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Abstract

Convertible debt represents 10% of all USA debt yet despite its ubiquity it still posses difficult modelling challenges. This paper investigates alternative convertible bond model specifications. The work reviews the literature on convertible debt valuation especially the methodologies adopted by practitioners. Inadequacies in the historical and current valuation methods are highlighted. The different features used in convertible bond contracts found on the International Security Markets Association database are catalogued for both the Japanese and USA markets. Fashions in the contracts that have changed through time are noted. Modal, average, maximum and minimum USA contract parameters for various features are used to establish realistic and representative convertible bond contracts. The motivation for analyzing the ISMA data is determine which contracts features are important before investigating model errors. The model errors themselves are a function of the contract in question and cannot therefore, be examined in abstract. The sensitivity of the modal convertible bond contract price to the method of modelling the spot interest rate and the intensity process is examined. The convertible bond price sensitivity to the input parameters reveals that accurately modelling the equity process and capturing the contract clauses in the numerical approximation appear crucial whereas the intensity rate and spot interest rate processes are of second order importance.

Keywords: Convertible bonds, modelling, interest rate process, intensity rate process.

JEL classification: G12 and G13

1 Introduction

Convertible debt represents 10% of all USA debt[1] but despite its ubiquity it still posses difficult modeling challenges. This paper investigates the seriousness of alternative convertible bond model miss-specification errors. Convertible bond indentures typically have complex contract clauses with embedded optionality and it can be argued that convertible bond prices are a function of many factors which demand the modeling of several correlated stochastic processes. For example: the the spot interest rate for the straight bond price component; the equity price for the option to convert the bond into shares, the intensity rate process (because companies which issue convertible debt typically have poor credit ratings[2]) and sometimes an FX rate if the bond is issued in one currency for conversion into equity in another currency. As practitioners avoid models with more than 2 factors[3] it is an empirical question as to which of the factors are the most important i.e., which of the competing practical models with 2 or less factors is the least miss-specified.

This work reviews the literature on convertible debt valuation and attempts to ascertain the current best practice. Inadequacies in the historical and current valuation methods are highlighted. The different features used in convertible bond contracts found on the International Security Markets Association database are cataloged for both the Japanese and USA markets. Fashions in the contracts that have changed through time are noted. Modal, average, maximum and minimum USA contract parameters for various features are used to establish realistic and representative convertible bond contracts. The motivation for analysing the ISMA data is to determine what contracts features are important before investigating model errors. The model errors themselves are a function of the contract in question and cannot therefore, be examined in the abstract. The sensitivity of the modal convertible bond contract price to the method of modeling of the spot interest rate, the intensity process and the method of discounting cash flows is examined. The different models are nested within an equity based convertible bond model with default modeled using Jarrow and Turnbull (1995) [27]. The framework nests the models of Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42], Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17], as special cases.

The paper is organized as follows: Section 2 analyzes the ISMA database for the frequency of occurrence of various contract features in both the USA and Japanese markets and representative parameter values for USA convertible bond contracts; Section 3 describes different models (both firm and equity value) for pricing convertible debt. Section 4 nests the models of Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42], Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17] in an equity based convertible bond model with default modeled using Jarrow and Turnbull (1995) [27]; Section 5 compares the price sensitivity of realistic convertible bond contracts to using different models; and finally conclusions are drawn in Section 6. Appendix A describes the type of contract features found in convertible bond deals. Appendix B gives a glossary of various convertible bond valuation terms.

2 ISMA Data

The convertible bond indenture description data was obtained from the International Securities Market Association (ISMA)[4]. The database was first produced in 1998 and only includes deals that were still alive at that time i.e. deals redeemed prior to 1998 are not always included. Also the database only includes bonds covered by ISMAs' rules which basically means all bonds that used to be called "Eurobonds". Therefore the database will not include convertible bonds issued in domestic markets. However, according to Philips (1997) [38] "the Eurobond market has become an increasingly important place for newly issued convertible bonds." Moreover, Calamos (1998) [9] describes the Eurobond market as "The third-largest convertible market in the world...". The database should therefore capture a representative cross-section of all convertible bond contracts.

[1] This is the average ratio of convertible debt to total debt between 1900 and 1993 according to Kang and Lee (1996) [29].

[2] Kang and Lee (1996) [29] find that convertible bonds are relatively high-risk, high-growth and highly leveraged firms. In their survey of convertible bond offerings out of a sample of 91 there were 17 rated Aa or A, 20 rated Baa, 24 rated Ba and 30 rated B.

[3] Practitioners dislike models with more than two factors because of the complexity of their implementation and the difficulty in estimating all the model parameters.

[4] ISMA is the self-regulatory body and trade association for the international securities market. Its purpose is to oversee the orderly functioning of the market and to represent the interests of its members on issues that affect the market.

2.1 Japanese Convertible Bond Contract Features

The ISMA database contains 348 Japanese convertible bond contract specifications however, 6 of the contract specifications were only provisional and so were discarded. Table 1 records the remaining 342 contracts and states the contract specifications as the number of occurrences per year. 99% of the convertible bonds have a call option of which 85% of the call prices are a function of time. 88% of the bonds have a hard no-call period and 91% have a soft no-call period. 60% of the bonds are stated to be callable on a change in tax status. 23% of the bonds have a put clause which generally states a single date and price at which the bond can be put back to the issuer. 78% of the bonds were issued in non-Japanese Yen currencies. In the 1980's the refuge currency was almost exclusively the US dollar whereas in the 1990's it was almost exclusively the Swiss Franc. Gemmill (1993) [19] attributes the large quantity of dollar denominated Japanese warrat and convertible bond issues in the 1980s to the regulation of rights issues and the favourable Yen / dollar exchange rate at the time. 56% of the bonds have refix clauses. The literature suggests these clauses were first introduced in Japan in 1991. In the ISMA database the first refix clause is observed in 1993. 27% of the bonds have refix clauses which are a function of the exchange rate between the domestic (Yen) and foreign currency (US dollar or Swiss Franc). 11% of the bonds have a soft no-call period trigger level which is a function of the domestic and foreign exchange rates. This feature first appears in the database in 1992. 1% of the bonds are original issue discount notes. 2% are exchangeable. 3% have mandatory conversion clauses either at maturity or for a percentage of the bonds during the life of the issue. 2% have non-fixed coupon or deferred interest features.

2.2 USA Convertible Bond Contract Features

The ISMA database contained 119 US convertible bond contract specifications. Table 2 records their contract characteristics as the number of occurrences per year. 72% of all the bonds have a hard no-call period which can be anything from one month to several years. The particular date when the bond becomes callable and the call price are stated in all the contracts. For 60% of the bonds the call price varies as a function of time (i.e., there is a call schedule set out in the indenture) typically a new call price is fixed each year. The soft call period is a feature in 41% of the bonds with the trigger price of the equity typically being 130% or 150% of the conversion price for 15 to 30 consecutive days from 5 to 30 days prior to call notice. Moreover, the call price for the soft call can also be allowed to vary with time for example, with annual fixing. 93% of the bond contracts have a clause which allows the bond to be called in the event of a change in the tax status. Normally this call feature is available after a stated date and with a stated call price, typically 100, but sometimes this also has a time varying call price which is fixed annually. 53% of the bonds have a put clause which typically allows the bond to be put at 100 if the issuing company ceases to be listed or is the subject of a take over. However, in 7% of the bonds the put clause has prices which vary as a function of time, again with the put price being fixed annually. 7% of the bonds have a discount on par. 3% of the bonds are denominated in currencies other than the US dollar. 14% of the bonds are exchangeable into stock other than that of the bond issuing company. 32% of the bonds have conversion prices which are a function of time (i.e. fixed between certain dates). 2 bonds had coupons which were not constant. 1 bond had a refix clause and 1 had a percentage of the notes which could be redeemed early at the option of the issuer.

2.3 USA Contract Parameters

Concentrating on the USA market the convertible bond indentures are analyzed below for their representative parameter values.

Analyzing the maturity of all the convertible bonds Table 3 shows that 89.1% have maturities of 5 , 7, 10 and 15 years with the individual percentages being 17.1%, 19.5%, 17.1% and 35.4%, respectively. The coupon frequencies are annual, semi-annual, quarterly and none (for zero coupon deals) with frequency of occurrence percentages of 34.1%, 59.8%, 1.2% and 4.9%, respectively. The callable convertible bonds fall into two categories: those 25.3% of all bonds with no schedule that are callable from inception at 100 in the event of a change in tax status; and those 12.7%, 40.5%, 2.5% and 7.6% of all bonds with hard no-call periods with schedules and prices starting 2, 3, 4 and 5 years from inception, respectively. A minority of bonds have soft no-call periods, see Table 4. The most common soft no-call periods are 3 and 5 years from

Table 1: Japanese Convertible Bond Contract Features

Year	Number of Issues	Hard Non-call Period	Soft Non-call Period	Trigger Level Function of FX Rate	Put Clause	Discount on Par	Cross Currency	Ex-change-able	Manda-tory Conver-sion	Variable Coupon or De-ferred Interest	Refix Clause	Refix Clause Function of FX Rate
1981	1	1	1				1					
1982	8	8	8		1		8					
1983	9	9	9				9					
1984	11	11	11				11					
1985	18	18	18				17					
1986	8	8	8				8					
1987	22	22	22		22		22					
1988	5	5	5				5					
1989	2	2	2				2					
1990	0											
1991	0											
1992	4	2	3	1			3					
1993	56	48	53	4		1	38	2	1	1	26	
1994	76	72	73	11			64	2		1	55	24
1995	31	29	31	14			29			1	27	22
1996	46	30	35	7	28		30	1	4	5	46	30
1997	17	14	11	1	9		7	1	4		17	6
1998	7	3	4		6		2				5	
1999	19	17	14		12	1	8				15	8
2000	2	2	2		2		2				2	2
Total	342	301	310	38	80	2	266	6	9	8	193	92
% of all Issues	100%	88%	91%	11%	23%	1%	78%	2%	3%	2%	56%	27%
Total 1991 - 95	167	151	160	30	0	1	134	4	1	3	108	46
% of sub-period	100%	90%	96%	18%	0%	1%	80%	2%	1%	2%	65%	28%
Total 1996 - 00	91	66	66	8	57	1	49	2	8	5	85	46
% of sub-period	100%	73%	73%	9%	63%	1%	54%	2%	9%	5%	93%	51%

Table 2: USA Convertible Bond Contract Features

Year	Number of Issues	Hard Non-call Period	Soft Non-call Period	Put Clause	Discount on Par	Cross Currency	Ex-change-able	Early Redemption	Variable Coupon or Deferred Interest	Refix Clause
1981	1									
1982	0									
1983	2	2								
1984	2	2								
1985	3	2								
1986	12	2	11	5		1	4			
1987	22	9	19	5		2	3			
1988	0									
1989	3	3	2	3			2			
1990	3	3	2			1	1			
1991	7	6	4	4	2					
1992	3	2		1	3					
1993	12	11	3	8			2			
1994	5	3		4	1		1			
1995	10	10	2	9			1			
1996	16	14	2	13					1	1
1997	13	12	4	9	1		1		1	
1998	5	5		2	1		2	1		
Total	119	86	49	63	8	4	17	1	2	1
% of all Issues	100%	72%	41%	53%	1%	3%	14%	1%	2%	1%
Total 1989 - 93	28	25	11	16	5	1	5	0	0	0
% of sub-period	100%	89%	39%	57%	18%	4%	18%	0%	0%	0%
Total 1994 - 89	49	44	8	37	3	0	5	1	2	1
% of sub-period	100%	90%	16%	76%	6%	0%	10%	2%	4%	2%

Table 3: USA convertible bond contract parameter values for maturity, coupon size, coupon frequency, the period of the hard no-call clause and the first call price. Soft no-call and put clauses appear in a minority of the bond indentures and have therefore, been excluded from this table. The tabulated results are based on 82 convertible bond indentures.

Metric	Maturity	Coupon Rate	Coupon Frequency	First Hard Call Time	First Hard Call Price	Conversion Price
Mode	15 yrs	6.00%	Semi-annual	3 yrs	100	38.80
Mean	10.35 yrs	5.55%	-	2.13 yrs	101.46	14.0
Maximum	20 yrs	8.00%	Quarterly	5.09 yrs	110	193.30
Minimum	3 yrs	0%	None	0 yrs	22.58	1.43

Table 4: Parameter values for USA convertible bond contracts with soft no-call provisions. Put clauses are only found in a minority of convertible bond contracts with soft no-call provisions and therefore, they are excluded from this table. The tabulated results are based on 35 convertible bond indentures.

Metric	Maturity	Coupon Rate	Coupon Frequency	First Hard Call Time	First Hard Call Price	First Soft Call Time	Soft Call Threshold	Conversion Price
Mode	15 yrs	6.00%	Annual	0 yrs	106	3 yrs	130.0%	14.00
Mean	13.03	5.94%	-	1.23 yrs	104.73	4.10 yrs	138.34%	43.81
Maximum	20 yrs	8.00%	Annual	5.03 yrs	110	14.01 yrs	200.0%	193.30
Minimum	4 yrs	2.00%	Semi-annual	0 yrs	100.0	1.03 yrs	127.0%	5.00

Table 5: Parameter values for USA convertible bond contracts with put provisions. The tabulated results are based on 13 convertible bond indentures.

Metric	Maturity	Coupon Rate	Coupon Frequency	First Hard Call Time	First Hard Call Price	First Soft Call Time	Soft Call Threshold	First Put Time	First Put Price	Conversion Price
Mode	15 yrs	5.75%	Annual	0 yrs	106	5 yrs	130.0%	7 yrs	100	–
Mean	11.70	5.54%	-	1.07 yrs	104.77	5.82 yrs	137.2%	4.91 yrs	114.90	64.95
Maximum	15 yrs	8.00%	Annual	3.05 yrs	110	14.01 yrs	200.0%	7 yrs	143.25	193.30
Minimum	5 yrs	2.00%	Semi-annual	0 yrs	100.0	2.17 yrs	127.0%	0 yrs	100.00	5.00

inception. The 3 year soft no-call period contracts tend to have 0, 1 and 2 year hard no-call periods and the 5 year soft no-call period contracts tend to have 0, 2 and 3 year hard no-call periods. The majority of bonds have put clauses. However, 54.9% of all bonds have put clauses typically at 100 that are available to the holder at any time from inception only in the event of the stock being de-listed or a change in control of the owner. A mere 15.9% of convertible bonds have put clauses with date and price schedules that are freely available to the holder, see Table 5.

The modal contract has: a maturity of 15 years, semi-annual 6% coupons and a hard no-call feature for the first 3 years.

2.4 Empirical Data Implications for Modeling

The empirical data on convertible bond contract clauses for the USA shows that to model realistic contracts requires the modeling of hard no-call schedules, soft no-call schedules, put schedules and conversion. Contracts of such complexity can only be solved by numerical methods. The optimal exercise strategy of these clauses is a free boundary problem and hence finite difference methods or trees are the algorithms of choice[5]. The soft no-call clauses are essentially Parisian options and are path dependent. Typically the equity price has to exceed a threshold level (or barrier) for a period of days before the bond becomes callable. Table 4 shows that for the modal contract with soft no-call clause the threshold is 130% higher than the conversion price of $14.00 and must be exceeded for 30 days (not tabulated). Avellaneda and Wu (1999) [2] show how Parisian options can be priced in trinomial trees. Their work builds on the work of Chesney, Jeanblanc-Picque and Yor (1997) [10] who calculate the density of excursion necessary for pricing Parisian options.

For the pricing Japanese convertible bonds the empirical data suggests that the above clauses must also be supplemented by the refix clause. Refix clauses allow the reseting of the conversion price and they are triggered when the average equity price trades below a threshold for a period of days. Like the Parisian option this is a path dependent feature which is difficult to price in a tree. However, as the monitoring period is typically only 4 or 5 days (not tabulated) and the life of the bond is typically of the order of 10 years then the monitoring period is likely to be collapsed onto one time step in the numerical approximation. Hence only the threshold needs to be checked at the relevant time step which is far simpler than modeling the path dependence.

3 Modeling Convertible Bonds

3.1 Firm Value Convertible Bond Models

The valuation of convertible bonds based on the modern Black-Scholes-Merton contingent claim pricing literature starts with Ingersoll (1977) [25] and Cox-Rubinstein (1985) [15]. In his paper Ingersoll develops arbitrage arguments to derive several results concerning the optimal conversion strategy (for the holder) and call strategy (for the issuer) as well as analytical solutions for convertible bonds in a variety of special cases. For example, an important result is that he decomposes the value of non-callable convertible bond CB into a discount bond K (with the same principal as the convertible bond) and a warrant with an exercise price equal to the face value of the bond i.e. $CB = K + \max(\gamma V_T - K, 0)$ where V_T is the value of the company at T and γ is the fraction of the equity that the bond holders posses if they convert (the dilution factor). His assumption of no dividends on the equity leads to the result that it is never optimal to convert prior to maturity. Ingersoll then generalizes his result to price convertible bonds with calls. In this case the convertible bond is decomposed into a discount bond, a warrant and an additional term representing the cost of the call which reduces the value of the callable convertible bond relative to the non-callable convertible bond.

[5]For single factor models this implies algorithms like the binomial tree of Cox, Ross and Rubinstein (1979) [14], the trinomial tree Parkinson (1977) [37] and the finite difference models of Brennan and Schwartz (1978) [6] and Courtadon (1982) [12]. For multiple factors this implies the multi-dimensional tree algorithms of Boyle, Evnine and Gibbs (1989) [4], Kamrad and Ritchken (1991) [28] and multi-dimensional finite difference methods like Alternating Direction Implicit method, see Morton and Myers (1998) [34].

Ingersoll is able to solve analytically for the price of the convertible bond because of his assumption of no dividends and no coupons. Brennan and Schwartz (1977) [5] use finite difference methods to solve the partial differential equation for the price of a convertible bond with call provisions, coupons and dividends. Later Brennan and Schwartz (1980) [7] numerically solved a two-factor partial differential equation for the value of the convertible bond. This modeled both the value of the firm and also the interest rate stochastically. Nyborg (1996) extends this model to include a put provision and floating coupons. Brennan and Schwartz found that often the additional factor representing stochastic interest rates had little impact on the convertible bond price.

Nyborg (1996) [36] introduces coupons into the convertible valuation by assuming that they are financed by selling the risk-free asset. In his simple but worthwhile extension he uses Rubinstein's (1983) [40] diffusion model to value the risky and risk-less assets of the firm separately and gets an analytical solution for the value of the convertible bond. Dividends can also be handled in this model if they are assumed to be a constant fraction of the risky assets. He also analyzes the impact of other debt in the capital structure of the firm (senior debt, junior debt and debt with a different maturity to the convertible bond[6]). When the coupons are financed through the sale of risky assets an analytical solution is no longer possible.

For pricing derivative securities such as convertible bonds subject to credit risk the above structural models view derivatives as contingent claims not on the financial securities themselves, but as compound options on the assets underlying the financial securities. In the Merton (1974) [33] increasing the volatility of the assets of the firm increases the credit spread with respect to the risk free rate. Varying the volatility of the assets of the firm stochastically has the result of varying the credit spread of the compound option stochastically. Geske's (1979) [20] compound option pricing model has the volatility of the equity being negatively correlated to the value of the firm. As the value of the firm decreases, the leverage increases and the volatility of the equity increases and vice versa. Thus the firm value models easily capture some appealing properties.

The papers of Ingersoll, Nyborg and Brennan and Schwartz assume that the value of the firm as a whole is composed of equity and convertible bonds and they model the value of the firm as a geometric Brownian motion. The more recent literature considers the convertible bond to be a security contingent on the equity and (for more complicated models) the interest rate rather than the value of the firm. The equity is then modeled as a geometric Brownian motion. One of the reasons for doing this is that the value of the firm is not observable unlike the value of the firm's equity which is traded in the market. The advantage of firm value models is that it is relatively easy to model the value of the convertible bond when the firm is in financial distress. In Figure 2 the Brennan and Schwartz (1977) [5] convertible bond prices can be seen to be a proportion of the share value of the firm where the par value of the outstanding bond is less than the aggregate value of the firm.

3.2 Equity Value Convertible Bond Models

In their Quantitative Strategies Research Notes, Goldman Sachs (1994) [41] consider the issue of which discount rate to use when valuing a convertible bond. They consider two extreme situations: Firstly where the stock price is far above the conversion price and the conversion option is deep in-the-money and is certain to be exercised. Here they use the risk-free rate as they argue that the investor is certain to obtain stock with no default risk. Second they consider the situation where the stock price is far below the conversion price and the conversion option is deep out-of-the-money. Here the investor owns a risky corporate bond and will continue to receive coupons and principal in the absence of default. The appropriate rate to use here is the risky rate which they obtain by adding the issuer's credit spread to the risk-less rate[7]. They use a simple one factor model with a binomial tree for the underlying stock price. However, at each node they consider the probability of conversion and use a discount factor that is an appropriately weighted arithmetic average

[6] Convertible debt is usually subordinate to other debt that the firm may have issued.

[7] When the authors talk about discounting using a risk free rate and a risky rate they appear to be using a short hand notation. All discounting is presumably performed at the risk free rate in accordance with standard financial theory. However, as the bond can default the expectation must be that only some fraction of its promised principal will be received. This manifests itself as the credit spread observed over the risk free rate. Hence when the authors talk of discounting the bond at the risky rate they are really talking about discounting the risky bond's expected future value at the risk free rate where the future value is some fraction of the riskless bond's future value.

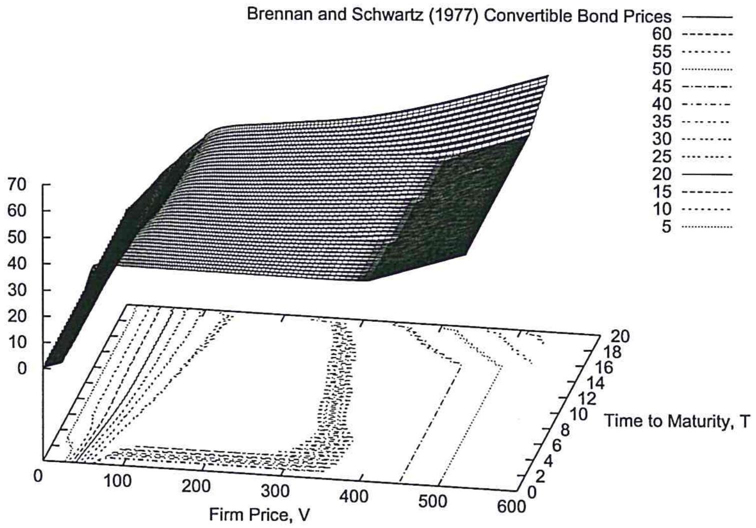


Figure 1: Brennan and Schwartz (1977) [5] convertible bond option prices for a contract with par value for the bond of 40, semi-annual coupon of 1.0, quarterly dividend of 1.0, convertible into 10% of the shares outstanding after conversion, firm variance of 0.001 per month, risk free rate of 0.005 per month and with a hard no-call period for the first 5 years followed by a call price of 43 for the next 5 years, 42 for the next five years and 41 for the final 5 years (this example was taken from Brennan and Schwartz (1977) [5].)

of the risk-less and risky rate. At maturity T the probability of conversion is either 1 or 0 depending on whether the convertible is converted or not. Backward induction is then used to determine the probability at earlier nodes, i.e. the conversion probability is the arithmetic average of the two future nodes. If at a node the bond is put then the probability is set to zero and if the bond is converted the probability is set to one. The methodology seems somewhat incoherent i.e., the investor is assumed to receive stock through conversion even in the event of default but the stock is not explicitly modeled as having zero value in this eventuality. Moreover, the intensity rate is not introduced into the drift of the stock as one would expect. Finally the model makes no mention of any recovery in the event of default on the debt.

The approach used by Goldman Sachs is formalized by Tsiveriotis and Fernandes (1998) [42]. In their paper they decompose the value of the convertible bond into a cash account and an equity account[8]. They then write down two coupled partial differential equations: The first equation for a holder who is entitled to all cash flows and no equity flows, that an optimally behaving holder of the corresponding convertible bond would receive, this is therefore discounted at the risky rate (as defined above). The second equation represents the value of the payments to the convertible bond related to payments in equity and is therefore discounted at the risk-free rate. The equations are coupled because any free boundaries associated with the call, put and conversion options are located using the PDE related to the equity payments and these are the boundary conditions used for the PDE related to the cash payments. The model outlined by Tsiveriotis and Fernandes is again a one factor model in the underlying equity. It is better than the Goldman Sachs model in the sense that the correct weighting (for example taking into account coupons) rather than a probability weighting is used for discounting the risky and risk-less components of the convertible bond price. Although, the Tsiveriotis and Fernandes model is more careful about modeling the cash and equity cash flows it suffers from the same theoretical inconsistencies as Goldman Sachs e.g. the intensity rate does not enter the drift on the equity process, the equity price is not explicitly modeled as jumping to zero in the event of default and any recovery from the bond is omitted.

Ho and Pfeffer (1996) [22] describe a two-factor convertible bond pricing model. Unlike the two factor model of Brennan and Schwartz the Ho and Pfeffer model can be calibrated to the initial term structure. The interest rate factor is modeled using the Ho and Lee (1986) [21] model. Ho and Pfeffer use a two dimensional binomial tree as their pricing algorithm. The authors appear to discount all cash flows at the risky (i.e., risk free plus credit spread) rate which implies the equity price goes to zero in the event of bond default and therefore, the intensity rate enters into the drift on the equity. However, this is implicit in their model and is not actually stated in the paper. Furthermore, any recovery on the bond in the event of default is omitted from the model. Moreover, from an empirical point of view, they use a constant spread over the risk free rate at all points to capture the credit risk. Goldman Sachs and Tsiveriotis and Fernandes are likewise guilty of this and it means that the credit spread is assumed fixed irrespective of whether the equity price is very high or very low. Empirically, the credit spread grows as equity prices deteriorate[9].

A better one factor model of interest rates is the extended Vasicek or Hull and White (1994) [23] and (1996)[24] model, as this is a mean-reverting interest rate model[10], unlike that of Ho and Lee (1986) [21]. Davis and Lischka (1999) [17] use this interest rate model and a Jarrow and Turnbull (1995) [27] style stochastic hazard rate to capture credit risk in their convertible bond pricing model. The Jarrow and Turnbull model can be calibrated so that the hazard rate reproduces the survival probabilities observed in the market. Davis and Lischka describe three possible models: the first has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek interest rate process and a deterministic

[8]cf. Ingersoll (1977) [25] with the decomposition into a bond and a warrant.

[9]Often an ad-hoc function is used by practitioners to allow the credit spread to vary inversely with the level of the equity. For example, the slope of a credit spread function is given by,

$$\beta = \frac{-1.0}{S_0} \ln \left(1 - \frac{1}{cs_0 + 1.0} \right) \tag{1}$$

where S_0 is some value of equity level S where the credit spread cs_0 is observed. The credit spread cs_j at other equity levels S_j is determined by,

$$cs_j = \left(\frac{1}{1 - \exp(-\beta S_j)} \right) - 1. \tag{2}$$

[10]For long maturity bonds like convertible bond contracts it would *a priori* appear important to capture the mean-reverting nature of interest rates.

intensity rate; the second model has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek intensity rate process and a deterministic interest rate; and the third model has a stochastic equity process (including the intensity rate in the drift), an extended Vasicek interest rate process and an intensity rate following a perfectly negatively correlated arithmetic Brownian motion process with respect to the equity process. The first and second models have considerable symmetry the only difference comes through the impact of the recovery rate. The third model is described as a $2\frac{1}{2}$ factor model. It is intuitively appealing and certainly preferable to modeling the intensity rate as an ad-hoc functions of the equity level. However, the arithmetic Brownian motion of the intensity process implies that the intensity rate can become negative. The inclusion of the intensity rate in the drift of the equity (in the event of no-default), a zero equity price in the event of default and the inclusion of a recovery rate makes these models more coherent with theory. The ability to correlate the intensity rate with the equity price is also appealing from an empirical point of view. However, their model is not implemented in Lischka's thesis (Lischka (1999) [30]), there are scant results in their working paper Davis and Lischka (1999) [17] and no comparisons with other models or evidence that this level of complexity is necessary.

Quinlan (2000) [39] highlights the difficulty of parameter estimation once a model has been selected: long-term equity implied volatilities do not exist, dividend forecasts must be estimated[11], determining the credit spread for subordinated debt can be difficult if the firm is not rated and correlations between the interest rate process and the equity process are difficult to measure and are non-stationary. Moreover, assumptions must be made about when the issuer will call a convertible, if it can be called. North American issuers will usually do this when parity rises $15-30\%$ above the call price. But there is no rule that applies in all cases and for example, this would most certainly not be the case for the Japanese market[12].

4 A Convertible Bond Pricing Model Nesting Other Models as Special Cases

[11] As convertible bonds have long maturities they are sensitive to both the dividend forecast and the method of modeling dividends.

[12] Nyborg (1995) [35] develops a signalling model for callable convertible debt where the choice of financing signals private information about the firm's prospects (mean returns). The existence of bankruptcy costs implies that equity is the preferred security in the absence of proprietary information. However, under asymmetric information riskier assets such as equity have worse adverse selection properties than less risky assets such as straight debt. The advantage of equity is that it has excellent insurance properties against financial distress unlike straight debt. Hence equity will be issued by pessimistic firms, while straight debt will be issued by more optimistic firms. Convertible debt will be issued by medium quality firms. In this model forcing conversion is a bad signal since it indicates a desire to insure against a deterioration in the equity price and the risk of ending up with unconvertible convertibles. On the other hand not forcing conversion is a good signal. Hence this model explains the empirical findings of Ingersoll (1977) [25] that forced conversion does not occur until the conversion value exceeds the call price by a median amount of 43.9%. In addition call protection periods and call notice periods also explain why firms allow the conversion value to rise well above the call price before calling the bond. The managers seek to avoid the danger of the equity price falling during the call notice period. However, despite its negative perception when invoked, callable convertible debt is preferred to non-callable convertible debt as it allows firms in some circumstances to force convertible bond holders into equity. Rather than being seen as delayed equity perhaps convertible debt may just have better adverse selection properties than equity.

4.1 The Reduced Form Default Model

The reduced form[13] approach to modeling credit was pioneered by Jarrow and Turnbull (1995) [27]. Their approach takes the firm credit spread[14] and the term structure of interest rates as inputs.

The default event is modeled as a point process with one jump to default in period $u \in [0, \tau]$. The indicator function denotes the jump process,

$$N(u) = \mathbf{1}_{\{\tau \leq \mathbf{u}\}}, \tag{3}$$

where the default event occurs at the stopping time τ. A compensating intensity process (also known as the arrival rate or hazard rate process) $\lambda(u)$ drives $N(u)$ such that,

$$N(u) - \int_0^u \lambda(s) ds \tag{4}$$

is a martingale. Let $N(u) = \sum_{n \geq 1} \mathbf{1}_{\{\tau \leq \mathbf{u}\}}$ and let the compensated process be $N(u) - \lambda u$ with the arrival rate λ constant, then $N(u)$ is a standard Poisson process[15]. Therefore the probability of i jumps occurring between time t and time u is,

$$\mathbf{P}\left[N(u) - N(t) = i\right] = \frac{(\int_t^u \lambda(s) ds)^i}{i!} \exp\left(-\int_t^u \lambda(s) ds\right), \forall i \in \mathbb{N}^+, \tag{5}$$

for any $u, t \in [0, \tau]$ such that $u > t$. Only the first jump in the time interval $[t, u]$ is relevant as the jump is into bankruptcy and therefore $i = 0$. The conditional probability that bankruptcy will not have occurred at time u i.e., the survival probability is therefore,

$$\mathbf{P}\left[N(u) - N(t) = 0\right] = \exp\left(-\int_t^u \lambda(s) ds\right). \tag{6}$$

Over a small time horizon the probability of default is, to a first order approximation, proportional to the intensity rate,

$$\mathbf{P}\left[N(u) - N(t) = 1\right] \approx \lambda(t)\Delta t. \tag{7}$$

4.2 Equity, Spot Interest Rate and Intensity Rate Processes

Following Davis and Lischka (1999) [17] a stochastic process is specified under the risk-neutral measure $\mathbb{Q}$ for the equity price, the interest rate and the intensity rate. However, the exact form of the interest rate and the intensity rate is undefined here so as to allow other models to be nested as special cases, see Table 10.

[13] Merton (1974) [33] examines the pricing of bonds when there is a significant probability of default. He uses a structural model of the firm to show that there is an isomorphic relationship between the levered equity of the firm and a call option. He assumes that default only occurs when the value of the firm is less than the value of the debt (this is a first passage time model of default). However, in reality this is unrealistic as default usually occurs long before the firm's assets are exhausted. As a result of this assumption Merton's model implies credit spreads which are much smaller than those observed in reality. However, care must be taken in making comparisons between credit spreads derived from a model under the risk-neutral measure and historic credit spreads observed under the objective measure. Black and Cox (1976) [3] develop a model where the firm defaults when the value of the firm's assets reaches some lower threshold. This feature allows their model to generate credit spreads consistent with those observed in the market. Longstaff and Schwartz (1995) [31] again developed a closed form structural model of firm default however, they extended the Black and Cox (1976) [3] model by introducing a Vasicek (1977) [43] style stochastic spot interest rate. By varying the correlation between the assets of the firm and the spot interest rate for firms in different industries and sectors they reproduce the empirical observed result that firms with similar default risks but in different sectors can have significantly different credit spreads.

[14] A firm may have a sufficient number of bond issues of the same seniority trading that a firm specific term structure of interest rates can be constructed. However, normally this is not the case and a term structure is derived from similar corporate bonds i.e., firms from the same sector and rating group.

[15] If the arrival rate remains time-inhomogeneous and is itself driven by a random variable then the process is a Cox or doubly stochastic process. Cox introduced the idea that intensity rate could be a function of state variables, see Cox and Miller (1996) [13].

Table 6: Nested convertible bond models: specification of the equity process, the spot interest rate process and the intensity (hazard) rate process.

Model Name	Equity Process	Interest Rate Process	Intensity Rate Process
Naive Model	Stochastic	Deterministic $r(t)$	None i.e. $\lambda(t) = 0$
Goldman Sachs (1994)	Stochastic	Deterministic $r(t)$	Deterministic $\lambda(t)$
Tsiveriotis and Fernandes (1998)	Stochastic	Deterministic $r(t)$	Deterministic $\lambda(t)$
Ho and Pfeffer (1996)	Stochastic	Stochastic (Ho and Lee) $c(r,t) = \theta(t)$ and $d(r,t) = \sigma_2$	Deterministic $\lambda(t)$
Davis and Lischka (1999)	Stochastic	Stochastic (extended Vasicek) $c(r,t) = (\theta(t) - \alpha_1 r(t))$ and $d(r,t) = \sigma_2$	Stochastic: $a(\lambda,t) = (\gamma(t) - \alpha_2\lambda(t))$, $b(\lambda,t) = \sigma_3$ and $dW(t)_3$; inversely correlated with equity process $a(\lambda,t) = \gamma(t)$, $b(\lambda,t) = \sigma_3$ and $dW(t)_1$; or deterministic $\lambda(t)$

4.2.1 Equity Process

Under the risk neutral measure $\mathbb{Q}$ the stock price is assumed to be given by the following stochastic differential equation,

$$dS(t) = (r(t) + \lambda(t) - q(t))S(t)dt + \sigma_1 S(t)dW(t)_1 - S(t_-)dN(t), \tag{8}$$

where $r(t)$ is the spot interest rate and $q(t)$ is the continuous dividend rate. When default occurs the stock price jumps to zero by subtracting the stock price immediately prior to default St_-. Conditional on default not having occurred the stock has the usual solution except the return is increased by $\lambda(t)$ to compensate for the risk of default,

$$S(t) = S(0)\exp\left[\int_0^t (r(s) + \lambda(s) - q(s))\,ds - \frac{1}{2}\sigma_1^2 t + \sigma_1 W_1(t)\right]. \tag{9}$$

4.2.2 Short Rate Process

Under the risk neutral measure $\mathbb{Q}$ the spot interest rate follows the following stochastic differential equation,

$$dr(t) = c(r,t)dt + d(r,t)dW(t)_2 \tag{10}$$

where $c(r,t)$ is the drift of the spot rate which can be mean reverting and $d(r,t)$ is the volatility of the spot rate. The price at time t of a bond maturing at time T is given by $P^T(t) = \mathbb{E}^{\mathbb{Q}}\left[\exp(\int_t^T -r(s)ds\right]$.

4.2.3 Intensity Rate Process

In order to model the volatility of credit spreads the intensity rate process and or the recovery rate process must be specified. As mentioned above the Jarrow and Turnbull (1995) [27] allows the intensity process to be an arbitrary random process. Jarrow, Lando and Turnbull (1997) [26] allow the intensity to be a function of state variables, namely, credit ratings. Ammann (2001) [1] in a hybrid model has intensity rate as a function of firm value. Das and Tufano (1996) [16] use a deterministic intensity rate but and allow the recovery rate to depend on the state of the economy. For the purposes of comparing convertible bond models the intensity process is here assumed evolve under the risk-neutral measure $\mathbb{Q}$ according to the following stochastic differential equation,

$$d\lambda(t) = a(\lambda,t)dt + b(\lambda,t)dW(t)_3 \tag{11}$$

where $a(\lambda,t)$ is the drift of the process which can be mean-reverting and $b(\lambda,t)$ is the volatility of the intensity rate. The recovery rate δ is assumed to be a predetermined fraction of the convertible bond notional K. Hence, in the event of default the price of the convertible bond jumps to the recovery value δK which is assumed to be invested at the risk free rate. The survival probability is determined by applying Itô's lemma to Equation 6.

Finally, the processes can be correlated such that, $\mathbb{E}[d\lambda(t), dr(t)] = \rho_{\lambda,r}dt$, $\mathbb{E}[dS(t), dr(t)] = \rho_{S,r}dt$ and $\mathbb{E}[d\lambda(t), dS(t)] = \rho_{\lambda,S}dt$ however, these may be degenerate for some models.

4.3 Convertible Bond Boundary Conditions

The value of the convertible bond must always be greater than or equal to the value of conversion[16] at times when it is convertible,

$$CB(t) \geq cr(t)S(t), \tag{12}$$

[16]The parity relationship of the convertible bond gives a minimum arbitrage boundary. If the convertible bond falls to a discount to parity then it is possible to buy the bond and simultaneously sell the underlying stock short, thus locking in an arbitrage profit. However, in practice both transaction costs and any accrued interest lost on conversion have to be taken into account. Moreover, shorting the underlying stock may not be possible or at least limited.

where $CB(t)$ is the value of the convertible bond at time t, $cr(t)$ is the conversion ratio which may follow a schedule and $S(t)$ is the value of the underlying equity. At maturity the convertible bond must be worth the principal amount K plus the final coupon c_T, if any, or the conversion price $cr(T)S(T)$,

$$CB(T) = \begin{cases} cr(T)S(T) & \text{if } cr(T)S(T) \geq K + c_T \\ K + c_T & \text{if } cr(T)S(T) < K + c_T \end{cases} \tag{13}$$

where T is the maturity of the convertible bond. If the bond is not callable or putable as $S \to \infty$,

$$CB(t) \sim cr(t)S(t), \tag{14}$$

and as $S \to 0$ the convertible bond price is bounded by the bond floor[17],

$$CB(t) \sim \mathbb{E}_t^Q \left[K + \sum_{i=1}^{n} c(t_i) \right] \tag{15}$$

where c is the coupon payable at times $t_i \in [t, T]$. The convertible bond value as $r(t) \to \infty$ and $r(t) \to 0$ depends on the process for $r(t)$ i.e., whether it is mean-reverting or not. If the the convertible bond is callable (the issuer's option),

$$CB(t) \leq cp(t), \tag{16}$$

where $cp(t)$ is the amount the bond can be called for by the issuing company. The value of the call price, $cp(t)$ can be time dependent according to a schedule in the indenture. If the convertible bond is putable (the holder's option),

$$CB(t) \geq pp(t), \tag{17}$$

where $pp(t)$ is the amount for which the bond can be put back to the issuing company. The value of the put price, $pp(t)$ can again be time dependent according to a schedule in the indenture. If the bond is trading in a region where it is contracted to be convertible, callable and putable then optimal conversion is given by,

$$CB(t) = \max\left(pp(t), cr(t)S(t), CB(t), \min\left(cp(t), CB(t)\right)\right), \tag{18}$$

other regions are special instances of this case.

Figure 2 shows the boundary conditions for a stylized convertible bond with conversion ratio of 1. Lowering the interest rate raises the bond floor and increasing the interest rate decreases the bond floor. If the volatility is increased the convertible bond price curve rises and vice versa. If the FX rate changes (for a cross-currency denominated bond) or the conversion ratio changes then the angle of the parity line changes. The premium tends to decrease with increasing share price. A call provision lowers the convertible bond price curve at the strike level. Whereas a put provision increases the convertible bond price at the strike level. As the stock price changes the convertible bond price has four regions of behavior; the first at very low stock prices is where the company is in financial distress and the stock price is viewed as a signal of financial strength and an estimate of default probability, the second region the convertible bond synthesizes straight debt and trades close to the bond floor, in the third region the convertible bond trades as a true hybrid instrument with a high premium and in the fourth region at very high stock prices the convertible bond synthesizes equity and trades close to parity.

4.4 The Treatment of Different Cash-flows

The different convertible bond models make different assumptions about the intensity rate $\lambda(s)$ and the recovery rate δ. Moreover, within each model different assumptions are made about the valuation of cash

[17] The second minimum arbitrage boundary is the so called ***investment value, straight value*** or ***floor*** of the bond. This is the value of the convertible bond if it were just a straightforward coupon bearing bond i.e. without the conversion features. If the interest rate changes this alters the level of the bond floor. High interest rates will lower the bond floor and low interest rates will raise the bond floor.

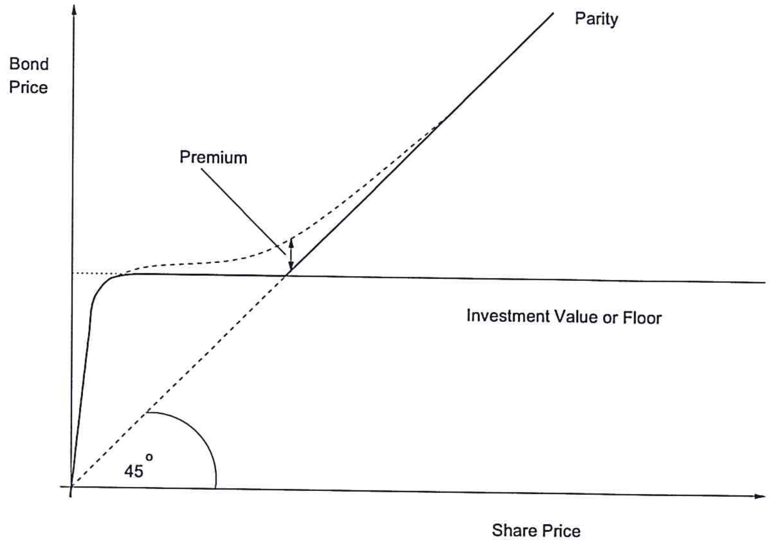


Figure 2: Stylized convertible bond price. The thick black line represents the lower bound of the convertible bond price. The lower bound is the parity price for high equity values or the bond floor for lower equity values. The dotted line represents the convertible bond price which trades at a premium to the lower bound in the hybrid region.

flows depending on whether they are related to equity or debt (cash). This is straight forward at certain times in the life of the convertible bond where the nature of the cash flow is clear cut. For example, at maturity it is known whether the convertible bond has been converted into equity or is a bond which pays cash to the holder. However, some of the alternative models attempt to capture what happens prior to maturity when the convertible bond is composed: partly of equity (including dividends) and partly of cash (including coupons); all of equity if converted; and all of cash if put. The value attributed to each cash flow is represented, using the Jarrow and Turnbull (1995) [27] methodology, by the following expression[18] with $i = 1, \dots, n$ cash-flows,

$$\begin{aligned} CB(u) = \mathbb{E}_u^Q & \left[\sum_{i=1}^{n} \left[\exp\left(- \int_u^{t_i} r(s) ds \right) \right. \right. \\ & \left[CB(t_i) \exp\left(- \int_u^{t_i} \lambda(s) ds \right) \right. \\ + & \left. \left. \left. CB(t_i) \delta \left(1 - \exp\left(- \int_u^{t_i} \lambda(s) ds \right) \right) \right] \right] \right] . \end{aligned} \tag{22}$$

The parameter values δ and $\lambda(s)$ are a function of the model and the nature of the particular cash-flow,

[18]Duffie and Singleton (1999) [18] make use of the approximation in Equation 7 for their default model. Using their methodology convertible bond cash-flows are discounted as,

$$CB(u) = \mathbb{E}_u^Q \left[\sum_{i=1}^{n} \left[\exp\left(- \int_u^{t_i} R(s) ds \right) CB(t_i) \right] \right], \tag{19}$$

where they define the $L = (1 - \delta)$ to be the fractional loss in market value and,

$$\exp(-R(t)) = (1 - \lambda(t)) \exp(-r(t)) + \lambda(t) \exp(-r(t))(1 - L). \tag{20}$$

They note that for small time periods this can be approximated as,

$$R(t) \approx r(t) + \lambda(t) L. \tag{21}$$

- The "naive" risk-free model assumes all cash flows are valued with, $\delta = 0$ and $\lambda(s) = 0$.
- Goldman Sachs (1994) define a probability of conversion ν such that all cash flows are weighted by ν are valued with $\delta = 0$ and $\lambda(s) = 0$ and then all cash flows weighted by $(1 - \nu)$ are valued with $\delta = 0$ and $\lambda(s) \neq 0$. The convertible bond price is the sum of the two probability weighted amounts.
- Tsiveriotis and Fernandes (1998) assume equity related cash-flows are valued with $\delta = 0$, $\lambda(s) = 0$ and debt (cash) related cash-flows are valued with $\delta = 0$, $\lambda(s) \neq 0$. The convertible bond price is the sum of the equity related cash-flows and the debt related cash-flows.
- Ho-Pfeffer (1996) assume all cash flows are valued with $\delta = 0$ and $\lambda(s) \neq 0$.
- Davis-Lischka (1999) assume equity related cash-flows are valued with $\delta = 0$, $\lambda(s) = 0$ and all debt (cash) related cash-flows are valued with a recovery rate such that $\delta \in [0,1]$ and $\lambda(s) \neq 0$. The convertible bond price is the sum of the equity related cash-flows and the debt related cash-flows.

The above framework for thinking about the different models in terms of equity and debt cash flows is in the spirit of Goldman Sachs (1994) and Tsiveriotis and Fernandes (1998) papers. However, a more illuminating framework for comparing the different models is presented in the next section.

4.5 An Analysis Using Margrabe's Model

A convertible bond can be thought of as a portfolio of a risky straight bond worth B at $t = 0$ which pays K at T_2 and an option to exchange the bond for equity[19] worth c at $t = 0$. Margrabe (1978) [32] shows that the price of a European option to exchange asset, S_2 for asset, S_1 at expiration, T_1 is given by,

$$c = Q_1 S_1 \exp((b_1 - r)T_1) N(d_1) - Q_2 S_2 \exp((b_2 - r)T_1) N(d_2), \tag{23}$$

$$d_1 = \frac{\ln(Q_1 S_1 / Q_2 S_2) + (b_1 - b_2 + \hat{\sigma}^2/2)T_1}{\hat{\sigma}\sqrt{T_1}}, \tag{24}$$

$$d_2 = d_1 - \hat{\sigma}\sqrt{T_1}, \tag{25}$$

and

$$\hat{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \tag{26}$$

where, Q_1 and, Q_2 are the quantities of asset, S_1 and, S_2, respectively.

The models of Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42], Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17] can be interpreted (with reference to a simplified convertible bond contract) using the philosophy of Margrabe as a tool. The modal contract is simplified by assuming that the exchange option is European with maturity at the end of the hard no-call region (i.e., at the end of the first 3 years, $T_1 = 3$) and that the bond pays no coupons. Later, Figures 3, 4 and 5 will show that the European assumption for the option style is reasonably accurate. Using Margrabe as a tool, S_1 can be interpreted as the price level of the equity, S, Q_1 the conversion ratio, cr, S_2 the bond price level, B and Q_2 the quantity of the bond which is unity. The price of the bond B at $t = 0$ is assumed to be related to the principal K that the bond pays at T_2 via $B = K\exp(-yT_2)$ where y is the continuously compounded bond yield. The option replicating portfolio can be seen (from Equation 23) to consist of $\exp((b_2 - r)T_1)N(d_2)$ of borrowed money and $cr\exp((b_1 - r)T_1)N(d_1)$ of equity. The values of b_1 and b_2 are model dependent. In the case where $S \to \infty$ then $N(d_1)$ and $N(d_2) \to 1$ i.e., the replicating portfolio for the option to exchange is composed of a long position in equity worth $Scr\exp((b_1 - r)T_1)$ and a short position in cash worth $B\exp((b_2 - r)T_1)$ or $K\exp(-yT_2)\exp((b_2 - r)T_1)$ which is exactly offset by the long risky bond. The convertible bond price,

[19] Alternatively, a convertible bond can be represented as a portfolio of the conversion value of equity plus an exchange option to put the equity and receive the risky bond.

CB will thus asymptotically go to $CB \to Scr\exp((b_1 - r)T_1)$ as $S \to \infty$ and if the option to exchange is American then $CB \to \max(Scr\exp((b_1 - r)T_1), Scr)$ as $S \to \infty$. Thus if there is a continuous dividend rate q then $b_1 = r - q$ and $CB \to crS$ for the American option to exchange. In the case where $S \to 0$ then $N(d_1)$ and $N(d_2) \to 0$ i.e., the option to exchange debt for equity is worthless and therefore, the replicating portfolio consists of a 0 long position in equity and a 0 short position in cash. The convertible bond price, CB is composed of a long position in the risky bond worth $K\exp(-yT_2)\exp((b_2 - r)T_1)$ and a worthless option to exchange, $c = 0$. Therefore, as $S \to 0$ then $CB \to K\exp(-yT_2)\exp((b_2 - r)T_1)$. If the yield curve is assumed flat then $y = b_2$ and $CB \to K\exp(-b_2T_2)\exp((b_2 - r)T_1)$.

Table 7 shows the values of b_1 and b_2 for the models of Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42], Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17]. The "naive" riskfree model assumes the forward bond price (and therefore also the cash hedge) grows at a conditional expectation adjusted rate which is here the riskfree rate, $b_1 = r$ and is discounted at the riskfree rate, r. The forward equity price grows at a conditional expectation adjusted rate which is here $b_2 = r - q$ and is discounted at the riskfree rate, r. The "naive" model is a straw man as it is clearly not realistic for the forward price of the risky bond to grow at the riskfree rate, r. Goldman Sachs (1994) [41] and Tsiveriotis and Fernandes (1998) [42] assume the forward bond price (and therefore, the cash hedge) grows at a conditional expectation adjusted rate which is here $b_2 = r + \lambda$ and is discounted at the riskfree rate, r. The equity grows at a conditional expectation adjusted rate which is here, $b_1 = r - q$ and is discounted at the riskfree rate, r. Although, these models are realistic in evolving the forward price of the risky bond at $r + \lambda$ they do not consider any recovery on the risky bond. Moreover, the forward equity price conditional on no-default does not include the intensity rate, λ. If a conditional expectation adjusted rate including the possibility of default is used for the risky bond of a company then to be consistent it must be used for the equity[20]. Ho and Pfeffer (1996) [22] also assume the forward bond price (and therefore, the cash hedge) grows at a conditional expectation adjusted rate of $b_2 = r + \lambda$ and is discounted at the riskfree rate, r. However, in their paper they appear to discount all cash flows at a risky rate (by which they mean $r + \lambda$) this implies they must have $b_1 = r - q + \lambda$ in order for their model not to be miss-specified but this is not stated. Finally, Davis and Lischka (1999) [17] assume the forward bond price (and therefore, the cash hedge) grows at a conditional expectation adjusted rate of $b_2 = r + \lambda L$[21] and is discounted at the riskfree rate, r. They assume that the forward equity price evolves at $b_1 = r + \lambda - q$ and is discounted at the riskfree rate, r. This is the most rigorous and coherent model relative to standard theorems of valuation. Conditional expectations prior to default on both debt and equity are adjusted to recognise the possibility of default and recovery is explicitly modeled.

Asymptotically, as noted above, when $S \to 0$ then $CB \to K\exp(-b_2T_2)\exp((b_2 - r)T_1)$ but this value is a function of b_2 which is model dependent. This indicates that the convertible bond price will be maximized using the "naive" model with $CB = K\exp(-rT_2)$, minimized at $CB = K\exp(-rT_2)\exp(-\lambda(T_2 - T_1))$ for Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42] and Ho and Pfeffer (1996) [22] and intermediate for Davis and Lischka (1999) [17] at $CB = K\exp(-rT_2)\exp(-\lambda L(T_2 - T_1))$. As $S \to \infty$ then for a European option to exchange one asset for another $CB \to Scr\exp((b_1 - r)T_1)$ which is maximized for Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17] at $Scr\exp((\lambda - q)T_1)$ and minimized for the "naive" model, Goldman Sachs (1994) [41] and Tsiveriotis and Fernandes (1998) [42] at $Scr\exp(-qT_1)$. For an American option to exchange one asset for another as $S \to \infty$ all the models will give $CB \to \max(Scr\exp((b_1 - r)T_1), Scr)$ which for a non-zero dividend rate q means the "naive" model, Goldman Sachs (1994) [41] and Tsiveriotis and Fernandes (1998) [42] will give $CB \to Scr$. For Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17] the situation is more complex and depends on the relative sizes of the intensity rate λ and the dividend rate q. If $\lambda < q$ then the option will be exercised early whereas if $\lambda \geq q$ then the option will not be exercised prior to maturity at T_1.

[20] One would expect the recovery on the bond to be greater than for the equity as the bond holders have the first claim on the company's assets.

[21] Davis and Lischka (1999) [17] use a Jarrow and Turnbull (1995) [27] model of default. However, their model has been restated here in terms of a Duffie and Singleton (1999) [18] (see footnote Equations 19, 20 and 21) to facilitate interpretation using Margrabe (1978) [32].

Table 7: Margrabe interpretation of the convertible bond models proposed by Goldman Sachs (1994) [41], Tsiveriotis and Fernandes (1998) [42], Ho and Pfeffer (1996) [22] and Davis and Lischka (1999) [17].

Model Name	b_1	b_2	Asymptote $S \to 0$	Asymptote (European) $S \to \infty$	Asymptote (American) $S \to \infty$
Risk-free	$r - q$	r	$K \exp(-rT_2)$	$Scr \exp(-qT)$	$\max(Scr, Scr \exp(-qT))$
Goldman Sachs (1994)	$r - q$	$r + \lambda$	$K \exp(-rT_2) \exp(-\lambda(T_2 - T_1))$	$Scr \exp(-qT)$	$\max(Scr, Scr \exp(-qT))$
Tsiveriotis and Fernandes (1998)	$r - q$	$r + \lambda$	$K \exp(-rT_2) \exp(-\lambda(T_2 - T_1))$	$Scr \exp(-qT)$	$\max(Scr, Scr \exp(-qT))$
Ho and Pfeffer (1996)	$r + \lambda - q$	$r + \lambda$	$K \exp(-rT_2) \exp(-\lambda(T_2 - T_1))$	$Scr \exp((\lambda - q)T)$	$\max(Scr, Scr \exp((\lambda - q)T))$
Davis and Lischka (1999)	$r + \lambda - q$	$r + \lambda L$	$K \exp(-rT_2) \exp(-\lambda L(T_2 - T_1))$	$Scr \exp((\lambda - q)T)$	$\max(Scr, Scr \exp((\lambda - q)T))$

5 Results

5.1 Surface Plots of Convertible Bond Prices

The impact of different model specifications on convertible bond prices is examined in this section by plotting the price of the modal contract for different equity levels, S and for different times, t for each model.

Figure 8 and Figure 9 show surface plots for the convertible bond price against equity level, S and time, t for the "naive" model (the simplest convertible bond model discussed) and Davis and Lischka (1999) [17] with a stochastic spot interest rate and a deterministic intensity rate (one of richest models discussed). By observation both plots appear virtually identical. At low equity prices the convertible bond synthesizes straight debt and trades close to the bond floor (the contour lines can be seen to wonder up and down "valleys" associated with the coupon payments on the bond) and at very high equity levels the convertible bond synthesizes equity and trades at parity (straight contour lines). At the front of the figures is a region (which lasts for 3 years in the modal contract) where the convertible bond has a hard no-call feature. Whether or not the holder of the convertible bond will choose to convert the bond to equity in this region depends on the yield advantage. Except in the case where there are dividends and the equity level is very high the holder of the convertible bond will optimally choose not to convert the bond and will therefore, enjoy a stream of coupon payments[22]. At high equity levels the dividend stream may be preferable to the coupon stream and the holder of the convertible bond will optimally choose to convert the bond to equity. The examples, here have a coupon rate of 6% (on a principal of 100) payable semi-annually and a continuous dividend rate of 3%. In the figures at high equity levels the yield advantage favors immediate conversion to equity. After 3 years the convertible bond becomes callable at 100 and therefore, conversion can be forced if it is optimal for the issuer. The call feature can be seen (in the contour lines) to suppress the convertible bond price which gets lower as the first 3 years comes to an end.

5.2 Asymptotic Analysis

Figures 3, 4 and 5 show convertible bond prices against equity levels, S for different convertible bond models. The different figures show slices at different time horizons, t through convertible bond price surfaces like those shown above. At low equity levels where the convertible bond synthesizes debt the prices differ primarily due to the different treatment of intensity, λ and recovery, δ rates i.e., for the "naive" model $\lambda = 0$ and $\delta = 0$; for Ho and Pfeffer, Tsiveriotis and Fernandes and Goldman Sachs $\lambda \neq 0$ and $\delta = 0$; and for Davis and Lischka $\lambda \neq 0$ and $\delta \neq 0$. In this region there is essentially no optionality and the prices can be verified as asymptotically correct by comparing them with the discounted straight bond cash flows. At high equity levels where the convertible bond trades at parity there is no optionality as conversion will have occurred. Again the prices are asymptotically correct. However, the different models produce varied prices in the hybrid region, as this is not clearly visible in the figure some comparative prices have been exhibited in Table 10. It is clear that the stochastic spot interest rate models of Ho and Pfeffer and Davis and Lischka produce very similar prices to the deterministic spot interest rate models. The stochastic intensity rate model of Davis and Lischka has lower prices in the hybrid region i.e., the yield advantage moves in favor of converting at lower equity levels than the other models.

Figure 3 at time, $t = 0$ shows convertible bond prices in the hard no-call period. In the hard no-call period the convertible bond has a large hybrid region where it has both debt and equity properties. Figures 4 and 5 at time $t = 3$ and $t = 3.75$ show convertible bond prices immediately prior and during the bond callable region, respectively. In these figures the hybrid region is very small for the modal contract and the convertible bond is either synthesizing debt or equity.

[22] As noted earlier Ingersoll was the first to observe that in the absence of dividends it is never optimal to convert prior to maturity.

Table 8: Convertible bond prices using the "naive" risk free model. Risk-free rate $r = 5\%$, par value of bond $K = 100$, coupon frequency is semi-annual, coupon rate 6%, dividend rate $q = 3\%$, conversion price $cr = 38.80$, volatility of equity $\sigma_1 = 30\%$, maturity $T = 15$ years, callable after 3 years at 100. A Cox, Ross and Rubinstein binomial tree was used with 200 time steps.

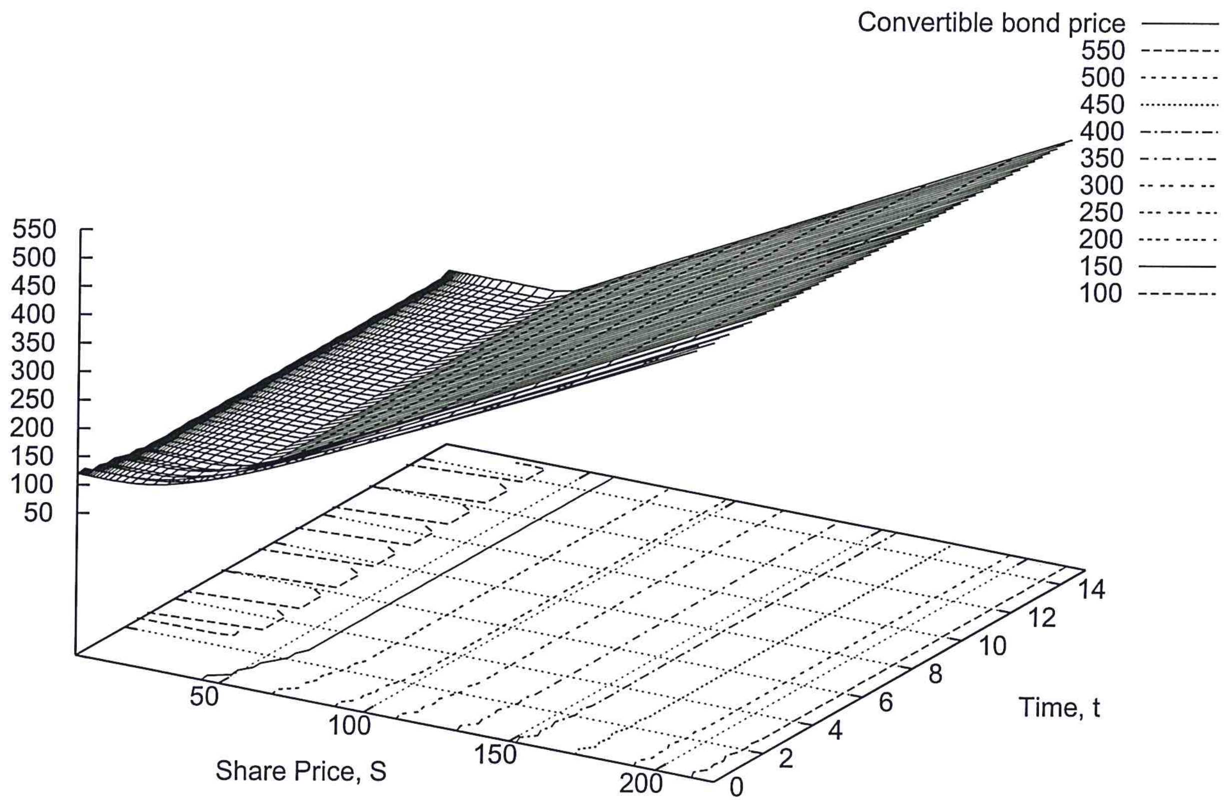


Table 9: Convertible bond prices using Davis and Lischka (1999) with correlated stochastic equity and interest rates and a Jarrow-Turnbull default with deterministic intensity rate. Risk-free rate $r = 5\%$, speed of interest rate mean-reversion $\alpha = 1\%$, level of interest rate mean reversion $\theta = 5\%$, interest rate volatility $\sigma_2 = 0.1$, intensity rate $\lambda = 3\%$, recovery rate $\delta = 0.3210$, correlation between interest rate and equity $\rho = -0.1$, par value of bond $K = 100$, coupon frequency is semi-annual, coupon rate 6%, dividend rate $q = 3\%$, conversion price $cp = 38.80$, volatility of equity $\sigma_1 = 0.3$, maturity $T = 15$ years, callable after 3 years at 100. A 2 dimensional trinomial tree (i.e., 9 probabilities originating from each node) was used with 200 time steps.

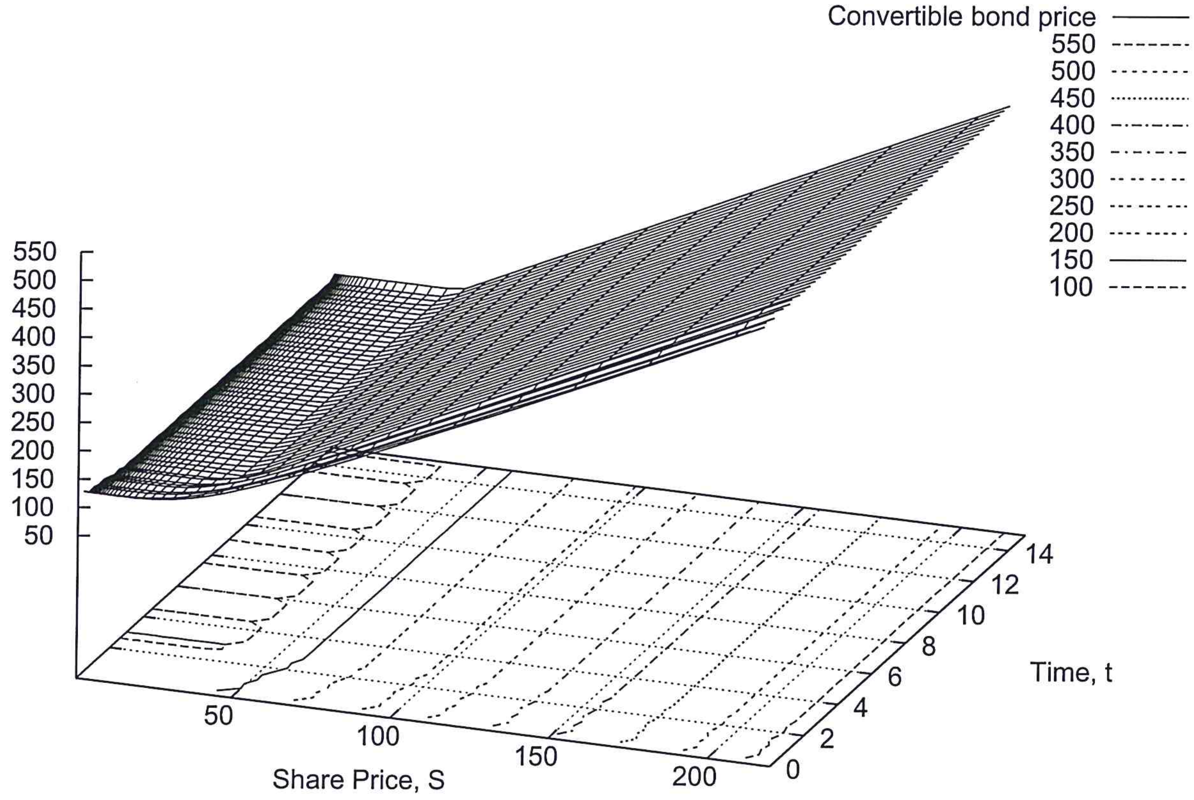


Figure 3: Time slice at $t = 0$ years of convertible bond prices calculated using various nested models.

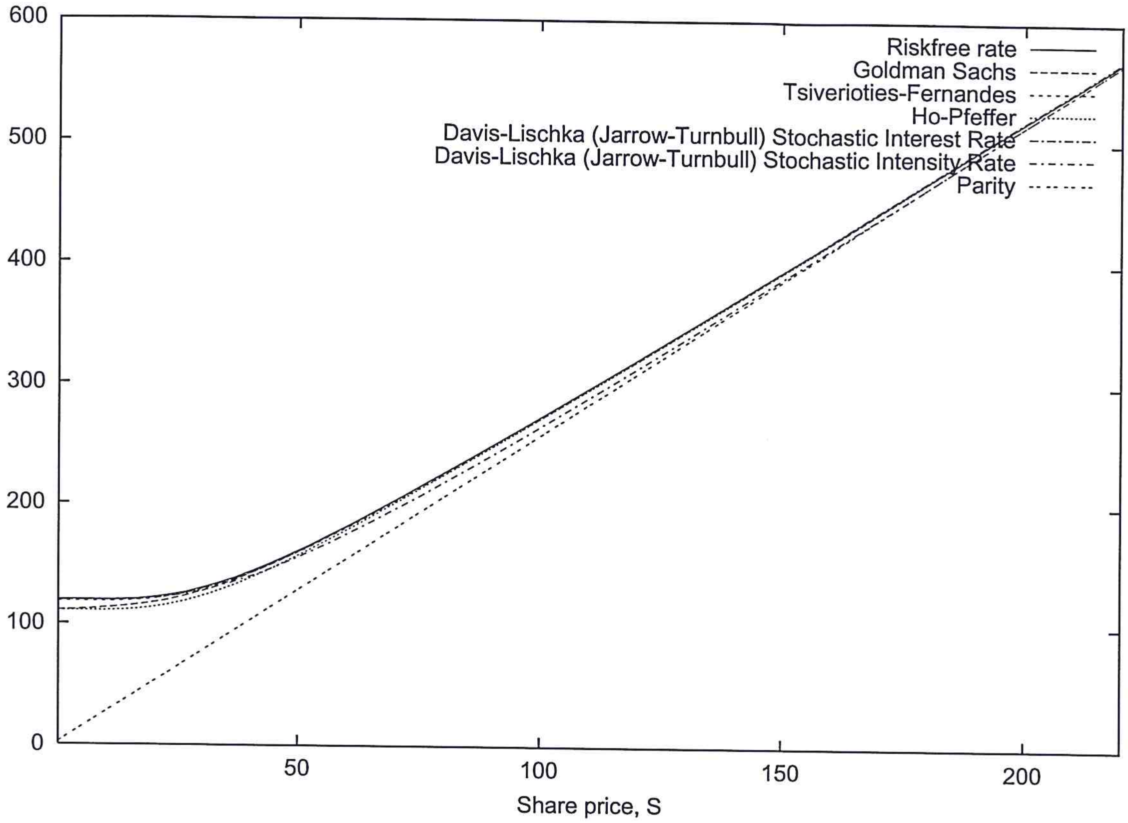


Table 10: Sample convertible bond prices for the modal contract at $t = 0$ for different convertible bond models. The underlying equity price is $S = 100$. The bond is in the hybrid region prior to becoming callable.

Model Name	Convertible Bond Price
Risk-free	272.9
Goldman Sachs (1994)	272.8
Tsiveriotis and Fernandes (1998)	271.7
Ho and Pfeffer (1996)	271.4
Davis and Lischka (1999) (stochastic interest rate)	272.7
Davis and Lischka (1999) (stochastic intensity rate)	264.8
Parity Value	257.7

Figure 4: Time slice at $t = 3$ years (immediately prior to the end of the hard no-call period) of convertible bond prices calculated using various nested models.

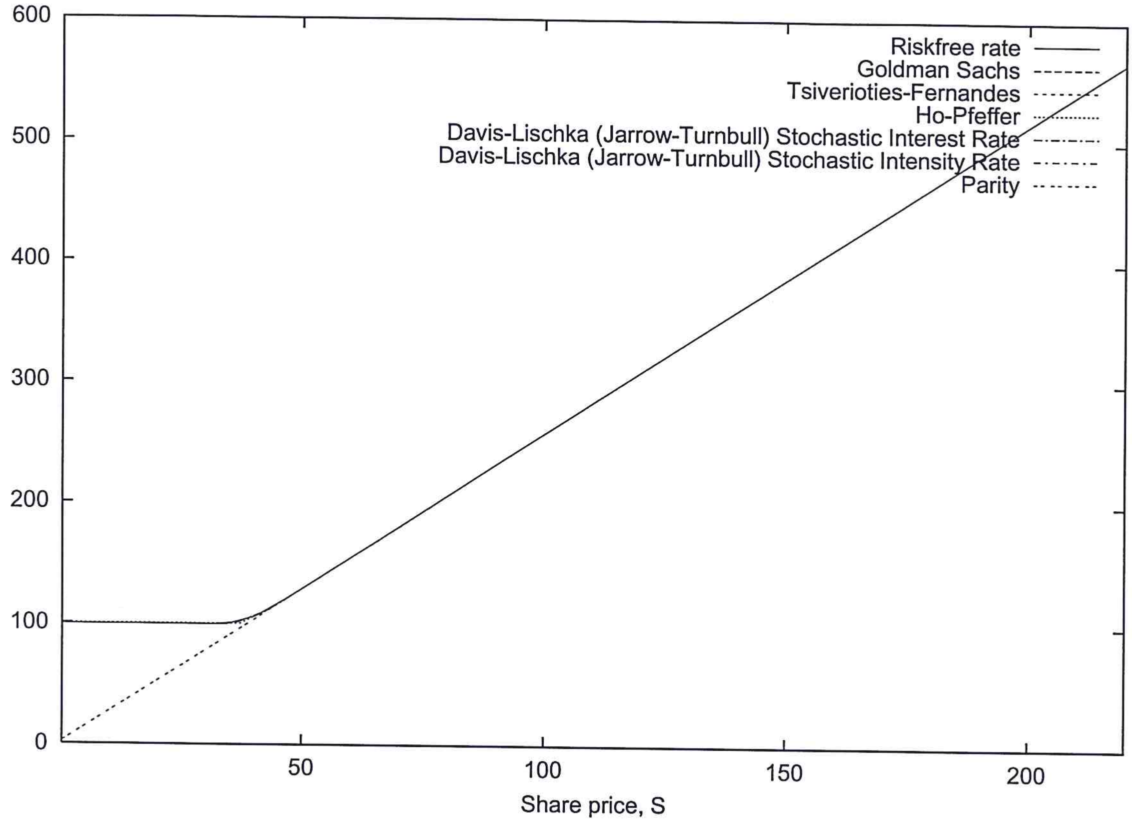
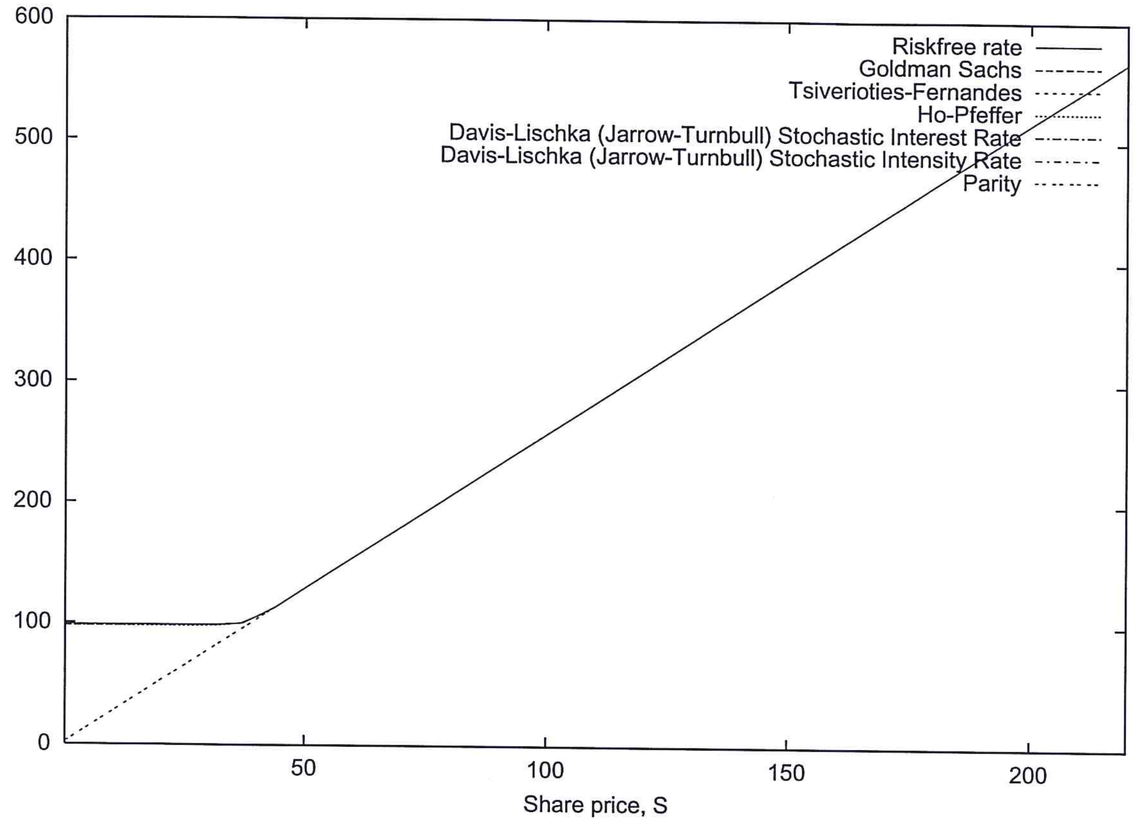


Figure 5: Time slice at $t = 3.75$ years (in the callable region) of convertible bond prices calculated using various nested models.



5.3 Model Sensitivities to Input Parameters

The following Figures 12, 13, 14, 15, 16, 17, 18 19, 20, 21, 22, 23 and 24 show the sensitivity of the convertible bond price for the Davis and Lischka model with respect to the model's input parameters. The sensitivities are numerical derivatives (or Greeks) computed by a multiplicative 1% increase and 1% decrease in the input parameter. Figure 12 shows the change in convertible bond price with respect to the equity level, S. For high equity levels, $\frac{\partial C}{\partial S}$ levels off at the conversion ratio, cr. In the hard no-call region where the hybrid region is large the transition from 0 to cr is smooth whereas in the call region where the hybrid region is small the transition is discontinuous. Figure 13 shows the change in convertible bond price with respect to the dividend rate, q. The convertible bond is most sensitive to a change in the dividend rate in the hybrid hard no-call region. An increase in dividend rate reduces the convertible bond price. Figure 14 shows the change in convertible bond price with respect to the conversion ratio, cr. Increasing the conversion ratio results in a relatively large increase in the convertible bond price. The conversion ratio increases smoothly in S in the hybrid hard no-call region and rapidly in the callable region. It is greatest when the convertible bond synthesizes equity. Figure 15 shows the change in convertible bond price with respect to the spot interest rate, r. The convertible bond is most sensitive to the interest rate in the hard no-call region and to a lesser extent when synthesizing debt. An increase in the interest rate results in a decrease in the convertible bond price. The Davis and Lischka model being used here has stochastic interest rates and the surface plot in the figure can be thought of as a slice (with interest rate level, r equal to the initial level of 5%) through a higher dimensional space where the interest rate as well as the equity level vary stochastically through time. Figure 16 shows the change in convertible bond price with respect to the level of interest rate mean reversion, θ. The convertible bond price is most sensitive to θ in the hard no-call region and where the convertible bond synthesizes debt. The figure shows clearly in the hard no-call region the point where the yield advantage to equity becomes preferable to debt as there is a distinct cut off above which the convertible bond has no sensitivity to θ. Unsurprisingly, because of the model, the shape of Figure 17 (which shows the change in convertible bond price with respect to the rate of mean-reversion, α) is very similar to Figure 16. Increasing either θ or α has the result of decreasing the convertible bond price. Figure 18 shows the change in convertible bond price with respect to the rate of spot interest rate volatility, σ_2. The convertible is most sensitive to σ_2 in the hard no-call region and where the convertible bond synthesizes debt. Figure 19 shows the the change in convertible bond price with respect to the correlation rate, ρ between the equity price process and the spot interest rate process. The figure is perhaps the least dramatic but shows that the convertible bond price is most sensitive to correlation in the hybrid region especially in the hard no-call region and perhaps also at the change over point for the yield advantage of debt and equity. Figure 20 shows the change in convertible bond price with respect to the intensity rate, λ. Once again the convertible bond is most sensitive to a change in λ in the hard no-call region where the convertible bond is synthesizing debt. Similarly, Figure 21 the change in convertible bond price with respect to the recovery rate, δ is greatest in the hard no-call region where the convertible bond is synthesizing debt. Figure 22 shows the change in convertible bond price with respect to the intensity volatility, σ_3. The convertible bond is most sensitive to σ_3 in the hard no-call hybrid region with a sudden cut off at the point where the yield advantage to equity becomes preferable to debt. Figure 23 and Figure 24 show the change in convertible bond price with respect to the call price, cp and call time ct, respectively. The convertible bond is obviously most sensitive to the call price in regions where the convertible bond is callable but has not yet been called. Whereas it is most sensitive to call time immediately prior to becoming callable and in the hybrid region.

In order to establish which model features have the greatest impact on the convertible bond price Table 11 shows the impact of the 2% perturbation (1% up and 1% down) on the model inputs for the modal convertible bond contract at $S = 100$ and $t = 0$. Correctly estimating the equity process appears very important as the equity level, S and to a lesser extent dividend rate, q and the equity volatility, σ_1[23] have a large impact on

[23]Brennan and Schwartz (1988) [8] were the first to point out that purchasers of convertible bond issues are likely to be much less concerned by the prospect of increases in the future risk of the company. For although an increase in risk would reduce the straight debt value of their bonds, it would also increase the value of the warrant element. Consequently, when there is doubt about the future policies of the company, the convertible is likely to be the preferred instrument. The relative insensitivity of the value of the convertible bond to the risk of the issuing company makes it easier for the bond issuer and purchaser to agree on the value of the bond even when they disagree on the risk of the company. This allows them to be issued on terms that look fair to the management even when the market rates the risk of the issuer higher than does the management of the issuing company.

the convertible bond price. Contract clauses such as the call time, ct and the conversion ratio, cr have the second most dramatic impact on the convertible bond price. The modeling of credit seems to be of second order importance. Finally, the stochastic modeling of the spot interest rate appears the least important model feature for the modal contract.

6 Conclusions

Convertible debt represents 10% of all USA debt but despite its ubiquity it still posses difficult modeling challenges. The reason for this is two-fold: firstly, the ISMA data shows that the indentures typically have complex clauses such as call, put and conversion schedules; and secondly, the convertible bond price is a function of many factors which, it can be argued, demand the modeling of several correlated stochastic processes.

The paper analyses the ISMA data and notes the relative frequency of contract clauses in the two most important convertible bond markets, Japan and the USA. The data shows that in the the USA hard no-call, soft no-call and put clauses are standard in bond indentures and in Japan refix clauses can be added to this profile. During the period covered by the database it is observed that refix clauses and soft no-call clauses have been introduced to meet (according to Calamos (1998) [9]) a perceived requirement by investors. Representative (average and modal) USA convertible bond contracts are established and extremum features noted. The literature on practical convertible bond pricing models is reviewed. A critique of the different models is made as well as their reinterpretation in terms of Margrabe's model. The models are empirically compared in terms of the modal contract. Table 10 shows that different modeling assumptions about the intensity rate and the recovery rate have a measurable impact on convertible bond prices. However, the ISMA database shows that typical convertible bond bid-ask spreads (not tabulated) are anywhere in a range from between 2% to 5%. Therefore, at the top end of this range all the model prices are within the bid-ask spread. Thus, although a theoretically coherent model is always a sensible prerequisite, it turns out that for the model inputs examined here the less coherent models do not perform adversely.

The results from this paper agree with those of Brennan and Schwartz (1980) [7] who find that modeling the interest rate as a stochastic rather than a deterministic factor is of secondary importance to modeling the firm value as a stochastic factor. Although, here the equity price is modeled rather than the firm value. It is also vitally important to model the call, put and conversion clauses carefully i.e., these contract features have a profound impact on the convertible bond price especially when the equity is trading close to the call and put prices. Therefore, the start date, end date and prices of these features must be captured accurately within the numerical approximation (e.g. the trinomial tree). It is in the hybrid region (when the bond is not callable or putable) where the convertible bond price is most sensitive to the correlation between the equity and interest rate, the interest rate, the level of interest rate mean-reversion, the rate of interest rate mean-reversion, the intensity rate and the volatility of the intensity rate. Once the bond is callable then conversion can be forced (if it is optimal for the bond issuer) leaving the holder with equity. This clearly has no sensitivity to the correlation or the other interest rate or intensity rate process parameters. However, the convertible bond price in this region is sensitive to the conversion ratio as this determines the quantity of shares one receives if conversion is forced. When the convertible bond is trading in the distressed region its price is most sensitive to a change in the recovery rate. The convertible bond price sensitivity to the input parameters reveals that accurately modeling the equity process and capturing the contract clauses in the numerical approximation appear crucial whereas the intensity rate and spot interest rate processes are of second order importance.

Table 11: Modal contract convertible bond price sensitivities to model input parameters. The Greeks are computed numerically at $t = 0$ and $S = 100$ by a multiplicative 1% increase and 1% decrease to the relevant input parameter. The model used is that of Davis and Lischka (1999).

Greek (Multiplicative change 1% increase 1% decrease)	Convertible Bond Price Change
$\frac{\partial C}{\partial S} = 2.388$	4.776
$\frac{\partial C}{\partial q} = -503.3$	−0.302
$\frac{\partial C}{\partial \sigma_1} = 44.50$	0.267
$\frac{\partial C}{\partial cr} = 92.65$	4.776
$\frac{\partial C}{\partial r} = -44.00$	−0.088
$\frac{\partial C}{\partial \theta} = -3.843$	−0.002
$\frac{\partial C}{\partial \alpha} = 5.000$	0.001
$\frac{\partial C}{\partial \sigma_2} = -1.000$	−0.002
$\frac{\partial C}{\partial \rho} = 0.000$	0.000
$\frac{\partial C}{\partial \lambda} = -21.67$	−0.013
$\frac{\partial C}{\partial \delta} = 0.9346$	0.006
$\frac{\partial C}{\partial \sigma_3} = 35.50$	0.071
$\frac{\partial C}{\partial cp} = 0.036$	0.072
$\frac{\partial C}{\partial ct} = 31.47$	1.888

Table 12: The surface $\frac{\partial C}{\partial S}$ plotted against equity levels, S and time, t.

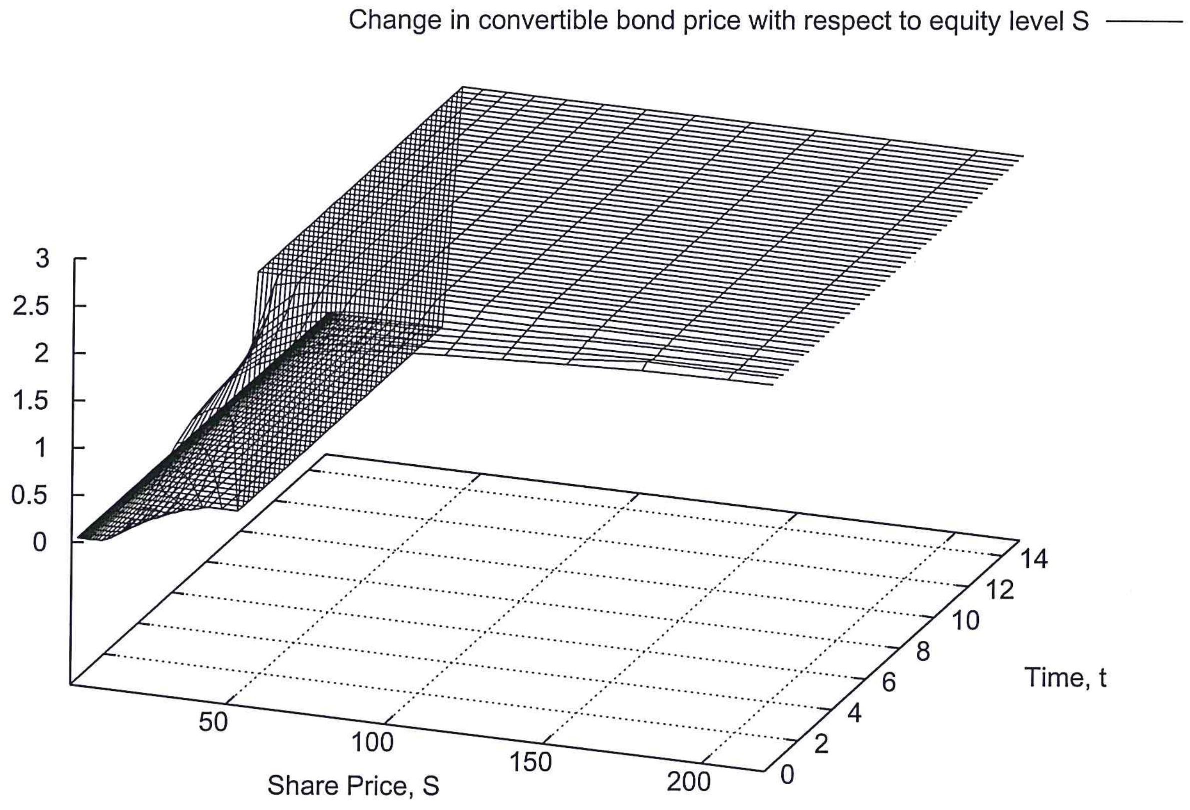


Table 13: The surface $\frac{\partial C}{\partial q}$ plotted against equity levels, S and time, t.

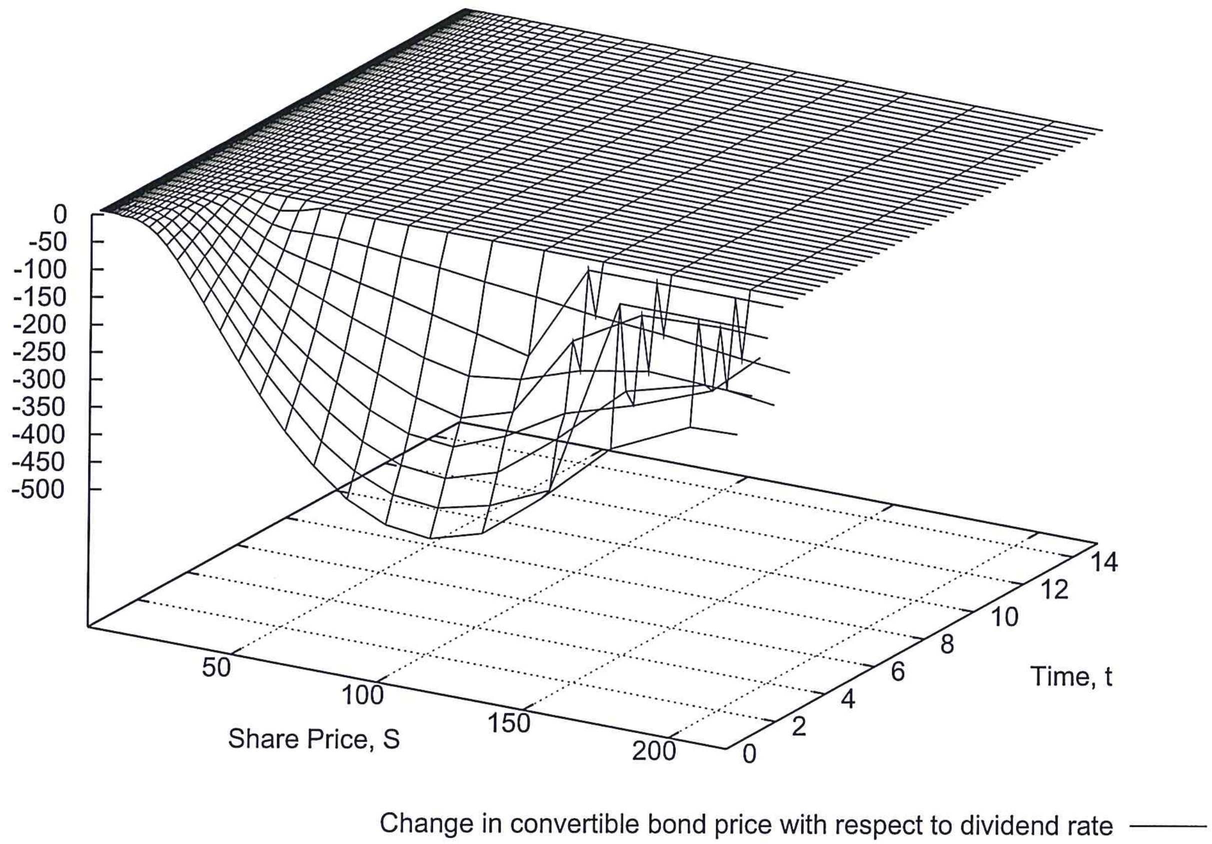


Table 14: The surface $\frac{\partial C}{\partial cr}$ plotted against equity levels, S and time, t.

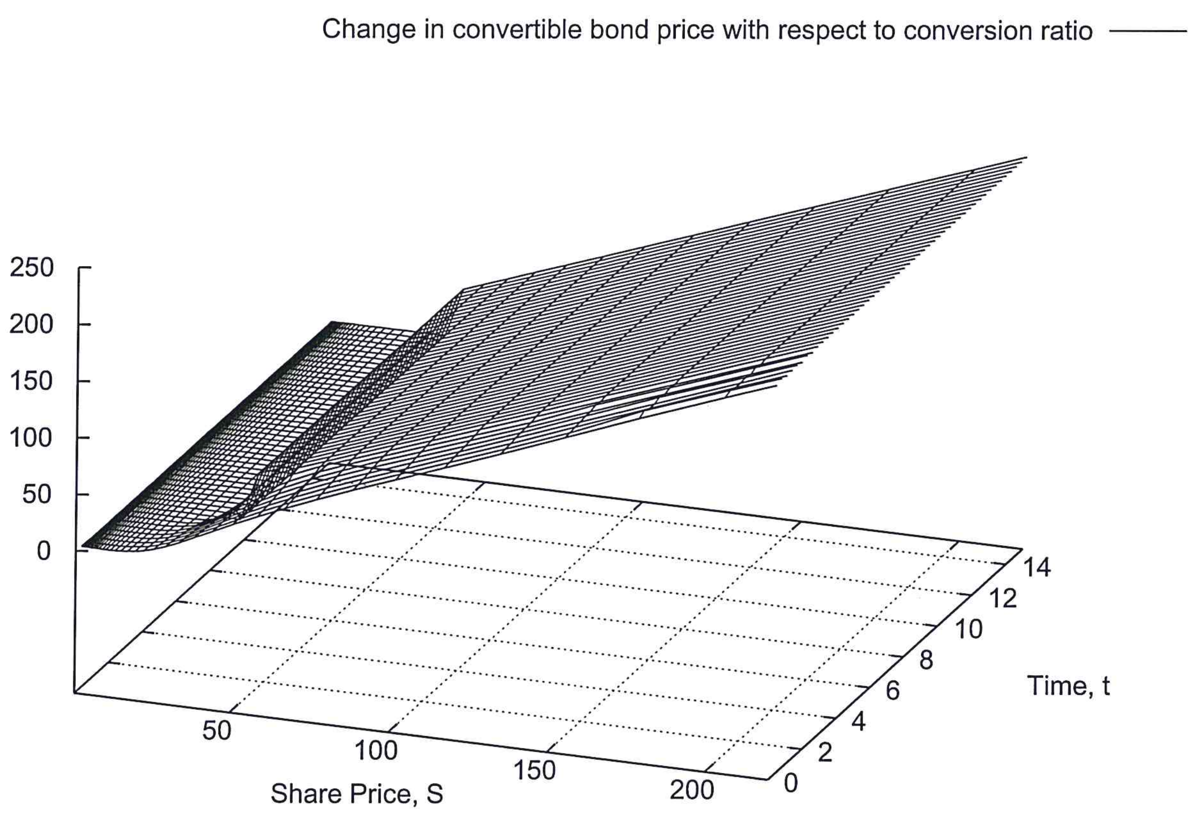


Table 15: The surface $\frac{\partial C}{\partial r}$ plotted against equity levels, S and time, t.

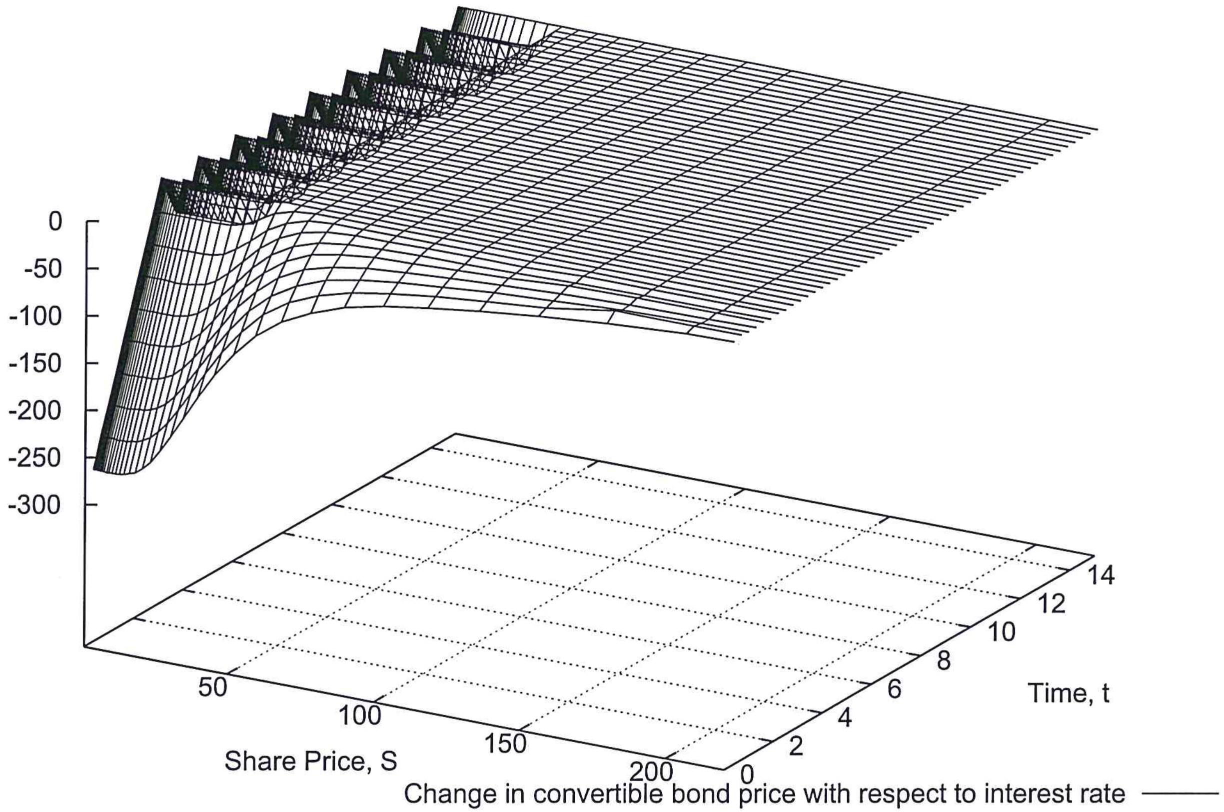


Table 16: The surface $\frac{\partial C}{\partial \theta}$ plotted against equity levels, S and time, t.

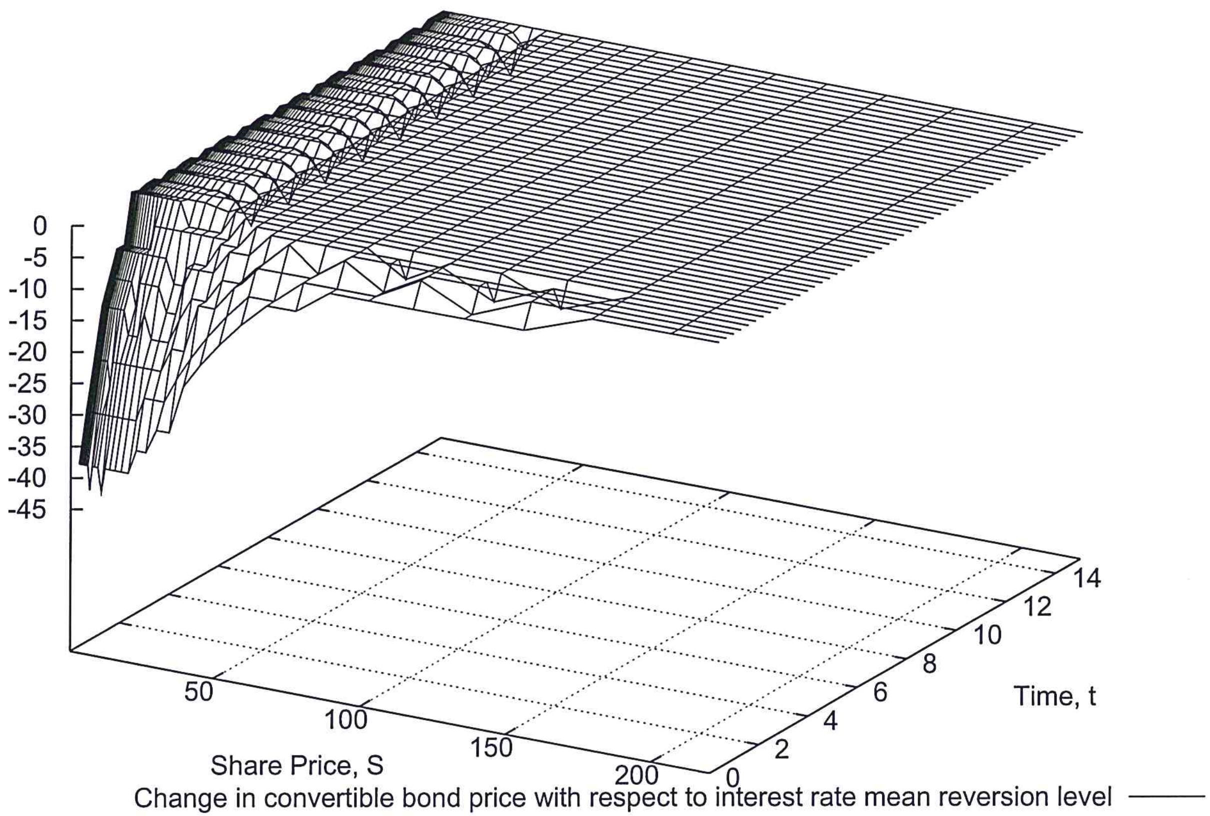


Table 17: The surface $\frac{\partial C}{\partial \alpha}$ plotted against equity levels, S and time, t.

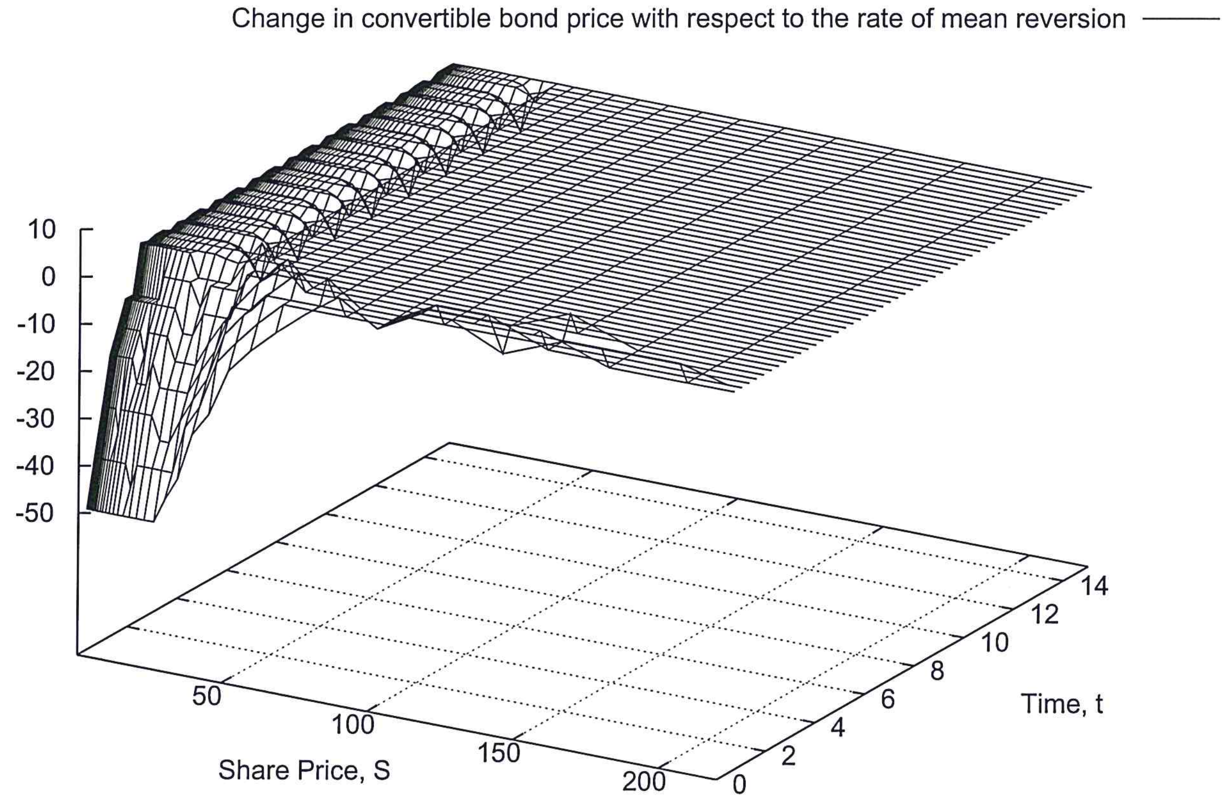


Table 18: The surface $\frac{\partial C}{\partial \sigma_2}$ plotted against equity levels, S and time, t.

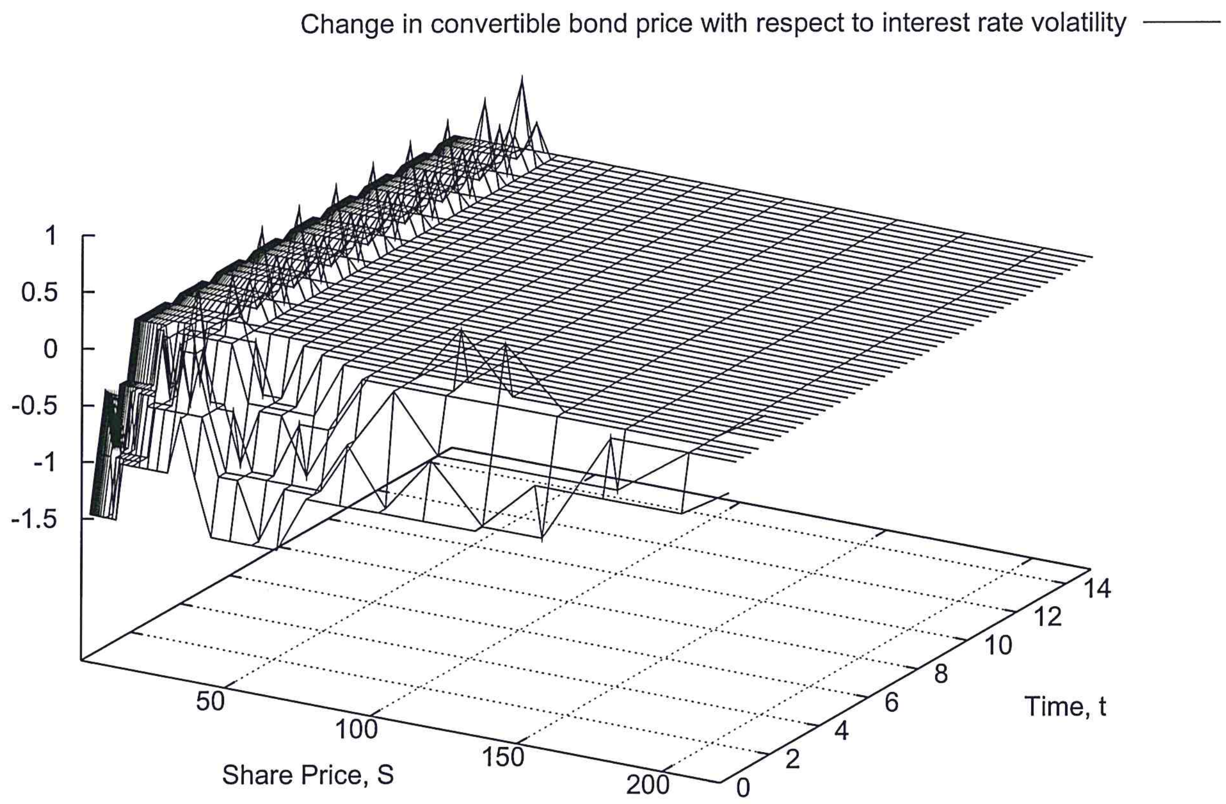


Table 19: The surface $\frac{\partial C}{\partial \rho_{12}}$ plotted against equity levels, S and time, t.

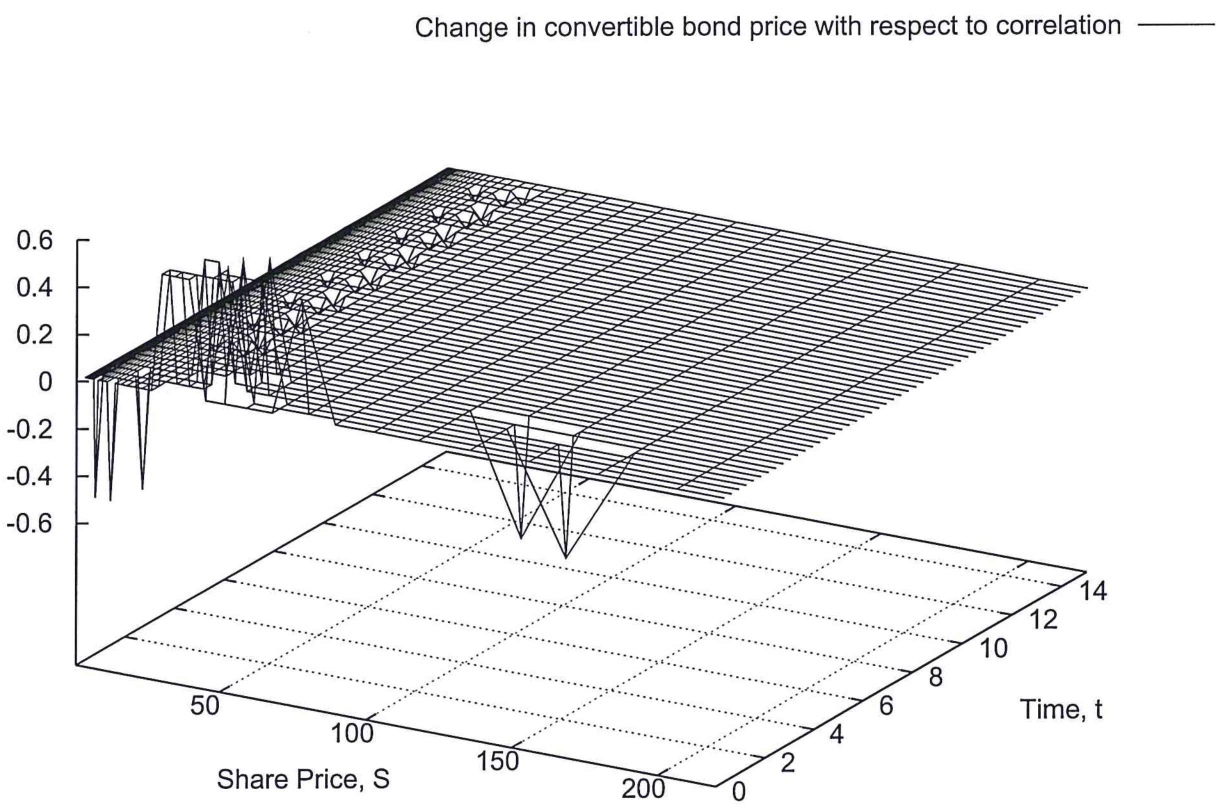


Table 20: The surface $\frac{\partial C}{\partial \lambda}$ plotted against equity levels, S and time, t.

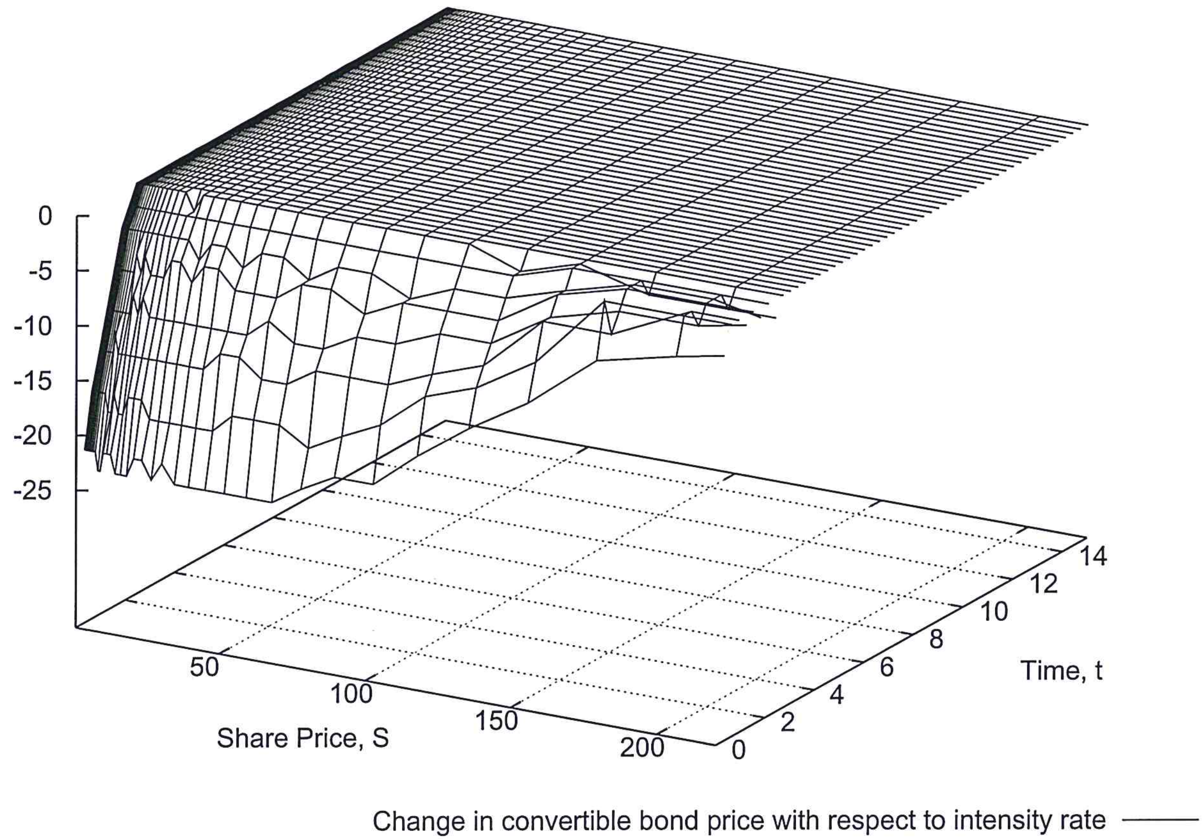


Table 21: The surface $\frac{\partial C}{\partial \delta}$ plotted against equity levels, S and time, t.

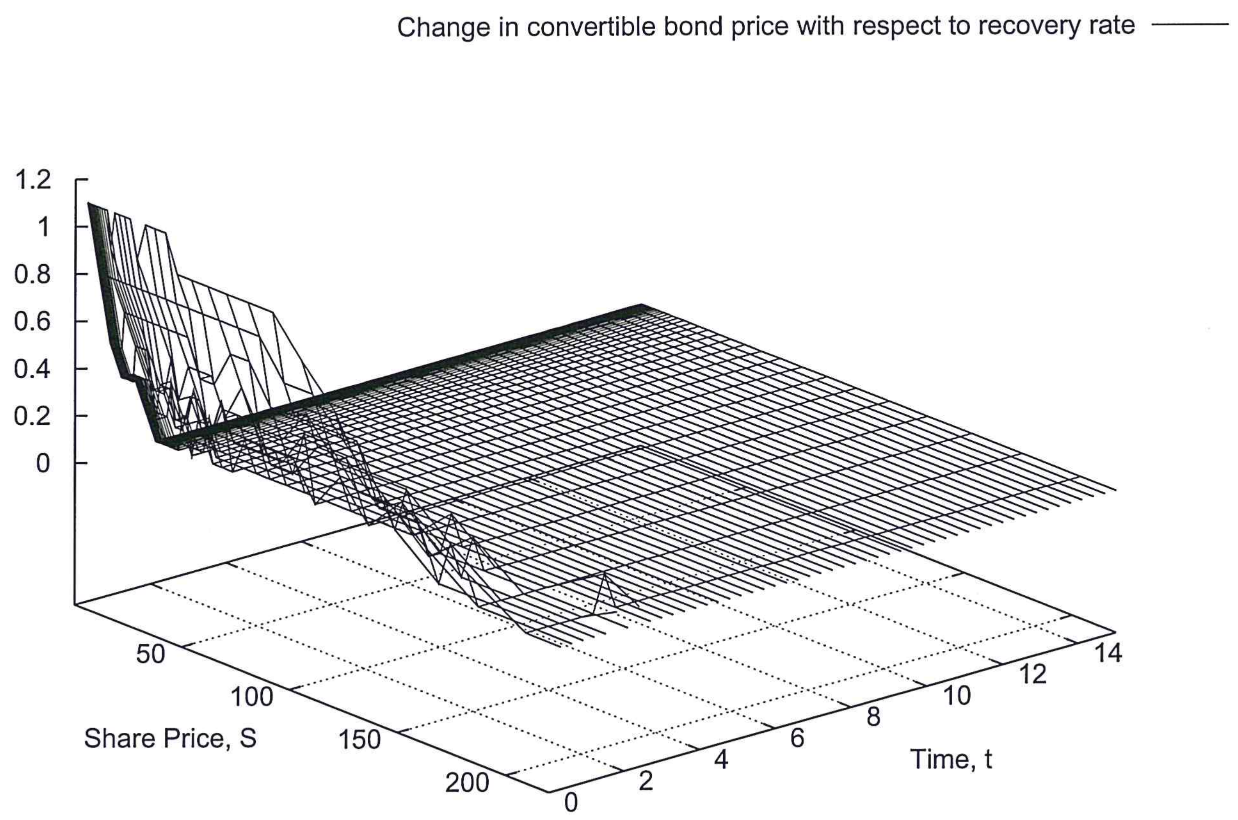


Table 22: The surface $\frac{\partial C}{\partial \sigma_3}$ plotted against equity levels, S and time, t.

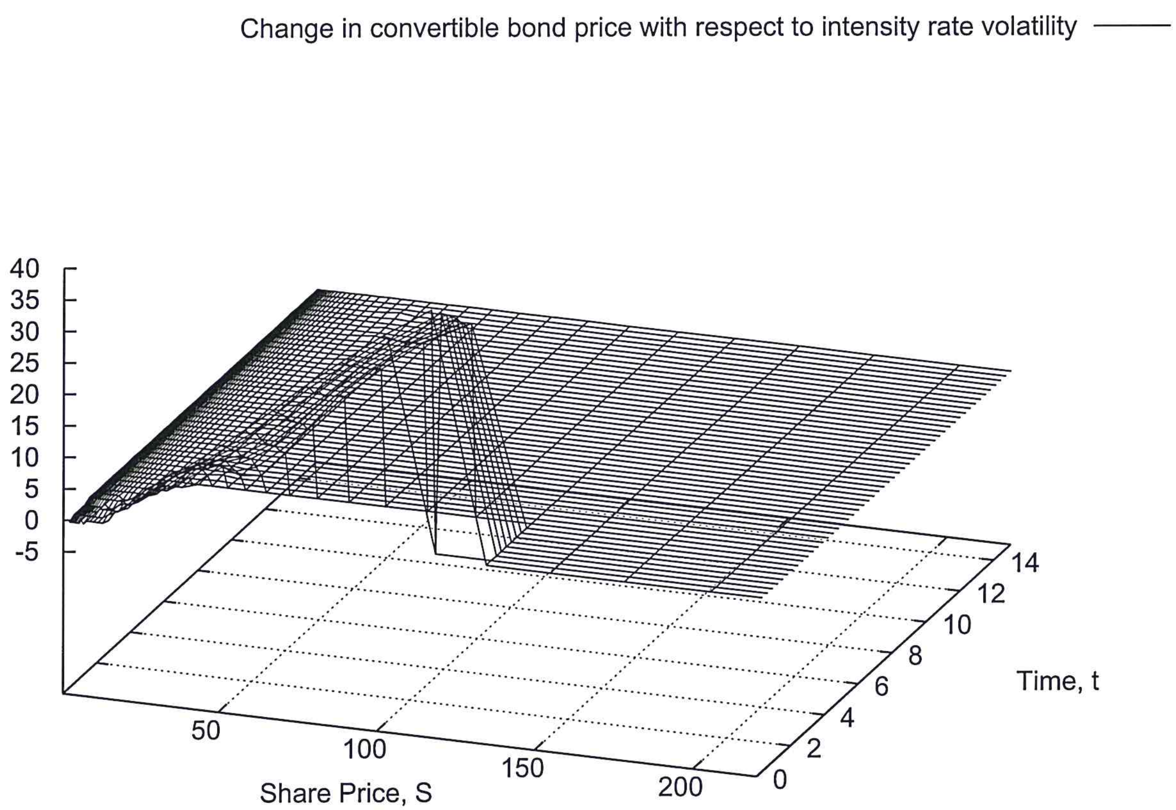


Table 23: The surface $\frac{\partial C}{\partial cp}$ plotted against equity levels, S and time, t.

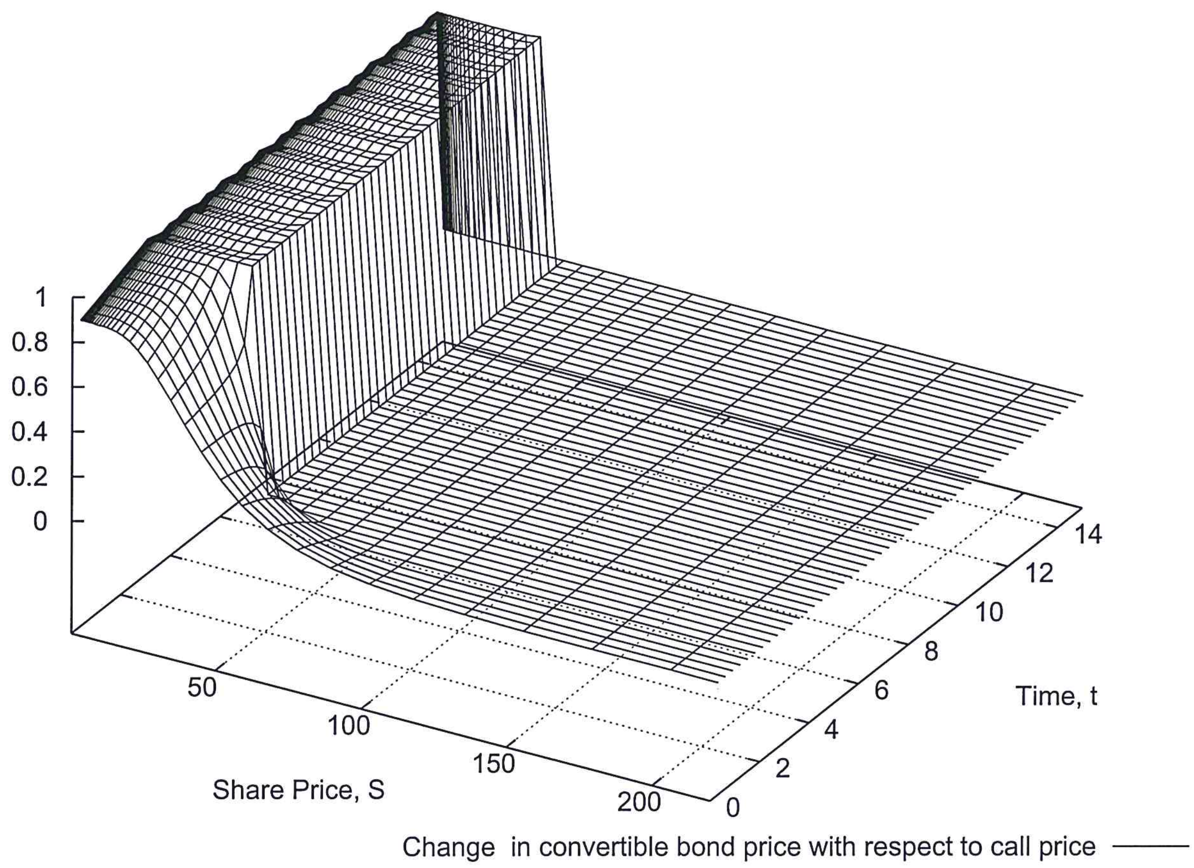
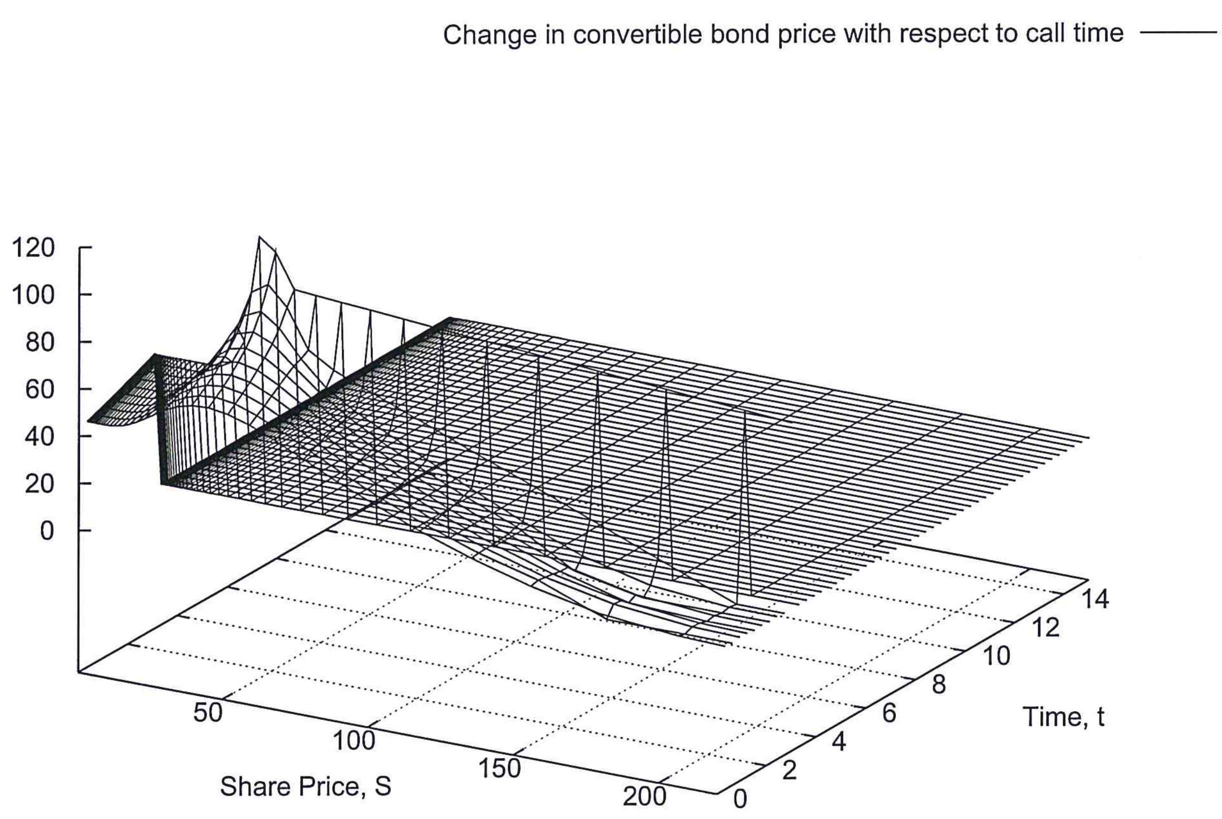


Table 24: The surface $\frac{\partial C}{\partial ct}$ plotted against equity levels, S and time, t.



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A Convertible Bond Contract Features

A.1 Convertible Bond Financing

Convertible bond debt can, to a unique degree, be structured to tailor the needs of the individual borrower. The different contract features outlined below can be used to create an instrument that behaves virtually like straight debt, straight equity or anywhere in between. Original issue discount, put features and high premium or coupon produce a convertible bond that synthesizes debt. Whereas mandatory conversion, call features and low premium or coupons produce a convertible bond that synthesizes equity.

The following subsections describe various common convertible bond contract features. The definitions are based around those in Philips [38], Calamos (1998) [9], Connolly [11] and Goldman Sachs (1994) [41].

A.1.1 Maturity

If the maturity of a convertible bond is increased then all other things being equal the value of the convertible bond is reduced as the bond floor is lowered. Therefore, longer dated convertible bonds are more equity orientated. There is more equity participation for upside moves, but similarly the protection of the floor is further beneath current levels. With certain characteristics, investors will want to pay more for the shorter dated convertible bond because the parity is not far above the bond floor and therefore the instrument is not merely synthesizing equity. There are many occasions, for example in Japan, where a company may issue half its debt as a five year deal and the remainder as a seven year deal.

A.1.2 Coupon

Increasing the coupon *ceteris paribus* causes the bond price to rise and vice versa if the coupon is decreased. The bond floor is reduced if the coupon is lowered and increased if it is raised. In the former case this lowers the convertible bond premium and in the later case it increases the the convertible bond premium. Typically the coupon frequency per year for bonds in the US is two (or semi-annual).

A.1.3 Principal

The principal is the face value of the convertible bond, usually the amount for which the bond can be redeemed at maturity. Although, sometimes a convertible can be redeemed at maturity for an amount greater than the principal of the bond. In the convertible bond literature a large face value is thought to signal management confidence in future returns.

A.1.4 Conversion Ratio

The conversion ratio is the number of shares of the underlying equity for which the convertible bond can be exchanged. It is not uncommon for there to be a conversion schedule which adjusts the conversion ratio during the life of the bond. Beyond this the ratio is usually changed only to account for stock dividends or splits of the underlying shares, so as to preserve the total equity value for which the convertible can be exchanged. Again in the convertible bond literature a high conversion ratio is thought to signal management confidence in the level of returns.

A.1.5 Call Provisions

Convertible bonds almost always have call features which allow the issuer to repurchase *call back* the bonds at a particular price, the *call price*[24]. The feature creates greater flexibility in the capital structure of the

[24] If the issuer calls the bond, the holder has a brief period (stated in the contract), usually 30 days, within which to convert the bond or surrender it. If the bond is surrendered the holder receives the call price in cash.

company[25]. There is often a period after issue called *hard non-call* protection where where the borrower cannot call the bond, this is typically a period of 3 to 5 years[26]. Furthermore, there exist *soft call*[27] (or *stock performance call*) provisions where the issuer may call the bond only if it trades for more than a *trigger* price (or *provisional call level*) say 130% of the conversion price for a period of time, for example 30 days[28]. Similar to this last feature are the *call-bond-lag* and the *call-parity-lag*. In the former case the call is delayed while Bond Lag $\times$ Call Price + Accrued Interest > Pure Bond value. In the later case the call is delayed until Parity > Parity Lag $\times$ Call Price + Accrued Interest. Almost invariably call notices are issued to induce the holder to convert. The call price is often allowed to vary with time and these prices and dates are set out in the *call schedule*. While call prices for coupon bonds generally decrease in steps until maturity, zero-coupon bonds have a call price that accretes at the call accretion rate. The call provision lowers the value of the convertible bond to the holder and reduces the expected life of the instrument.

A.1.6 Put Provisions (or Holder's Option)

Convertible bonds with put features are less common than those with call features particularly in the developed world. The put provision allows the holder to put the bond back to the company at a particular price the *put price* on a given date as described in the *put schedule*. This is desirable when the convertible bond's share price is very low and will lead to earlier redemption when the convertible has no option value. The put provision increases the value of the convertible bond to the holder. Zero-coupon bonds have put prices that grow in time at an accretion rate, usually the same rate as the call accretion rate. On the final date the bond may be redeemed for the principal.

A.1.7 Original Issue Discount Note

Convertible bonds which are not issued at par i.e. 100% but for example, at 75% of par are termed "original issue discount". These deals more closely resemble straight debt. The parity level is lower for an original issue discount note than for the comparable par issue bond.

A.1.8 Exchangeable Bonds

Exchangeable bonds are convertible bond issued by one company for conversion into shares of another company. This may be a useful way for companies to divest interests in other companies. The Italian and Malaysian governments have used this route when selling state interests in companies.

A.1.9 Cross Currency Denominated Convertible Bonds

There are many non-domestic convertible bonds which are convertible into shares in one currency but redeemable into cash in another currency. As the exchange rate is constantly changing through time so therefore is the conversion price. This has a profound effect on the price behavior prior to expiry.

A.1.10 Refix (Reset) Clauses

In the early 1990's Japanese companies began to issue convertible bonds with *refix clauses*. These were designed to make the issues more attractive to the investment community. A refix clause alters the conversion ratio (shares per bond) or conversion price, subject to the share price level on certain days between issue and expiry. Refix clauses add value to the holder of the convertible bond and therefore increase the up front premium paid for the bond. The reset feature protects investors from a decline in the share price. If the

[25] It is generally argued that companies insert such clauses so that they may refinance at lower rates, but this is erroneous, as investors would pay less for the bond with this feature. However, the feature does create greater flexibility in the capital structure of the company.

[26] Hard call protection periods were first introduced after 1970 to protect the investor holding a convertible bond from the issuer calling the bond immediately after issue. It therefore gurantees some the investor at least 2 or 3 years of income.

[27] Soft call protection clauses were introduced after 1982.

[28] The threshold has the interpretation of a barrier and the pre-specified period for which the barrier must be exceeded before it can be called implies this is a Parisian option. Avellaneda and Wu (1999) [2] demonstrate how to price these options in a trinomial tree.

average share price for a predermined period (usually 4 or 5 days) trades below a predetermined threshold price then the conversion price is decreased (conversion ratio increased) subject to a predetermined maximum reduction (increase).

A.1.11 Other Non-Standard Clauses

Convertible bond coupons can be allowed to change with time (step up / step down coupons). The conversion terms can state that the holder will receive a combination of shares and cash instead of just shares. The investor may or may not be entitled to the accrued interest when converting to a common dividend after conversion. Make-whole call provisions (*screw* clauses) force the issuer to pay for the lost interest from the first few years if the convertible is called during this time, even if the investor decides to convert.

A.1.12 Non-Traditional Convertible Bonds

Recent years have seen the introduction of many non-traditional convertibles. The largest class of these is the class of new mandatory convertibles known as equity-linked securities. In the US they have names like DECS, PRIDES, PERCS, ELKS, ACES and YEELDS. Many of these are preferred securities, but some are issued as debt. In Europe Reverse Convertibles have similar features[29]. Equity-linked securities are structured so as to offer investors an enhanced yield in return for a reduced or capped upside potential. They are often convertible only at maturity, and the conversion price is set in a way that depends on the stock price, or the average stock price over some number of days prior to maturity. It is important to note that, because of the mandatory conversion, these securities offer no downside protection. Because of their mandatory nature these new securities are not true convertible bonds and therefore this criterion is used for their exclusion from this survey.

[29]The payoff from a Reverse Convertible or a PERCS (Preferred Equity Redemption Cumulative Stock) is like the payoff from a covered call. The payoff from DECS (Dividend Enhanced Convertible Stock) or PRIDES (Preferred Redeemable Increased Dividend Equity Security) is like the payoff from a long stock position plus a short call option at one strike plus a fraction of a long call option at a higher strike.

B Glossary of Valuation Terms

The conversion price of a convertible bond is generally set at a level above the current equity price for example 5%, 10%, 15% or 20%. The conversion ratio for a convertible bond issued in the domestic currency is given by,

Conversion Ratio = Bond Denomination / Conversion Price,

the bond denomination is the face value or principal amount of the deal. For a convertible bond issued in a foreign currency it is given by,

Conversion Ratio = (Bond Denomination × Fixed FX) / Conversion Price.

The Conversion Price is normally 25 − 30% higher than the market price at issue. It is common to find convertible bonds denominated in one currency but convertible into the currency of the underlying equity. The fixed exchange rate to be used is normally specified in the convertible bond contract or indenture.

The term parity or intrinsic value is often used in the convertible bond industry to describe the value of the underlying share expressed as a percentage of the face value of the bond in the domestic currency. If the bond and equity are in the domestic currency then,

Parity = Share Price / Conversion Price,

or

Parity = Share Price × Conversion Ratio.

If the convertible bond is issued in a foreign currency then,

Parity = (Share Price × Conversion Ratio)/ Bond Principal in Currency of Equity.

The Premium of a convertible bond measures as a percentage how much more an investor is willing to pay for the bond than the shares it converts into. It is defined as,

Premium=(Market Price of convertible bond / Parity)-1.

Other things being equal convertible bonds have a low premium at high share prices and a high premium at low share prices. Also as the maturity date of the convertible bond approaches the value of the embedded option to convert decreases with a corresponding fall in the premium.

The current yield of a convertible bond is the coupon payment on the bond expressed as a percentage of its market price. It is defined as,

Current Yield = Coupon / Market Price of the Convertible Bond.

A convertible bond is said to have yield advantage if the current yield exceeds the dividend yield. The yield advantage can also be thought of as the extent to which the convertible bond trades above the parity. This relationship is expressed as,

Yield Advantage=Current Yield − Dividend Yield.

Break even is defined as the number of years it takes for the premium on the bond to be recouped by the current yield advantage. Simple break even is calculated as,

Break Even = (Bond Price − Parity)/Yield Advantage.