehat{\alpha}_1 - 1)/\widehat{\lambda}_1 = (1.3368 - 1)/46.4211 = 0.0073$, and $-(\widehat{\alpha}_2 - 1)/\widehat{\lambda}_2 = -(1.1187 - 1)/34.1302 = -0.0035$, respectively. Likewise, $\phi$ is easily deduced from model parameters. Since the expectation of a Gamma distribution (25) is $\frac{\alpha}{\lambda}$, $\phi = \frac{\alpha_1}{\lambda_1}$ and $E\left[R_m(t) - R_f(t)\,|R_m(t) < R_f(t)\right] = -\frac{\alpha_2}{\lambda_2}$. The probability of returns being below the risk-free rate is explicitly measured via the parameter estimation of $p$. Hence, for daily data, we get $\phi = 1.19/191.05 = 0.62\%$ and $E\left[R_m(t) - R_f(t)\right] = 0.53(0.62\%)+0.47(-0.64\%) = 0.03\%$. The similar numbers for lower frequency data are, for weekly, $\phi = 1.38\%$ and a market risk premium of 0.09% whilst, for monthly data, we get $\phi = 2.88\%$ and a market risk premium of 0.42%. The comparison here with those of the CMTN and normal distributions confirms that despite the fact that the segmented fittings and the probabilities differ, the distribution-wide parameter (i.e. the market risk premium) is identically estimated for all three frequencies.

We saw earlier that using the log-likelihoods values, CMTN specifies the excess market returns better than the normal distribution. More general comparisons between all three distributions are possible with the Akaike Information Criterion (AIC)[6]

$$AIC = 2(LL - N) \tag{27}$$

where $LL$ is the log-likelihood of the estimation and $N$ the number of parameters to be estimated. This was introduced in Akaike (1973) and is well-discussed in Judge, Griffiths, Carter-Hill, and Lee (1985) and Maddala (1992); the higher the AIC, the better the fit. Table 5 reports the likelihood, model selection criteria and estimated values of $\phi$ for all fitted distributions. For the daily data, the AIC was 16905 for the normal distribution against 17075 for MG, indicating that the advantages we shall gain in using CMTN, which has five parameters, is not offset by loss in estimation efficiency. This is also clearly reflected for weekly data (2644 for normal, 2652 for CMTN, and 2654 for MG). However, for monthly data, most affected by Central Limiting effects, the normal distribution (436.8) and the CMTN distribution (438.2) both dominate the MG distribution (431.3), this being perfectly consistent with the Bera-Jarque statistics in Table 1.

Although the market risk premiums are the same for all three distributions, there are some differences in the way the asymmetry is modelled, as also evident from differences in the estimate of $\phi$. The additional benefit demonstrated through the rejection of normality in favour of CMTN and/or MG is clear through the values of AIC. As the MG distribution fits the data better than the normal distribution, but has very different properties around zero and in the tails than the CMTN, it forms an appropriate alternative distributional assumption for excess market returns, especially for high frequency data. Given the

[6] We note that there are alternative approaches to model selection than using the Akaike criterion. One could decide that the number of parameters is secondary and simply look at likelihood. Alternatively, if concerned with sample sizes, one could use the Schwartz Bayesian criterion $SB = LL - \frac{N}{2}\log n$ where $n$ is the sample size, introduced in Schwartz (1978). As the AIC is the most common model selection criterion used in the literature and we make no prior assumption about which criterion should be preferred, we stay with the AIC in this paper.

parameters are identically estimated but the MG have higher AIC, we shall thus also apply MG for the test of our main hypothesis (9) and (11).

When we assume that the joint marginal density function of $R_m^+(t), R_m^-(t)$, and $\delta(t)$ follows MG - as in (26) - and the conditional density function of $R_p(t)$, $pdf\ (R_p(t)|R_m^-(t),\ R_m^+(t), \delta(t))$, is normal as in (13), we can obtain the joint probability density function of $\{R_p(t), R_m^+(t), R_m^-(t), \delta(t)\}$ using the equation suggested in (14). This forms the second alternative joint distribution in our tests of the main hypothesis, and the derivation of the relevant likelihood functions for testing our main hypothesis (9) and (11), can be found in the Appendix.

This completes our theoretical derivations of the tests which will be used in our empirical analysis. To summarise, we have shown how the original tests of Harlow and Rao (1989) and Eftekhari and Satchell (1996) depend upon an untested restriction (8) which, if not satisfied, implies the alternative model in their tests, the LPM-CAPM, is misspecified. In addition, we have showed that by formulating a test for this maintained hypothesis, we nest both the CAPM and LPM-CAPM in the general unrestricted framework (4), and give the form of the specific hypothesis, (9) and (11), which need to be tested to distinguish between them. Consequently, the test allows the data to tell us whether to use the standard risk and performance measures of Sharpe (1966), Treynor (1965) and Jensen (1972), which derive from CAPM, the Sortino and Price (1994) performance criteria, which is justified by LPM-CAPM, or the more general asymmetric measures corresponding to the unrestricted version of (4), which were introduced and formally analysed in Pedersen and Satchell (1999). As the testing requires an explicit assumption to be made on the marginal distribution of excess market returns, we have considered the particular features of the daily, weekly and monthly returns on the FTSE All-share, and provided evidence for appropriate distributional assumptions. Finally, by combining marginal and conditional distributions, we have derived the log-likelihood functions for our main hypothesis tests, which we now apply to equity returns with different frequencies and for differently-sized companies.

## 5. APPLICATION TO SMALL AND LARGE UK COMPANIES DATA

We now examine equity data for small and large UK companies (and for different frequencies) and use the results to comment upon some stylized facts and whether, for some types of financial returns data, we find evidence favouring symmetric and asymmetric market risk measures.

Larger companies generally have a greater number of shareholders, larger volume of trade and higher frequency of trading. Monthly returns are additions of many trades and, by the Central Limit Theorem[7], approach normality. Small companies often have fewer shareholders and thinner markets with low trading frequency and volume, so that monthly returns do not exhibit the same degree of central limiting and, consequently, should be "less normal". This we dub the "frequency effect". Further, Cosh and Hughes (1995) argue that small companies fail more often than larger due to their youth, inexperience and often quite narrow product ranges, which would imply that investors need to account for bankruptcy risk. This, together with takeover, merger and acquisition speculation, encourages the presence of more outliers in returns, which skews the distribution and/or gives fat tails. (For a more detailed examination of the general structures of small companies, one should consult works of Hughes and Storey (1994) and Storey, Keasey, Watson, and Wynarczyk (1987).) This second effect, we label the "size effect".

Based on these observations, we have two hypotheses: (1) as company size decreases, we should move from CAPM to LPM-CAPM to asymmetric model as the preferred structure, and (2) as frequency increases, we should likewise be moving from CAPM through LPM-CAPM to (4). We shall examine these claims in detail. Further, we consider if - in some particular equity data groups - we can conclude that conventional models based on CAPM are suitable, and - where not - how serious the problem is. Presenting the LPM-CAPM as an alternative model, we then also see if this provides adequate coverage for the areas where issues arise or if one needs (4), with it's lack of theoretical foundation, to capture appropriate risk in returns. Finally, we shall comment upon the obvious consequences of our results for risk management based on CAPM-type measures.

**5.1. Data.** The UK stock market is covered by indices generated by FTSE International. In order to be admissible for these indices, securities must satisfy conditions of investibility, size and liquidity[8]. The largest eligible companies comprising 98%-99% of total market capitalisation constitute the FTSE All-share for the following year. The All-share is further split into several sub-indices; the FTSE100 contains the largest 100 companies; the FTSE250, the next 250 largest companies; the FTSE350 comprises both of these, and the FTSE SmallCap contains those companies too small to be included in the FTSE350. Finally, the FTSE Fledgling index (henceforth Fledgling) covers those companies too small to enter the All-share index.

Dimson and Marsh (1998) analysed the returns of all UK companies with market

[7]Strictly speaking, for central limiting arguments to apply, we implicitly assume that the distribution of single return moves has finite variance. This is an assumption we make unreservedly.

[8]Details of these and the criteria that individual companies must satisfy are given on the FTSE International website on the World Wide Web: www.ftse.com.

capitalisation under £188m; effectively complementary to the FTSE350, this covered the FTSE SmallCap, Fledgling and Alternative Investment Market companies. We refer the reader to their report for details and further references, but give two relevant findings here: firstly, Dimson and Marsh find evidence that the volume of trading in shares of small companies is far below that of FTSE100 companies; in fact 97% of the value of the All-share is invested in companies which have market capitalisation over £188m. In addition (see Dimson and Marsh (1998), page 62, Exhibit 51), the correlation between small companies and the All-share has steadily increased as smaller businesses have become more sensitive to fluctuations in the domestic market, which has resulted in beta with the FTSE All-share becoming larger and more significant.

Our sample period for the daily, weekly, and monthly FTSE100, FTSE250, and FTSE SmallCap constituents is from 1 August 1991 to 31 July 2001, which is the same as in Tables 1 to 5 for the analysis of the FTSE All-share index returns. Therefore, during the sample period, we have 2525 daily returns, 521 weekly returns, and 120 monthly returns for each stock[9]. Note that since the constituents of the indices have changed during the sample period, we used stocks included in the indices as of the 4th of September 2001. The number of stocks available at the beginning of our sample period is less than the number of stocks in the indices at the end of our sample period. Our results may be affected by this exclusion of stocks in the early period of our sample. The number of stocks available to us for the whole sample period are 77, 163 and 197 for FTSE100, FTSE250 and FTSE SmallCaps respectively[10]. We now examine this data in relation to our size and frequency effects.

**The "Size Effect".** As we are interested in the relative properties of large and highly liquid stocks to those of small and less trading stocks, we investigated the properties of the FTSE100, FTSE250 and FTSE SmallCap stocks separately in Tables 6a-6c in the Appendix. As reported by Dimson and Marsh, we confirm that average returns on top companies are larger than those on the middle and small companies, suggesting the stable engine of FTSE All-share growth over the last 10 years having been through Large-Cap stocks. For example, the monthly average return for the FTSE100 stocks has a mean

[9] The series is the Total Return Index from Datastream, which assumes dividends are re-invested. Total returns, $r_t$, are calculated as $r_t = \ln\left[\frac{P_{t+1}(1+DY_{t+1})}{P_t}\right]$, where $P_t$ = price at time $t$ and $DY_t$ = dividend yield at time $t$. When selecting suitable proxies for small and large companies, it is clear that no one definition is necessarily correct. In particular, over a given time horizon, companies may change size to become "large" or "small". Our chosen sample, however, contains those companies which have been among the top 100 companies for ten years, which seems as good a criterion as any.

[10] For the daily returns, the number of equities for the FTSE250 is 159.

of 1.20% and inter-quartile range of (0.7%, 1.5%), whilst those for the FTSE250 and FTSE SmallCaps have means of 1% and 0.8% and interquartile ranges of (0.6%, 1.3%) and (0.3%, 1.3%) respectively.

However the real top performers were typically smaller companies, along with the most spectacular failures, as can be seen from comparing the Maximum and Minimum rows and the interquartile ranges for the three different size related groups. As expected, the estimates of skewness and kurtosis show that the FTSE SmallCap stocks are more skewed and fat-tailed; for example, for the case of monthly returns, 77% of companies reject normality at the 5% level, versus 61%-62% amongst the larger companies, a difference which is more pronounced as the significance level of the test increases. In general, Bera-Jarque statistics for the FTSE SmallCap stocks' returns are much larger than those for the FTSE250 stocks' returns which are again larger than the FTSE100 stocks' returns. The main source of the trend seems to be kurtosis; that is, the smaller the firm, the more leptokurtotic their returns.

**The "Frequency Effect".** As mentioned previously, the Central Limit Theorem would dictate that lower frequency returns should be "more normal", since they are temporally aggregated returns of higher frequency returns (e.g. daily returns). This effect is found regardless of the type of data and appears more prevalent than the size effect. Indeed, by glancing at Tables 6a-6c, it is apparent that - regardless of size - the data becomes strongly non-normal at higher frequencies. For instance, the Bera-Jarque statistics for daily returns for even FTSE100 companies having an interquartile range of (1614, 9255), compared with (65, 432) and (2.7, 34.8) for weekly and monthly frequencies respectively.

Overall, Table 6 suggest that both size and frequency are important factors to decide if returns are normal, with frequency having the stronger effect. However, as we will conduct our analysis at all frequencies individually, we shall of course test statistically if the economic reasonings for smaller companies having more non-normal returns are powerful enough to merit use of a separate model. In summary, we expect our tests to confirm that CAPM is generally not appropriate for high frequency returns of all companies, whilst low frequency returns of large companies may be well explained with CAPM. For smaller companies with low frequencies, it is harder to form a precise hypothesis and the data will have to do the talking. Also, as for the question of which asymmetric model (LPM-CAPM or the general asymmetric model (4)) is best to apply where CAPM is unsuitable, a directional bias should be expected, which would favour the asymmetric model where data are more extremely non-normal, the LPM-CAPM filling the gap between these and those for which the CAPM is favoured.

## 6. RESULTS

We now present the results of the tests of the main hypotheses using the two distributions, CMTN and GM, as discussed in the previous section. The number of rejections of $H_1$ and $H_2$ at the 10% significance level are summarised in Table 7. As explained in the previous section, when $H_1$ is rejected, returns are better specified with the asymmetric model in (4). Hence, this tells us the theoretically justified models are incapable of capturing the risk in the returns. On the other hand, when $H_1$ is not rejected, we further test $H_2$. When $H_2$ is rejected, we choose the LPM-CAPM and finally, if $H_2$ is not rejected, we favour CAPM.

Firstly, we note that the results in Tables 7a-7c are almost identical for the two margional distributional assumptions (GM and CMTN). This suggests that our test is robust for different distributions. From now on, we hence explain our findings referring to the numbers in Table 7 pertaining to the MG distribution. We first look at the case of daily returns. For the FTSE SmallCap stocks, about 50% of cases still support the CAPM whilst about 27% and 23% support the asymmetric and LPM-CAPM models, respectively. Whilst this result may be surprising given that the mean-variance CAPM model is widely used without thorough specification tests, it should be noted that joint normality is not *necessary* but merely *sufficient* for CAPM. However, it is noteworthy that this result informs us that some 50% of small companies would be wrongly valued/priced etc., if daily data was used in a CAPM. For the larger companies such as the constituents of FTSE100, we find that 71% support the CAPM whilst the asymmetric model is chosen less than 10% of cases. Thus, size plays a role in determining the applicability of CAPM for daily data. In total, 40% of firms are better modelled using alternatives to CAPM, of which LPM-CAPM is preferred in over two thirds of the cases.

For weekly returns, Table 7b shows a similar pattenr except the acceptance rate of CAPM is increased by about 10-15% for all three size groups. When the frequency of returns is monthly, Table 7c shows that the percentage not rejecting the CAPM is about 80% across different company sizes, whilst about 13% of stocks are best specified with LPM-CAPM models and the asymmetric model again only chosen for 6% of stocks. Thus, although not a large percentage, the CAPM is still misused in about 20% of cases, even when we are talking about FTSE100 companies and monthly data, which is the most typical frequency used by analysts and corporate financiers today. We also note that the "size effect" seems most prevalent with daily returns. Figures 1a-1c summarises these findings.

Finally, we note that Table 7c also reveals that when (monthly) returns are normal, CAPM is appropriate in about 88% of cases[11]. Hence, the common rule of thumb that *"whenever you are dealing with normal equity returns, the CAPM is the right model"*, appears to be supported by our evidence although some cases are still modelled better with asymmetric risk regardless of the normality of returns. More importantly, however, it is clear that whenever we are dealing with non-normal returns - which includes *any* analysis with weekly and daily frequency data and a significant chunk of smaller company and monthly data, a customised testing procedure - a model of which is introduced in this paper - is necessary to adequately determine which model should be applied when performing risk analysis, pricing or asset valuations.

## 7. CONCLUSION

Determining an "appropriate" data-generating function which captures essential features of portfolio returns is fundamental for empirical risk measurement, valuations and performance measurement. Whilst studies have sporadically attempted to illustrate the levels of differences between the CAPM and classes of "asymmetric" risk models, the fundamental belief still appears to be that using the CAPM gives an accurate enough picture of the necessary features of risk and return for most equity data. In this paper, we have analysed more precisely which data and company types may more readily be thrown into a CAPM and which require more careful analysis, focusing on company size and data frequency in particular.

Using a recent paper by Knight, Satchell, Tran (1995), which addressed the problem of splitting distributions, we have presented an extension to the empirical testing procedure of Harlow and Rao (1989) and Eftekhari and Satchell (1996). This extension, which relies upon accurately modelling excess market returns using asymmetric distributions, enables us to nest the lower partial moment CAPM of Bawa and Lindenberg (1977) - and so also the traditional CAPM - in a more general asymmetric response model. Consequently, we are able to distinguish between three different models using standard econometric techniques.

We have analysed daily, weekly, and monthly returns of three different size UK company groups. Our results confirm that daily and weekly returns require most attention, in that non-traditional frameworks need to be applied in a large number of cases. On the other hand, CAPM works well for monthly returns of large companies, though some

[11] Note that conditional normality of portfolio returns is not sufficient for the CAPM to be valid.

doubt reigns over the use of monthly data on smaller firms. Finally, we confirm that when dealing with normal equity returns, the CAPM is indeed the preferred model, being chosen in 88% of cases. Most strikingly of our findings, however, is that when dealing with non-normal returns, which includes all analysis with daily and weekly data, separate testing is needed to determine the "correct" model. For these cases, the CAPM is still "favourite", typically chosen in 55-80% of cases. Moreover, the LPM-CAPM chosen in a further 15%-30% of cases, indicating that it is a solid alternative to CAPM in most cases where CAPM is rejected. This does provide some comfort for proponents of equilibrium-based "beta"-type risk measures, whilst promoting the LPM-CAPM. Indeed, only in the most extreme cases, some 6-7% on average, did both CAPM and LPM-CAPM get rejected in favour of the general asymmetric model (4). This indicates that - since LPM-CAPM nests CAPM - one could apply LPM-CAPM to all data, knowing that CAPM is automatically implied when appropriate, and capture the risk characteristics in almost all equity returns studied.

We hope the findings of this paper will fuel the ongoing debate between proponents and opponents of symmetric, downside and asymmetric risk measures in the academic literature. Further, since it is clear that, in some cases, CAPM per se is not in practice the best model, we hope that practitioners also will pay more attention to the fact that accurate financial risk analysis, equity valuation and asset pricing in general, may contain inaccurate measurements which can be improved. Indeed, there has been a more general acceptance of general risk and performance measures such as those introduced in Sortino and Price (1994) and Pedersen and Satchell (1999), which are generalisations of the conventional measures in Sharpe (1966), Treynor (1965) and Jensen (1972). Our findings in this article will hopefully also help these constructs gain further popularity amongst analysts of markets with highly skewed and kurtotic returns.

## 8. APPENDIX

**PROOF OF LEMMA 1**

By the continuity of individual truncated normal distributions, discontinuity can occur only at zero. To eliminate this, we need

$$\left.\frac{pe^{-\frac{1}{2}\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2}}{(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right])\sigma_1\sqrt{2\pi}}\right|_{R_m^+(t)=0} = \left.\frac{(1-p)\,e^{-\frac{1}{2}\left(\frac{R_m^-(t)-\mu_2}{\sigma_1}\right)^2}}{(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right])\sigma_2\sqrt{2\pi}}\right|_{R_m^-(t)=0}$$

or indeed

$$\frac{pe^{-\frac{1}{2}\left(\frac{\mu_1}{\sigma_1}\right)^2}}{(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right])\sigma_1\sqrt{2\pi}} = \frac{(1-p)\,e^{-\frac{1}{2}\left(\frac{\mu_2}{\sigma_1}\right)^2}}{(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right])\sigma_2\sqrt{2\pi}} \tag{28}$$

which can be rearranged to give (21) ■

**PROOF OF LEMMA 2**

If $x = [z\,|z > 0]$ has a p.d.f. given by (18), then

$$E\,[x] = \frac{1}{1-\Phi_x(0)}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty xe^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx\right]$$

Letting $y = \frac{x-\mu}{\sigma}$, so that $x = \sigma y + \mu$ and $dx = \sigma dy$, noting that $\Phi_x(0) = \Phi\left(-\frac{\mu}{\sigma}\right)$, where $\Phi(.)$ is the cumulative density of a standard normal variable, we get

$$\begin{aligned}
E\,[x] &= \frac{1}{1-\Phi(-\frac{\mu}{\sigma})}\left[\frac{1}{\sqrt{2\pi}}\int_{-\frac{\mu}{\sigma}}^\infty (\sigma y+\mu)\,e^{-\frac{1}{2}y^2}dy\right] \\
&= \frac{1}{1-\Phi(-\frac{\mu}{\sigma})}\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\frac{\mu}{\sigma}}^\infty ye^{-\frac{1}{2}y^2}dy + \frac{\mu}{\sqrt{2\pi}}\int_{-\frac{\mu}{\sigma}}^\infty e^{-\frac{1}{2}y^2}dy\right] \\
&= \mu\left(\frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right]}\int_{-\frac{\mu}{\sigma}}^\infty e^{-\frac{1}{2}y^2}dy\right) + \frac{\sigma}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right]}\left[-e^{\frac{1}{2}y^2}dy\right]_{-\frac{\mu}{\sigma}}^\infty \\
&= \mu + \frac{\sigma e - \frac{\mu^2}{2\sigma^2}}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right]}
\end{aligned}$$

since $\frac{1}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right]}\int_{-\frac{\mu}{\sigma}}^\infty e^{-\frac{1}{2}y^2}dy = 1$. Similarly, if $x = [z\,|z \le 0]$ has a p.d.f. given by (19), then

$$E\,[x] = \frac{1}{\Phi_x(0)}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^0 xe^{-\frac{1}{2}y^2\left(\frac{x-\mu}{\sigma}\right)^2}dx\right]$$

Again letting $y = \frac{x-\mu}{\sigma}$, so that $x = \sigma y + \mu$ and $dx = \sigma dy$, noting that $\Phi_x(0) = \Phi\left(-\frac{\mu}{\sigma}\right)$, where $\Phi(.)$ is the cumulative density of a standard normal variable, we get

$$\begin{aligned}
E[x] &= \frac{1}{\Phi_x(-\frac{\mu}{\sigma})}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu}{\sigma}}(\sigma y+\mu)\,e^{-\frac{1}{2}y^2}dy\right] \\
&= \frac{1}{\Phi(-\frac{\mu}{\sigma})}\left[\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu}{\sigma}} ye^{-\frac{1}{2}y^2}dy+\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu}{\sigma}} e^{-\frac{1}{2}y^2}dy\right] \\
&= \mu\left(\frac{1}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)}\int_{-\infty}^{-\frac{\mu}{\sigma}} e^{-\frac{1}{2}y^2}dy\right)+\frac{\sigma}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{\mu}{\sigma}\right)\right]}\left[-e^{\frac{1}{2}y^2}dy\right]_{-\infty}^{-\frac{\mu}{\sigma}} \\
&= \mu-\frac{\sigma e-\frac{\mu^2}{2\sigma^2}}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)}
\end{aligned}$$

since $\frac{1}{\sqrt{2\pi}\Phi\left(-\frac{\mu}{\sigma}\right)}\int_{-\infty}^{-\frac{\mu}{\sigma}} e^{-\frac{1}{2}y^2}dy=1$ ■

**THE KEY LOG-LIKELIHOOD FUNCTIONS UNDER CMTN**

With the conditional distribution modelled by (13), the "unrestricted" model (4) under the *CMTN* assumption has log-likelihood given by

$$\begin{aligned}
&-T\ln(2\pi)-T\ln\sigma-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(R_p(t)-\beta_{1p}R_m^-(t)-\beta_{2p}R_m^+(t)-\pi\delta(t)\right)^2-T_1\ln\sigma_1 \\
&+T_1\ln\left[\frac{\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right]-T_1\ln\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right) \\
&-\frac{1}{2}\sum_{t=1}^{T}\delta(t)\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2+T_2\ln\left[\frac{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right] \\
&\qquad -T_2\ln\sigma_2-T_2\ln\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)-\frac{1}{2}\sum_{t=1}^{T}[1-\delta(t)]\left(\frac{-R_m^-(t)-\mu_2}{\sigma_2}\right)^2 \qquad (29)
\end{aligned}$$

where $T_1=\sum_{t=1}^{T}\delta(t)$ and $T_2=\sum_{t=1}^{T}[1-\delta(t)]$. In order to get $H_1$ - see (9) - in terms of the parameters of this model, we recall that, by Lemma 2,

$$E\left[R_m^+(t)|R_m^+>0\right]=\mu_1+\frac{\sigma_1 e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sqrt{2\pi}\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)} \qquad (30)$$

where $\Phi$ is the c.d.f. of the standard normal distribution. Consequently, under *CMTN*, $H_1$ becomes

$$\pi=\left(\mu_1+\frac{\sigma_1 e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sqrt{2\pi}\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)}\right)\left(\beta_{1p}-\beta_{2p}\right) \qquad (31)$$

The "restricted" log-likelihood under $H_1$ is derived by substituting (31) in (29) and thus given by

$$-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(R_p(t)-\beta_{1p}R_m^-(t)-\beta_{2p}R_m^+(t)-\delta(t)\left(\mu_1+\frac{\sigma_1 e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sqrt{2\pi}\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)}\right)(\beta_{1p}-\beta_{2p})\right)^2$$
$$-T\ln(2\pi)-T\ln\sigma-T_1\ln\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)-\frac{1}{2}\sum_{t=1}^{T}\delta(t)\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2-T_1\ln\sigma_1$$
$$+T_1\ln\left[\frac{\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right]-T_2\ln\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)-T_2\ln\sigma_2$$
$$-\frac{1}{2}\sum_{t=1}^{T}[1-\delta(t)]\left(\frac{-R_m^-(t)-\mu_2}{\sigma_2}\right)^2+T_2\ln\left[\frac{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right] \tag{32}$$

We use the Likelihood Ratio Test in (22) to test $H_1$. The model in (29) is unrestricted one whilst (32) is the restricted model. When testing $H_2$, (32) is the "unrestricted" model, and the "restricted" log-likelihood is found by further imposing $\beta_{1p}=\beta_{2p}$, giving

$$-T\ln(2\pi)-T\ln\sigma-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(R_p(t)-\beta_{1p}\left[R_m(t)-R_p(t)\right]\right)^2-T_1\ln\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)$$
$$-\frac{1}{2}\sum_{t=1}^{T}\delta(t)\left(\frac{R_m^+(t)-\mu_1}{\sigma_1}\right)^2+T_1\ln\left[\frac{\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right]$$
$$-T_1\ln\sigma_1-T_2\ln\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)-\frac{1}{2}\sum_{t=1}^{T}[1-\delta(t)]\left(\frac{-R_m^-(t)-\mu_2}{\sigma_2}\right)^2-T_2\ln\sigma_2$$
$$+T_2\ln\left[\frac{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}}{\sigma_2\left(1-\Phi\left[-\frac{\mu_2}{\sigma_2}\right]\right)e^{-\frac{\mu_1^2}{2\sigma_1^2}}+\sigma_1\left(1-\Phi\left[-\frac{\mu_1}{\sigma_1}\right]\right)e^{-\frac{\mu_2^2}{2\sigma_2^2}}}\right] \tag{33}$$

## THE KEY LOG-LIKELIHOOD FUNCTIONS UNDER MG

The joint likelihood function for $\{R_p(t), R_m^+(t), R_m^-(t), \delta(t)\}$ at time $t$ is the product of the marginal likelihood (26) and conditional likelihood (13). By taking logarithms of this product, summing over a sample of size $T$, we have a log-likelihood of

$$
\begin{aligned}
-T\ln\sqrt{2\pi} - T\ln\sigma - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(R_p(t) - \beta_{1p}R_m^-(t) - \beta_{2p}R_m^+(t) - \pi\delta(t)\right)^2 \\
+T_1\ln p + T_1\alpha_1\ln\lambda_1 + (\alpha_1 - 1)\sum_{t=1}^{T}\delta(t)\ln R_m^+(t) \\
-\lambda_1\sum_{t=1}^{T}\delta(t)R_m^+(t) - T_1\ln(\Gamma(\alpha_1)) + T_2\ln(1-p) + T_2\alpha_2\ln\lambda_2 \\
+(\alpha_2 - 1)\sum_{t=1}^{T}[1-\delta(t)]\ln\left[-R_m^-(t)\right] - \lambda_2\sum_{t=1}^{T}[1-\delta(t)]\left[-R_m^-(t)\right] - T_2\ln(\Gamma(\alpha_2))
\end{aligned} \tag{34}
$$

for the unrestricted model corresponding to the general asymmetric model - i.e. (4) when $H_1$ is rejected - under the assumption of MG[12]. Since $E\left[R_m^+(t)\,|R_m^+(t)>0\right] = \frac{\alpha_1}{\lambda_1}$, $H_1$, the test of LPM-CAPM versus the general asymmetric model (4) - defined in (9) - becomes

$$H_1 : \pi = \frac{\alpha_1}{\lambda_1}\left(\beta_{1p} - \beta_{2p}\right) \tag{36}$$

which is tested against

$$H_{1A} : \pi \neq \frac{\alpha_1}{\lambda_1}\left(\beta_{1p} - \beta_{2p}\right) \tag{37}$$

The restricted likelihood under $H_1$ is calculated by substituting (36) into (34), which gives

$$
\begin{aligned}
-T\ln\sqrt{2\pi} - T\ln\sigma - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(R_p(t) - \beta_{1p}R_m^-(t) - \beta_{2p}R_m^+(t) - \frac{\alpha_1}{\lambda_1}\left(\beta_{1p} - \beta_{2p}\right)\delta(t)\right)^2 \\
+T_1\ln p + T_1\alpha_1\ln\lambda_1 + (\alpha_1 - 1)\sum_{t=1}^{T}\delta(t)\ln R_m^+(t) \\
-\lambda_1\sum_{t=1}^{T}\delta(t)R_m^+(t) - T_1\ln(\Gamma(\alpha_1)) + T_2\ln(1-p) + T_2\alpha_2\ln\lambda_2 \\
+(\alpha_2 - 1)\sum_{t=1}^{T}[1-\delta(t)]\ln\left[-R_m^-(t)\right] - \lambda_2\sum_{t=1}^{T}[1-\delta(t)]\left[-R_m^-(t)\right] - T_2\ln(\Gamma(\alpha_2))
\end{aligned} \tag{38}
$$

If we reject (36), the data do not support an equilibrium model (i.e. LPM-CAPM or CAPM), but favour the general asymmetric model (4). If (36) is not rejected, we test

[12]It is difficult to say much about this distribution analytically; indeed, the density function corresponding to this likelihood is not readily available. However, in Pedersen (1998), Corollary 1 establishes that its moment-generating function is

$$\Psi_{R_p(t)}(s) = e^{\frac{1}{2}s^2\sigma^2}\left[p\left(\frac{\lambda_1}{\lambda_1 - s\beta_2}\right)^{\alpha_1} + (1-p)e^{s\pi}\left(\frac{\lambda_2}{\lambda_2 + s\beta_1}\right)^{\alpha_2}\right] \tag{35}$$

where $\lambda_1 > s\beta_2$ and $\lambda_2 > -s\beta_1$. From this, its moments can be recovered by evaluating the derivatives of the logarithm of (**??**) at $s = 0$ (see, for instance Degroot (1989), page 202). As the first four central moments take up ten pages of algebra, they are omitted but available on request from the author.

LPM-CAPM against CAPM by considering $H_2 : \beta_{1p} = \beta_{2p}$ using the Likelihood Ratio test where (38) is the unrestricted likelihood, and the restricted likelihood, obtained by substituting $\beta_{1p} = \beta_{2p}$ in (38), is

$$
\begin{gathered}
-T \ln \sqrt{2\pi} - T \ln \sigma - \frac{1}{2\sigma^2} \sum_{t=1}^{T} \left(R_p(t) - \beta_{1p}\left[R_m(t) - R_f(t)\right]\right)^2 \\
+T_1 \ln p + T_1 \alpha_1 \ln \lambda_1 + (\alpha_1 - 1) \sum_{t=1}^{T} \delta(t) \ln R_m^+(t) - \lambda_1 \sum_{t=1}^{T} \delta(t) R_m^+(t) \\
-T_1 \ln(\Gamma(\alpha_1)) + T_2 \ln(1-p) + T_2 \alpha_2 \ln \lambda_2 + \\
(\alpha_2 - 1) \sum_{t=1}^{T} [1 - \delta(t)] \ln\left[-R_m^-(t)\right] - \lambda_2 \sum_{t=1}^{T} [1 - \delta(t)]\left[-R_m^-(t)\right] - T_2 \ln(\Gamma(\alpha_2)) \qquad (39)
\end{gathered}
$$

If $H_2$ is rejected, we can then conclude that the most suitable model describing the data is LPM-CAPM and therefore $\beta_{1p}$ is the "correct" risk measure. If $H_2$ is not rejected, we have illustrated strong support from CAPM.

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**Table 1 Returns on FTSE All-Share Index**

| Frequency | Maximum | Minimum | Mean | Standard Deviation | Skewness | Excess Kurtosis | Bera-Jarque |
|---|---|---|---|---|---|---|---|
| Daily | 0.057 | -0.039 | 0.000 | 0.009 | -0.069 | 2.500* | 659.298* |
| Weekly | 0.101 | -0.062 | 0.002 | 0.019 | 0.262* | 1.991* | 92.022* |
| Monthly | 0.100 | -0.110 | 0.009 | 0.039 | -0.420 | 0.114 | 3.589 |

Notes: The sample period for the daily, weekly, and monthly FTSE All-Share Index returns starts from 1 August 1991 to 31 July 2001. During the sample period, we have 2525 daily returns, 521 weekly return, and 120 monthly returns.
* represents significance at 5% level.

**Table 2 Maximum Likelihood Estimates of Normal Distribution on the Excess Market Returns**

| Frequency | Parameter | Estimate | t-statistic |
|---|---|---|---|
| Daily | $\mu$ | 0.0003 | 1.616 |
| | $\sigma$ | 0.0085 | 71.062 |
| Weekly | $\mu$ | 0.0009 | 1.048 |
| | $\sigma$ | 0.0191 | 32.279 |
| Monthly | $\mu$ | 0.0042 | 1.184 |
| | $\sigma$ | 0.0386 | 15.492 |

Notes: The excess market returns are calculated with the FTSE All-share Index and 3 month Treasury bill. See notes in Table 1 for other detailed explanation on data.

**Table 3 Maximum Likelihood Estimates of Continuous Mixed Truncated Normal Distribution on the Excess Market Returns**

| Frequency | Parameter | Estimate | t-statistic |
|---|---|---|---|
| Daily | $\mu_1$ | -0.0215 | -3.262 |
| | $\sigma_1$ | 0.0143 | 9.444 |
| | $\mu_2$ | -0.0622 | -2.066 |
| | $\sigma_2$ | 0.0216 | 4.817 |
| Weekly | $\mu_1$ | -0.0236 | -1.522 |
| | $\sigma_1$ | 0.0271 | 5.857 |
| | $\mu_2$ | -0.0255 | -1.499 |
| | $\sigma_2$ | 0.0266 | 5.428 |
| Monthly | $\mu_1$ | 0.0098 | 0.942 |
| | $\sigma_1$ | 0.0322 | 5.582 |
| | $\mu_2$ | -0.0953 | -0.659 |
| | $\sigma_2$ | 0.0689 | 1.979 |

Notes: See notes in Table 2 for the data.

**Table 4 Maximum Likelihood Estimates of Mixed Gamma Distribution on the Excess Market Returns**

| Frequency | Parameter | Estimate | t-statistic |
|---|---|---|---|
| Daily | $p$ | 0.5287 | 53.223 |
| | $\alpha_1$ | 1.1878 | 28.910 |
| | $\lambda_1$ | 191.0516 | 23.395 |
| | $\alpha_2$ | 1.1499 | 27.370 |
| | $\lambda_2$ | 179.8201 | 22.002 |
| Weekly | $p$ | 0.5585 | 25.675 |
| | $\alpha_1$ | 1.1452 | 13.539 |
| | $\lambda_1$ | 82.7621 | 10.874 |
| | $\alpha_2$ | 1.3321 | 11.888 |
| | $\lambda_2$ | 85.8034 | 9.834 |
| Monthly | $p$ | 0.6000 | 13.416 |
| | $\alpha_1$ | 1.3368 | 6.650 |
| | $\lambda_1$ | 46.4211 | 5.505 |
| | $\alpha_2$ | 1.1187 | 5.510 |
| | $\lambda_2$ | 34.1302 | 4.404 |

Notes: See notes in Table 2 for the data.

**Table 5 Summary of Fits to the FTSE All-Share Index Returns**

| Frequency | Distribution | Log-Likelihood Value | AIC | $\phi$ |
|---|---|---|---|---|
| Daily | Normal | 8454.28 | 16904.56 | 0.0069 |
| | MG | 8542.36 | 17074.72 | 0.0062 |
| | CMTN | 8536.26 | 17064.51 | 0.0063 |
| Weekly | Normal | 1323.97 | 2643.93 | 0.0155 |
| | MG | 1332.06 | 2654.12 | 0.0138 |
| | CMTN | 1329.94 | 2651.89 | 0.0150 |
| Monthly | Normal | 220.41 | 436.83 | 0.0323 |
| | MG | 220.63 | 431.26 | 0.0288 |
| | CMTN | 223.12 | 438.24 | 0.0296 |

Notes: See notes in Table 2 for the data.

## Table 6 Summary Statistics of FTSE100 and FTSE250 Constituents

**A. Daily Returns**

| | Statistics | Maximum | Minimum | Mean | Inter-quartile Lower | Inter-quartile Upper | 5% significance (in %) |
|---|---|---|---|---|---|---|---|
| FTSE100 Constituents (77 Equities) | Maximum | 0.358 | 0.066 | 0.141 | 0.102 | 0.160 | |
| | Minimum | -0.050 | -0.778 | -0.166 | -0.179 | -0.104 | |
| | Mean | 0.002 | 0.000 | 0.001 | 0.000 | 0.001 | |
| | Standard Deviation | 0.032 | 0.012 | 0.020 | 0.018 | 0.022 | |
| | Skewness | 1.665 | -7.902 | -0.128 | -0.214 | 0.300 | |
| | Excess Kurtosis | 219.384 | 1.451 | 12.011 | 3.879 | 9.341 | |
| | Bera-Jarque | 5089882.353 | 232.504 | 90445.577 | 1614.250 | 9254.809 | 100% |
| FTSE250 Constituents (159 Equities) | Maximum | 0.616 | 0.043 | 0.148 | 0.095 | 0.178 | |
| | Minimum | -0.043 | -1.143 | -0.165 | -0.197 | -0.083 | |
| | Mean | 0.002 | -0.001 | 0.000 | 0.000 | 0.001 | |
| | Standard Deviation | 0.042 | 0.008 | 0.017 | 0.012 | 0.019 | |
| | Skewness | 3.609 | -15.556 | -0.014 | -0.228 | 0.597 | |
| | Excess Kurtosis | 573.784 | 2.540 | 26.514 | 7.371 | 21.965 | |
| | Bera-Jarque | 34739356.485 | 696.499 | 514498.987 | 5840.233 | 50850.484 | 100% |
| FTSE Small Cap Constituents (197 Equities) | Maximum | 1.900 | 0.047 | 0.232 | 0.112 | 0.281 | |
| | Minimum | -0.047 | -1.204 | -0.251 | -0.311 | -0.109 | |
| | Mean | 0.002 | -0.001 | 0.000 | 0.000 | 0.001 | |
| | Standard Deviation | 0.074 | 0.006 | 0.020 | 0.012 | 0.024 | |
| | Skewness | 18.577 | -18.173 | -0.023 | -0.684 | 1.121 | |
| | Excess Kurtosis | 666.143 | 2.703 | 59.235 | 17.493 | 62.290 | |
| | Bera-Jarque | 46824837.378 | 769.516 | 1204213.433 | 32720.363 | 412085.546 | 100% |

**B. Weekly Returns**

| | Statistics | Maximum | Minimum | Mean | Inter-quartile Lower | Inter-quartile Upper | 5% significance (in %) |
|---|---|---|---|---|---|---|---|
| FTSE100 Constituents (77 Equities) | Maximum | 0.446 | 0.099 | 0.201 | 0.151 | 0.234 | |
| | Minimum | -0.078 | -0.833 | -0.219 | -0.252 | -0.155 | |
| | Mean | 0.008 | 0.000 | 0.003 | 0.002 | 0.003 | |
| | Standard Deviation | 0.072 | 0.028 | 0.044 | 0.039 | 0.048 | |
| | Skewness | 1.498 | -3.854 | -0.119 | -0.237 | 0.143 | |
| | Excess Kurtosis | 54.497 | 0.080 | 4.636 | 1.720 | 4.442 | |
| | Bera-Jarque | 65760.878 | 2.324 | 1696.915 | 65.407 | 432.022 | 99% |
| FTSE250 Constituents (163 Equities) | Maximum | 1.182 | 0.082 | 0.230 | 0.157 | 0.257 | |
| | Minimum | -0.059 | -1.170 | -0.218 | -0.252 | -0.134 | |
| | Mean | 0.008 | -0.003 | 0.002 | 0.001 | 0.003 | |
| | Standard Deviation | 0.108 | 0.018 | 0.042 | 0.033 | 0.048 | |
| | Skewness | 2.714 | -5.731 | 0.185 | -0.085 | 0.471 | |
| | Excess Kurtosis | 87.306 | 0.672 | 6.765 | 2.674 | 6.785 | |
| | Bera-Jarque | 168318.199 | 9.813 | 3334.236 | 159.949 | 1069.210 | 100% |
| FTSE Small Cap Constituents (197 Equities) | Maximum | 2.255 | 0.078 | 0.315 | 0.158 | 0.377 | |
| | Minimum | -0.065 | -1.098 | -0.299 | -0.377 | -0.155 | |
| | Mean | 0.008 | -0.006 | 0.002 | 0.001 | 0.003 | |
| | Standard Deviation | 0.171 | 0.018 | 0.050 | 0.032 | 0.062 | |
| | Skewness | 6.403 | -7.393 | 0.238 | -0.278 | 0.751 | |
| | Excess Kurtosis | 122.560 | 1.153 | 12.771 | 4.882 | 13.929 | |
| | Bera-Jarque | 330826.775 | 28.912 | 9288.084 | 517.501 | 4270.626 | 100% |

**C. Monthly Returns**

| | Statistics | Maximum | Minimum | Mean | Inter -quartile | | 5% significance |
|---|---|---|---|---|---|---|---|
| | | | | | Lower | Upper | (in %) |
| | Maximum | 0.829 | 0.154 | 0.263 | 0.196 | 0.286 | |
| FTSE100 | Minimum | -0.132 | -0.928 | -0.286 | -0.356 | -0.202 | |
| Constituents | Mean | 0.035 | -0.001 | 0.012 | 0.007 | 0.015 | |
| (77 Equities) | Standard Deviation | 0.145 | 0.058 | 0.085 | 0.070 | 0.091 | |
| | Skewness | 1.047 | -2.785 | -0.258 | -0.567 | 0.062 | |
| | Excess Kurtosis | 17.477 | -0.433 | 1.807 | 0.285 | 2.386 | |
| | Bera-Jarque | 1682.356 | 0.016 | 56.283 | 2.740 | 34.772 | 61% |
| | Maximum | 1.182 | 0.095 | 0.290 | 0.198 | 0.338 | |
| FTSE250 | Minimum | -0.120 | -1.170 | -0.307 | -0.348 | -0.206 | |
| Constituents | Mean | 0.034 | -0.014 | 0.010 | 0.006 | 0.013 | |
| (163 Equities) | Standard Deviation | 0.231 | 0.042 | 0.092 | 0.072 | 0.104 | |
| | Skewness | 1.604 | -2.818 | -0.217 | -0.473 | 0.059 | |
| | Excess Kurtosis | 17.557 | -0.348 | 1.887 | 0.520 | 2.171 | |
| | Bera-Jarque | 1636.193 | 0.077 | 60.472 | 3.346 | 30.751 | 62% |
| | Maximum | 1.735 | 0.110 | 0.390 | 0.225 | 0.451 | |
| FTSE Small Cap | Minimum | -0.122 | -1.522 | -0.394 | -0.483 | -0.221 | |
| Constituents | Mean | 0.036 | -0.025 | 0.008 | 0.003 | 0.013 | |
| (197 Equities) | Standard Deviation | 0.337 | 0.042 | 0.112 | 0.079 | 0.138 | |
| | Skewness | 2.532 | -2.939 | -0.070 | -0.447 | 0.330 | |
| | Excess Kurtosis | 22.435 | -0.305 | 2.897 | 0.880 | 3.453 | |
| | Bera-Jarque | 2689.439 | 0.157 | 110.702 | 6.769 | 68.805 | 77% |

Notes: The sample period for the daily, weekly, and monthly stock returns starts from 1 August 1991 to 31 July 2001. During the sample period, we have 2525 daily returns, 521 weekly return, and 120 monthly returns.

## Table 7 Percent of Rejections for the H1 and H2

**A. Daily Returns**

| | | Bera-Jarque | | H1 Rejected | | H1 Not Rejected but H2 Rejected | | Neither H1 nor H2 Rejected | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Asymmetric Model | | LPM-CAPM | | CAPM | |
| FTSE100 Constituents | MG | Rejected | 77 | 7 | (9.1%) | 15 | (19.5%) | 55 | (71.4%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 77 | 7 | (9.1%) | 15 | (19.5%) | 55 | (71.4%) |
| | CMTN | Rejected | 77 | 7 | (9.1%) | 16 | (20.8%) | 54 | (70.1%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 77 | 7 | (9.1%) | 16 | (20.8%) | 54 | (70.1%) |
| FTSE250 Constituents | MG | Rejected | 159 | 23 | (14.5%) | 42 | (26.4%) | 94 | (59.1%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 159 | 23 | (14.5%) | 42 | (26.4%) | 94 | (59.1%) |
| | CMTN | Rejected | 159 | 24 | (15.1%) | 41 | (25.8%) | 94 | (59.1%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 159 | 24 | (15.1%) | 41 | (25.8%) | 94 | (59.1%) |
| FTSE Small Cap Constituents | MG | Rejected | 197 | 54 | (27.4%) | 44 | (22.3%) | 99 | (50.3%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 197 | 54 | (27.4%) | 44 | (22.3%) | 99 | (50.3%) |
| | CMTN | Rejected | 197 | 55 | (27.9%) | 42 | (21.3%) | 100 | (50.8%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 197 | 55 | (27.9%) | 42 | (21.3%) | 100 | (50.8%) |

**B. Weekly Returns**

| | | Bera-Jarque | | H1 Rejected | | H1 Not Rejected but H2 Rejected | | Neither H1 nor H2 Rejected | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Asymmetric Model | | LPM-CAPM | | CAPM | |
| FTSE100 Constituents | MG | Rejected | 76 | 5 | (6.6%) | 9 | (11.8%) | 62 | (81.6%) |
| | | Not Rejected | 1 | 0 | (0.0%) | 0 | (0.0%) | 1 | (100.0%) |
| | | Total | 77 | 5 | (6.5%) | 9 | (11.7%) | 63 | (81.8%) |
| | CMTN | Rejected | 76 | 4 | (5.3%) | 9 | (11.8%) | 63 | (82.9%) |
| | | Not Rejected | 1 | 0 | (0.0%) | 0 | (0.0%) | 1 | (1.3%) |
| | | Total | 77 | 4 | (5.2%) | 9 | (11.7%) | 64 | (83.1%) |
| FTSE250 Constituents | MG | Rejected | 163 | 7 | (4.3%) | 32 | (19.6%) | 124 | (76.1%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 163 | 7 | (4.3%) | 32 | (19.6%) | 124 | (76.1%) |
| | CMTN | Rejected | 163 | 5 | (3.1%) | 32 | (19.6%) | 126 | (77.3%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 163 | 5 | (3.1%) | 32 | (19.6%) | 126 | (77.3%) |
| FTSE Small Cap Constituents | MG | Rejected | 197 | 20 | (10.2%) | 49 | (24.9%) | 128 | (65.0%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 197 | 20 | (10.2%) | 49 | (24.9%) | 128 | (65.0%) |
| | CMTN | Rejected | 197 | 22 | (11.2%) | 46 | (23.4%) | 129 | (65.5%) |
| | | Not Rejected | 0 | 0 | (0.0%) | 0 | (0.0%) | 0 | (0.0%) |
| | | Total | 197 | 22 | (11.2%) | 46 | (23.4%) | 129 | (65.5%) |

**C. Monthly Returns**

| | | Bera-Jarque | | H1 Rejected | | H1 Not Rejected but H2 Rejected | | Neither H1 nor H2 Rejected | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Asymmetric Model | | LPM-CAPM | | CAPM | |
| FTSE100 Constituents | MG | Rejected | 49 | 3 | (6.1%) | 7 | (14.3%) | 39 | (79.6%) |
| | | Not Rejected | 28 | 2 | (7.1%) | 1 | (3.6%) | 25 | (89.3%) |
| | | Total | 77 | 5 | (6.5%) | 8 | (10.4%) | 64 | (83.1%) |
| | CMTN | Rejected | 49 | 3 | (6.1%) | 7 | (14.3%) | 39 | (79.6%) |
| | | Not Rejected | 28 | 2 | (7.1%) | 1 | (3.6%) | 25 | (89.3%) |
| | | Total | 77 | 5 | (6.5%) | 8 | (10.4%) | 64 | (83.1%) |
| FTSE250 Constituents | MG | Rejected | 112 | 11 | (9.8%) | 17 | (15.2%) | 84 | (75.0%) |
| | | Not Rejected | 51 | 0 | (0.0%) | 8 | (15.7%) | 43 | (84.3%) |
| | | Total | 163 | 11 | (6.7%) | 25 | (15.3%) | 127 | (77.9%) |
| | CMTN | Rejected | 112 | 10 | (8.9%) | 18 | (16.1%) | 84 | (75.0%) |
| | | Not Rejected | 51 | 0 | (0.0%) | 9 | (17.6%) | 42 | (82.4%) |
| | | Total | 163 | 10 | (6.1%) | 27 | (16.6%) | 126 | (77.3%) |
| FTSE Small Cap Constituents | MG | Rejected | 161 | 11 | (6.8%) | 21 | (13.0%) | 129 | (80.1%) |
| | | Not Rejected | 36 | 0 | (0.0%) | 3 | (8.3%) | 33 | (91.7%) |
| | | Total | 197 | 11 | (5.6%) | 24 | (12.2%) | 162 | (82.2%) |
| | CMTN | Rejected | 161 | 14 | (8.7%) | 21 | (13.0%) | 126 | (78.3%) |
| | | Not Rejected | 36 | 0 | (0.0%) | 3 | (8.3%) | 33 | (91.7%) |
| | | Total | 197 | 14 | (7.1%) | 24 | (12.2%) | 159 | (80.7%) |

Notes: The sample period for the daily, weekly, and monthly stock returns starts from 1 August 1991 to 31 July 2001.
During the sample period, we have 2525 daily returns, 521 weekly return, and 120 monthly returns.
10% level of significance is used to test statistics. The numbers in the brackets represent represent the percentage of the number of cases chosen for a model among the rejected (not rejected) case.

**Figure 1A Percent of the Choice of the Three Models for Daily Returns Obtained with MG Distribution**

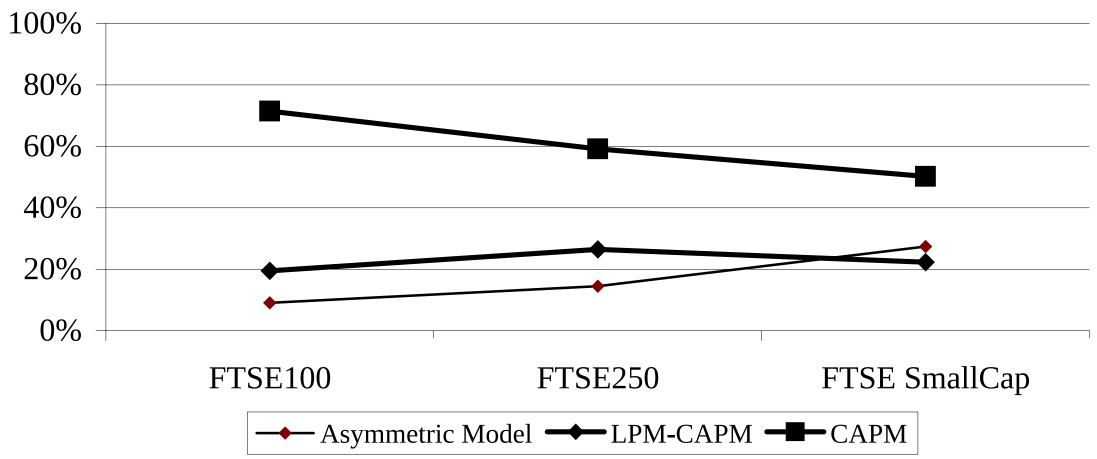


**Figure 1B Percent of the Choice of the Three Models for Weekly Returns Obtained with MG Distribution**

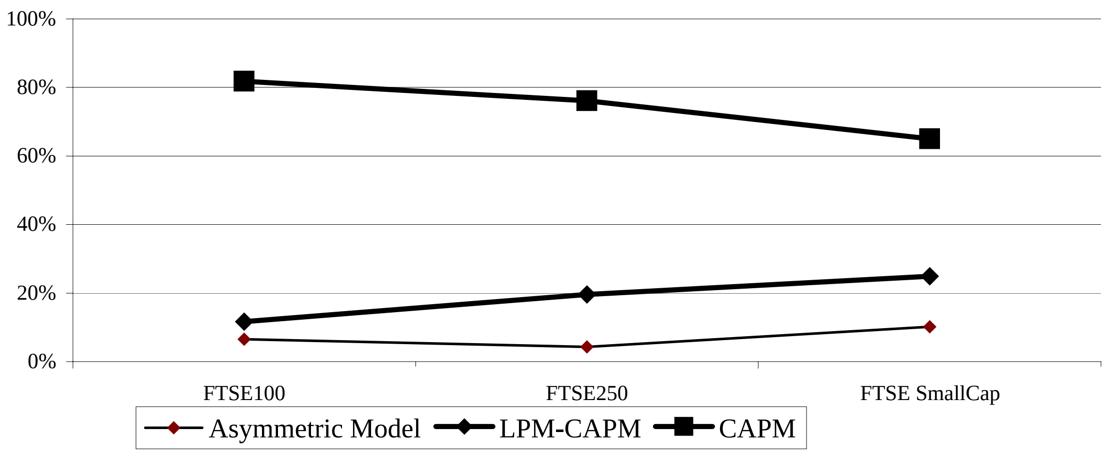


**Figure 1C Percent of the Choice of the Three Models for Monthly Returns Obtained with MG Distribution**