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Pricing of Implied Volatility Derivatives

Abstract

Starting from the description of the real mechanism on the basis of which implied volatilities are actually quoted by traders in option markets, we construct the risk neutral dynamics of the implied volatility to price implied volatility futures and forward starting compound options. These are exotic derivative contracts whose payoff depends on the future implied volatility. In particular, we obtain the risk neutral drift restriction that must be satisfied by each single stochastic implied volatility on the volatility surface invariant both to time to maturity and to (forward) moneyness. Also, we show that the instantaneous spot volatility is a point of this volatility surface, so we can apply our technique to determine its risk neutral dynamics.

Key words: implied volatility, instantaneous spot volatility, futures options, VOLAX, forward starting option, compound option.

JEL Classification: G12, G13.

1 Introduction

In this paper we construct the risk-neutral dynamics for the implied volatility surface in order to price exotic implied volatility contracts. These derivatives allow investors to hedge against changes in volatility. Indeed, as reported in Grunbichler and Longstaff (1996), "investors would now be able to manage their risk in two dimensions, price risk and volatility risk, an opportunity previously out of reach for all but the large and sophisticated option users."

Examples of such derivatives are options and forward contracts on the CBOE's Market Volatility Index (VIX), the VOLAX futures contract listed in the Deutsche Terminborse (DTB) since January 1998 and the forward starting (at the money forward) compound option that has been recently treated in the OTC markets. The VIX is calculated from eight near the money short-term volatilities on the S&P100. The underlying of the VOLAX futures is the VDAX, an implied volatility of a synthetic at the money straddle with three months to expiry. The settlement price of this volatility futures contract is based on the implied volatilities of the DAX index options. A forward starting at the money compound option is an implied volatility option because basically it is an option on the implied volatility of the underlying option. On the expiry date the holder of the option will compare the at the money implied volatility, that makes the Black-Scholes price equal to the futures-option market price, with the implied volatility that makes the Black-Scholes price equal to the strike of the compound option and will decide whether it is worth exercising.

Therefore, as opposed to volatility contracts like variance swaps whose payoff depends on the realized variance of the asset, the payoff of exotic implied volatility derivatives depends on the implied volatility of an option written on the reference asset.

A simplistic approach to price these exotic contracts is to use the forward implied volatility computed from the current term structure of implied volatilities. This approach is flawed from a theoretical point of view. Indeed, as pointed out by Britten-Jones and Neuberger (2000), using the forward implied volatility is equivalent to assuming that the realized variance is simply the square of the implied volatility and this is true only when we have a flat smile.

Even more founded stochastic volatility models that proxy the implied volatility by the instantaneous volatility, see for example Grunbichler and Longstaff (1996), neglect the feature of the contracts whose underlying is the implied volatility. Moreover, as found out in an extensive empirical study by Canina and Figlewski (1993), the implied volatility has virtually no correlation with the future volatility, and it does not incorporate the information contained in recent observed volatility.

In order to price implied volatility derivatives, a model for the implied volatility surface is required. Therefore, similarly to the idea pursued by Brace et al. (1997) with their market model for interest rates options, we reproduce in a formal setting the option traders' practice of quoting implied volatilities in moneyness and time to expiration terms. By convention the implied volatility is then transformed in a price by the Black-Scholes and Black formulas.

These two formulas have been extremely successful despite the numerous flaws and limitations of assumptions on which their derivation relies. In particular, the assumption of constant volatility has always been questionable and from an empirical point of view it is systematically contradicted. In order to overcome this limitation, the financial literature has proposed stochastic volatility models, such as Hull and White (1987), Stein and Stein (1991) and Heston (1993). These models assume different equilibrium mean reverting stochastic processes for the instantaneous variance of the spot price and derive an option formula with two state variables, i.e. the stochastic spot price and the stochastic spot volatility. Unfortunately, these models cannot in most cases be fitted to really quoted market option prices for different strikes and different maturities, see Das and Sundaram (1999). Furthermore, it is required the specification of a functional form for the volatility risk premium and usually this specification is arbitrary and is chosen mainly on the basis of analytical tractability.

For these reasons, Derman and Kani (1998) built a risk neutral model consistent with market option prices for both the term structure and the strike structure of local volatilities, pursuing the same fashion followed to derive the risk neutral model of the term structure of forward interest rates by Heath et al. (1992). The great improvement of the framework developed by Derman and Kani (1998) relies on the virtual possibility of exactly calibrating all option prices quoted in the market during every point in time. However, it is extremely difficult to be implemented owing to the non-markovianity of the modeled local volatilities, which prevents the model itself from being of immediate practical use.

To overcome the intrinsic complexity of the framework developed by Derman and Kani (1998), Schönbucher (1999) and Ledoit and Santa-Clara (1999) derived much more practical implied volatility market models, which represent the counterpart of what Brace et al. (1997) developed when modeling directly risk neutral market dynamics for LIBOR and swap rates. The great breakthrough of the approach followed by Schönbucher (1999) and Ledoit and Santa-Clara (1999) is that for the first time in the financial literature the implied volatility, i.e. the volatility that plugged in the Black-Scholes formula gives an option price matching an actual market option price, is modeled as an input rather than as an output. This original way of modeling is justified by the fact that the implied volatility is a different way to quote an option price, using the Black-Scholes formula as a market convention. Then, quoting different implied volatilities for different strikes and maturities is a simple way to cope with the fallacies of the Black-Scholes assumptions. Given that we accept the view that the presence of a smile helps to take into account deviations of real markets from model assumptions, implied volatilities assume therefore the dignity of independent variables.

In other words, according to this view the option price depends on two underlying state variables: the spot price of the underlying security and the implied volatility. In particular, Schönbucher (1999) models the absolute implied volatility of which he finds the risk neutral drift restriction. On the other hand, in a way closer to the market practice, Ledoit and Santa-Clara (1999) model indirectly the relative (in terms of maturities and moneyness of the options) implied volatilities with respect to absolute option prices in order to find risk neutral drift restrictions similar to the ones calculated by Schönbucher (1999).

In this paper with the aim of pricing implied volatility derivatives we derive directly the risk neutral drift restriction on the relative implied volatilities by making option prices relative in terms of times to maturity and moneyness of the options themselves. In this way, our state variables are the real implied volatilities that are actually quoted by the option market makers, who ignore spot levels and expiration dates when making quotes. For this reason, our model is a market model of implied volatilities close to market practice, because we model just the variables that are quoted in the market.

In the second section we explain how implied volatilities are quoted in option markets by traders and we show how the futures-option prices, where the implied volatilities are plugged, are invariant to time to maturity and relative strike. Then, in the third section we determine the functional form that has to be taken by the drift of the implied volatility to be consistent with risk neutral invariant futures-option prices. Also in the third section, we show that, as time to maturity goes to zero, the at the money implied volatility converges to the instantaneous spot volatility. In this way, we are able to find the risk neutral dynamics of the instantaneous spot volatility that, plugged in the underlying asset dynamics, allows for exotic option pricing consistently with the implied volatility surface evolution. This result is due to the double nature of the instantaneous volatility that both affects the underlying asset dynamics and belongs to the implied volatility surface related to the plain vanilla options quoted on that asset. Finally, in section four we show how our methodology can be simply implemented to price exotic derivatives such as implied volatility futures and forward starting compound options. An interpolation procedure for the implied volatility surface is introduced, allowing for Monte Carlo simulations of the implied volatility dynamics. We compare our results with the prices obtained for the same derivatives when, not coherently with the absence of arbitrage, the square of the at the money implied volatility multiplied by the time to maturity is used as a proxy of the total future variance of the underlying security price.

2 Description of the market mechanism to quote implied volatilities

As said in the introduction, the pricing formulas by Black and Scholes (1973) for options and Black (1976) for futures-options are often used as market conventions. All the parameters of these two option formulas are directly observed in financial markets except the volatility term. As a consequence, according to this accepted market practice, we use the Black-Scholes and the Black formulas to compute the volatility parameter which is implied in quoted option prices, and we refer to the obtained variable as to the implied volatility.

Despite this view of financial practice commonly held in the academia, what happens in financial markets is just the opposite. In other words option traders do not quote option prices directly. Instead, since there is an increasing monotone relation between volatilities and option prices (higher option prices correspond to higher volatilities regardless of being calls or puts), traders behave as if the formulas by Black and Scholes (1973) and Black (1976) were the "true" option pricing formulas and make quotes just of the unknown volatility terms. For this reason, option trading is also commonly referred to as volatility or to be more precise implied volatility trading. This means that the common market convention of using the formulas by Black and Scholes (1973) and Black (1976) produces the birth of a new market variable, the market implied volatility, which is not necessarily related to the instantaneous volatility or to the future total variance of the underlying security price. Therefore, we believe that in order to produce what we can call a market model for implied volatilities close to market practice, we should develop a methodology which takes correctly into account how traders actually quote market implied volatilities.

In particular, when quoting market implied volatilities, option traders ignore completely both the absolute level of the spot and the maturity date. In fact, option traders quote implied volatilities in moneyness (with respect to the futures price) and time to expiration terms.

Table 1 can be of help to understand the real mechanism that actually happens when market implied volatilities are quoted. Traders always quote an implied volatility for every single time to maturity and every single strike value relatively to the futures price at that particular expiration.

[INSERT HERE TABLE 1]

For example according to Table 1 - referring to hypothetically quoted implied volatilities on the DAX according to the market practice adopted by traders - the implied volatility for an option with expiration in one year and current absolute strike level at 5% less than the one year futures price (whatever it is) is quoted 32.00%.

In this way, it is easy to see that as time goes by the trader is not always quoting the implied volatility of the same option, because every day the one year futures prices in the previous example changes. Therefore, the strike price that was originally at 5% less than the earlier futures price is expressed as a different percentage, not necessarily equal to 5%, of the new futures price. Moreover, as time is approaching to expiration the one year implied volatilities become no longer applicable to the option which had one year time to expiration when it was initially priced.

3 The model

In the previous section we have seen that each market implied volatility is quoted in such a way that it remains invariant to changes of the futures price and to changes of the time to maturity. In other words, an implied volatility is quoted for each time to maturity and each relative futures price with the same time to maturity. Practitioners typically use the formula by Black (1976) to turn the quoted implied volatilities in the corresponding option prices.

In order to take into account this quoting mechanism, we will construct a model directly for $\Sigma_t(\tau, m)$, the implied volatility at time t of a call option with time to maturity τ and moneyness m. $\Sigma_t(\tau, m)$ as a function of τ and m is commonly referred to as the implied volatility surface. If we fix $m = \overline{m}$, the function $\Sigma_t(\tau, \overline{m})$ gives the volatility term structure. If we fix $\tau = \overline{\tau}$, the function $\Sigma_t(\overline{\tau}, m)$ gives the volatility strike structure or smile curve.

Since traders quote in percentage terms of futures prices[2], we will consider futures-options where both the option and the futures contracts share the same time to maturity. Moreover, we take into account the futures-style daily marking-to-market mechanism, so that the price of futures-options is simply the accrued value of standard options. This feature allows us to bypass the difficult task of determining the appropriate dividend yield for the index[3].

Duffie and Stanton (1992) and Liu (1990) show that, when dealing with futures-style options under the assumption of deterministic interest rates, options can be priced thanks to an adjustment of the formula by Black (1976). Therefore, in order to emphasize the dependence of the pricing formula on relative parameters, time to maturity τ and moneyness m, the futures-style option price $C_t(\tau, m, \Sigma_t(\tau, m))$ can be written as:

$$\begin{aligned} C_t(\tau, m, \Sigma_t(\tau, m)) &= : F_t(\tau) N(d_1) - mF_t(\tau) N(d_2) \\ &= F_t(\tau) \Psi_t[\tau, m, \Sigma_t(\tau, m)], \end{aligned} \tag{1}$$

where

$$\begin{aligned} d_1 &= \frac{-\ln m + \frac{1}{2}(\Sigma_t(\tau, m))^2 \tau}{\Sigma_t(\tau, m)\sqrt{\tau}}, \\ d_2 &= d_1 - \Sigma_t(\tau, m)\sqrt{\tau}; \end{aligned}$$

$F_t(\tau)$ is the futures price at time t relative to a hypothetical futures contract with constant *time to maturity* τ, i.e. without time decay; m is the *relative strike* (moneyness) expressed

[2]We assume that the interest rate r is constant, so that futures prices and forward prices are equal. In the rest of the paper we use the forward and futures terms as synonymous.

[3]Note that in several markets index options are effectively futures-options where the futures-style margining system is employed. Examples are the options dealt on commodities at the CME or the Hang Seng Index options mentioned by Duan and Zhang (2001).

in terms of $F_t(\tau)$; $\Psi_t[\tau, m, \Sigma_t(\tau, m)] = N(d_1) - mN(d_1)$ is the relative futures-option price with constant time to maturity; $\Sigma_t(\tau, m)$ is the implied volatility as we have defined above. Sometimes, we will use the short-hand notation F_t for $F_t(\tau)$, Σ_t for $\Sigma_t(\tau, m)$ and C_t for $C_t(\tau, m, \Sigma_t(\tau, m))$.

Note that the price of a European call option without futures-style margining can be obtained by discounting the above formula.

If we denote by $\hat{F}_t = \hat{F}_t(T)$ the futures price at time t relative to a futures contract with constant *time of maturity* T, by $\hat{\Sigma}_t = \hat{\Sigma}_t(T, X)$ the market implied volatility at time t for an option expiring at time T and *absolute strike* X and by $\hat{C}_t = \hat{C}_t(T, X, \hat{\Sigma}_t)$ the price of the corresponding futures-option, we can relate them to F_t, Σ_t and C_t as follows:

$$\begin{aligned}
\hat{F}_t(T) &= F_t(T-t) && (2)\\
\hat{\Sigma}_t(T, X) &= \Sigma_t\left(T-t, X/\hat{F}_t(T)\right) && (3)\\
\hat{C}_t(T, X, \hat{\Sigma}_t) &= C_t\left(T-t, \frac{X}{\hat{F}_t(T)}, \hat{\Sigma}_t\right)\\
&= \hat{F}_t(T)\,\Phi_t\left[T, \frac{X}{\hat{F}_t(T)}, \hat{\Sigma}_t(T, X)\right], && (4)
\end{aligned}$$

where

$$\Phi_t\left[T, \frac{X}{\hat{F}_t(T)}, \hat{\Sigma}_t(T, X)\right] = N\left(\hat{d}_1\right) - \frac{X}{\hat{F}_t(T)} N\left(\hat{d}_2\right)$$

is the relative futures-option price with constant time of maturity and

$$\begin{aligned}
\hat{d}_1 &= \frac{-\ln\frac{X}{\hat{F}_t} + \frac{1}{2}\left(\hat{\Sigma}_t(T, X)\right)^2 (T-t)}{\hat{\Sigma}_t(T, X)\sqrt{T-t}},\\
\hat{d}_2 &= \hat{d}_1 - \hat{\Sigma}_t(T, X)\sqrt{T-t}.
\end{aligned}$$

The reason that makes preferable to use expression (1) instead of (4) is that the implied volatility actually quoted by option traders is expressed in terms of fixed time to maturity and moneyness. By using equations (2) and (4) we can relate the absolute prices and the relative prices, since they are equal at time t, but some differences will arise when taking their dynamics, as it happens in interest rate modeling as we switch from the Heath-Jarrow-Morton fixed time of maturity parametrization to the Brace-Gatarek-Musiela fixed time to maturity parametrization and vice versa.

Let us now introduce the hypothesis relative to our state variables, represented by the asset price and the complete set of implied volatilities for different fixed levels of moneyness and time to maturity. In particular, for sake of simplicity, we assume to work in a non-dividend framework and we assume that the price dynamics of the asset underlying both to the futures contract and to the futures-option is described by the following SDE:

$$dS_t = rS_t dt + \sigma_t S_t dB_t^S, \tag{5}$$

where σ_t is the stochastic instantaneous spot volatility, B_t^S is the Brownian motion that rules the uncertainty of the spot price process, r is the risk-free rate.

Also, we assume that the set of market implied volatilities for all different fixed times to maturity τ and fixed relative strike m are described by the following set of SDE's:

$$d\Sigma_t(\tau, m) = \alpha_t(\tau, m, \Sigma_t)dt + \beta_t(\tau, m, \Sigma_t)dB_t^{\Sigma}, \tag{6}$$

where $\alpha_t(\tau, m, \Sigma_t)$ and $\beta_t(\tau, m, \Sigma_t)$ are respectively the drift and diffusion terms, possibly stochastic, of the market implied volatility; $B_t^{\Sigma} := B_t^{\Sigma_t(\tau,m)}$ is the Brownian motion that rules the uncertainty of each market implied volatility. Again, sometimes we will use the short notation α_t for $\alpha_t(\tau, m, \Sigma_t)$ and β_t for $\beta_t(\tau, m, \Sigma_t)$. We assume that the drift and the diffusion terms of the market implied volatility $\hat{\Sigma}_t$ at time t for an option expiring at time T and absolute strike X are respectively $\hat{\alpha}_t(T, X, \hat{\Sigma}_t)$ and $\hat{\beta}_t(T, X, \hat{\Sigma}_t)$.

Moreover, we allow for a correlation between the Brownian motion that rules the uncertainty of the market implied volatility and the Brownian motion that rules the uncertainty of the underlying security price, i.e.:

$$\tilde{\mathbb{E}}_t \left(dB_t^S dB_t^{\Sigma} \right) = \rho_t^{S,\Sigma}(\tau, m, \Sigma_t) dt =: \rho_t dt.$$

Since when quoting an implied volatility, traders disregard completely the absolute level of the underlying security price, we could assume zero correlation between the implied volatilities and the stock price without any loss of generality. However, some empirical evidence seems to suggest a negative correlation[4].

We need to make some important remarks on the dynamics in (6). First of all, we are assuming that the process defined above is ruled under the martingale measure. For this reason, the drift term α_t cannot be freely chosen. The idea we will pursue to obtain the expression of this term under the risk-neutral measure is to find the price process for a derivative contract characterized by fixed time to maturity and fixed moneyness and then to derive the no-arbitrage condition that it has to satisfy. Similarly to the procedure that allows to find the drift of forward rates in the Heath-Jarrow-Morton model, we will obtain an expression for α_t. Secondly, the stochastic component could be generalized to a multifactor model, eventually with jumps, such as in Cont et al. (2002). Finally, in specifying the process for different implied volatilities in a multifactor framework and not in a single factor environment as we have done above, we should ensure that the entire volatility surface moves consistently. In particular, we should guarantee that call and put spreads have a positive value and that long term options are more valued than short time options, see Merton (1973). Basically these conditions concern the steepness of the smile curve and term structure of volatilities. Unfortunately, this requirement translates into a rather complex set of conditions that should be imposed on all the risk neutral implied volatilities considered altogether and require to be set via latent variables (not directly quoted in option markets) that, borrowing from the interest rate terminology, are usually called forward implied volatilities[5]. However, we leave the treatment of this important aspect for further research.

As a consequence of the assumption on the stock price dynamics, we have the well known result that, under the risk neutral measure, the futures price relative to a futures contract with expiration at fixed time T is a martingale, see Duffie and Stanton (1992). So its dynamics are given by the following SDE:

$$d\hat{F}_t(T) = \sigma_t \hat{F}_t(T)\, dB_t^S. \tag{7}$$

Since we are interested in quantities expressed in relative terms, we can use the relationship that holds at time t, $F_t(\tau)=\hat{F}_t(t+\tau)$, in order to obtain the dynamics of $F_t(\tau)$,

[4] In fact, we remind that rather often above all in equity option markets there seems to be a negative correlation, since it usually happens that a decrease of the equity market is followed by an increase of implied volatility levels and viceversa. See Dumas et al. (1998).

[5] A term structure of volatility is well behaved if it is ensured that forward implied volatilities are always positive. The forward implied volatility is defined by the expression (22) given in section 4.

by following the slight variation of Itô's lemma suggested by Bjork (1998) at page 272:

$$dF_t(\tau) = \left.\frac{\partial \hat{F}_t(T)}{\partial T}\right|_{T=t+\tau} dt + d\hat{F}_t(T).$$

Given the assumption of constant interest rates, since the futures price is $\hat{F}_t(T) = S_t e^{r(T-t)}$, if we calculate the partial derivative with respect to T, we obtain the risk neutral dynamics of the futures price $F_t(\tau)$:

$$dF_t(\tau) = rF_t(\tau)\,dt + \sigma_t F_t(\tau)\,dB_t^S. \tag{8}$$

The result in (8) implies that in a risk neutral framework any futures price with fixed time to maturity grows at the risk free rate.

In order to obtain the relative option price dynamics from the standard option price dynamics we need a generalization of Itô's lemma, i.e. the Itô-Venttsel formula that can be found in Appendix A. By means of this tool, we are able to take into account the fact that, switching from the absolute to the relative parametrization, the absolute parameters T and X turn into random variables T_t and X_t, expressed in terms of the relative pair τ (time to maturity) and m (forward moneyness), i.e.:

$$\begin{aligned} T_t &= t+\tau, & dT_t &= dt; \\ X_t &= mF_t(\tau), & dX_t &= m\sigma_t\hat{F}_t(T)\,dB_t^S. \end{aligned} \tag{9}$$

Since at time t the following relation holds:

$$C_t(\tau, m, \Sigma_t) = \hat{C}_t\left(t+\tau, mF_t(T_t - t), \hat{\Sigma}_t\right) = \hat{C}_t\left(t+\tau, m\hat{F}_t(T_t), \hat{\Sigma}_t\right),$$

by applying the Itô-Venttsel formula, we get the following relationship between standard and relative option price dynamics:

$$\begin{aligned} dC_t(\tau, m, \Sigma_t) \;=\; & \Bigg\{ d\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) + \frac{\partial}{\partial T}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) dT_t \\ & + \frac{\partial}{\partial X}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) dX_t + \frac{1}{2}\frac{\partial^2}{\partial X^2}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)\langle dX_t\rangle \\ & + \frac{\partial}{\partial X}\left(\frac{\partial \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial \hat{F}}\sigma_t\hat{F}_t(T_t)\right) m\sigma_t\hat{F}_t(T_t)\,dt \\ & + \hat{\rho}_t(T_t, X_t, \hat{\Sigma}_t)\frac{\partial}{\partial X}\left(\frac{\partial \hat{C}_t(T_t, X_t, \hat{\Sigma}_t)}{\partial \hat{\Sigma}}\hat{\beta}_t\left(T_t, X_t, \hat{\Sigma}_t\right)\right) \cdot \\ & \quad m\sigma_t\hat{F}_t(T_t)\,dt \Bigg\}\Bigg|_{\substack{T_t = t+\tau \\ X_t = mF_t(\tau)}} \end{aligned} \tag{10}$$

Note that the last two terms in the previous expression arise from the application of the Itô-Venttsel formula, and would not appear in a standard Itô's derivation. After replacing the partial derivatives appearing in expression (10) with their expressions given in Appendix B, we obtain the expected value of the relative option price dynamics under the risk neutral measure:

$$
\begin{aligned}
\tilde{\mathbb{E}}_t \left[dC_t\left(\tau, m, \Sigma_t\right)\right] = \quad & \Big\{ rC_t + n\left(d_1\right) F_t \tfrac{\Sigma_t}{2\sqrt{\tau}} - \tfrac{n(d_1) F_t \sigma_t^2}{2\Sigma_t \sqrt{\tau}} \\
& + \rho_t \beta_t \sigma_t d_1 n\left(d_1\right) F_t + n\left(d_1\right) F_t \sqrt{\tau} \tfrac{\partial \Sigma_t}{\partial \tau} \\
& + m \rho_t \sigma_t n\left(d_1\right) F_t \sqrt{\tau} \tfrac{\partial \beta_t}{\partial m} \\
& + \Big[m \sigma_t n\left(d_1\right) F_t \sqrt{\tau} \Big(\tfrac{\sigma_t}{\Sigma_t} \tfrac{d_1}{\sqrt{\tau}} + \\
& + \tfrac{1}{2} \rho_t \beta_t d_1 d_2 \Big) \Big] \tfrac{\partial \Sigma_t}{\partial m} + \\
& + \tfrac{m^2 \sigma_t^2}{2} \tfrac{1}{\Sigma_t} d_1 d_2 n\left(d_1\right) F_t \sqrt{\tau} \left(\tfrac{\partial \Sigma_t}{\partial m}\right)^2 \\
& + \tfrac{m^2 \sigma_t^2}{2} n\left(d_1\right) F_t \sqrt{\tau} \tfrac{\partial^2 \Sigma_t}{\partial m^2} \Big\} \, dt,
\end{aligned}
\tag{11}
$$

where $n(x) = \exp\left(-x^2/2\right)/\sqrt{2\pi}$. The first term appearing in the r.h.s. of the previous expression shows that the growth rate of the "relative" option price equals the growth rate of futures prices without time decay; to obtain the whole expected dynamics of relative futures-option prices, according to expression (11), we shall correct this factor. In particular, the behavior of relative option prices depends on the deformations of the implied volatility surface via the partial derivatives $\partial\Sigma_t/\partial\tau$, $\partial\Sigma_t/\partial m$ and $\partial^2\Sigma_t/\partial m^2$. The meaning of these partial derivatives will be exploited in the following section.

3.1 Risk neutral drift restriction on each single market implied volatility

In this section we are going to determine an expression for the risk neutral drift $\alpha_t(\tau, m, \Sigma_t)$ of each market implied volatility. In the previous section we have found the risk neutral drift of each futures-option price invariant to time to maturity and relative strike. On the other hand, we observe that by means of expression (1), we can express the futures-option price in terms of the stochastic market implied volatility $\Sigma_t(\tau, m)$ with dynamics given by (6), and of the invariant futures price $F_t(\tau)$ with dynamics given by (8).

Therefore, a straightforward application of the two dimensional Itô's Lemma leads to obtain the drift component of the stochastic process $C_t(\tau, m, \Sigma_t)$:

$$
\begin{aligned}
& rF_t \Psi_t\left[\tau, m, \Sigma_t(\tau, m)\right] + F_t \alpha_t \frac{\partial \Psi_t\left[\tau, m, \Sigma_t(\tau, m)\right]}{\partial \Sigma} + \\
& \frac{1}{2} F_t \beta_t^2 \frac{\partial^2 \Psi_t\left[\tau, m, \Sigma_t(\tau, m)\right]}{\partial \Sigma^2} + \sigma_t F_t \beta_t \rho_{t,\tau} \frac{\partial \Psi_t\left[\tau, m, \Sigma_t(\tau, m)\right]}{\partial \Sigma},
\end{aligned}
\tag{12}
$$

where the partial derivatives of $\Psi_t := \Psi_t\left[\tau, m, \Sigma_t(\tau, m)\right]$ with respect to Σ_t are given by:

$$
\begin{aligned}
\frac{\partial \Psi_t}{\partial \Sigma} &= n\left(d_1\right)\sqrt{\tau}; \\
\frac{\partial^2 \Psi_t}{\partial \Sigma^2} &= n\left(d_1\right)\frac{d_1 d_2}{\Sigma_t}\sqrt{\tau}.
\end{aligned}
$$

We have now all that we need in order to find the risk neutral drift restriction of each stochastic market implied volatility. Indeed, if we equate expression (12) to the drift in expression (11), and we solve with respect to the unknown drift $\alpha_t(\tau, m, \Sigma_t)$ we obtain Theorem 1, indeed the main contribution of the present paper.

Theorem 1 *Let $C_t(\tau, m, \Sigma_t)$ be a futures-option price with fixed relative strike m and fixed time to maturity τ, written on $F_t(\tau)$. The risk neutral drift $\alpha_t^*(\tau, m, \Sigma_t)$ of the* *corresponding stochastic market implied volatility $\Sigma_t(\tau, m)$ is given by:*

$$
\begin{aligned}
\alpha_t^*(\tau, m, \Sigma_t) &= \frac{1}{2\tau\Sigma_t(\tau, m)}\left[(\Sigma_t(\tau, m))^2 - \sigma_t^2\right] \\
&+\rho_t(\tau, m, \Sigma_t)\beta_t(\tau, m, \Sigma_t)\sigma_t\left(\frac{d_1}{\sqrt{\tau}} - 1\right) - \frac{d_1 d_2}{2\Sigma_t(\tau, m)}(\beta_t(\tau, m, \Sigma_t))^2 \\
&+m\rho_t(\tau, m, \Sigma_t)\sigma_t\frac{\partial\beta_t(\tau, m, \Sigma_t)}{\partial m} + \frac{\partial\Sigma_t(\tau, m)}{\partial\tau} \\
&+m\sigma_t\left(\frac{\sigma_t}{\Sigma_t(\tau, m)}\frac{d_1}{\sqrt{\tau}} + \rho_t(\tau, m, \Sigma_t)\beta_t(\tau, m, \Sigma_t)d_1 d_2\right)\frac{\partial\Sigma_t(\tau, m)}{\partial m} \\
&+\frac{m^2\sigma_t^2}{2\Sigma_t(\tau, m)}d_1 d_2\left(\frac{\partial\Sigma_t(\tau, m)}{\partial m}\right)^2 + \frac{m^2\sigma_t^2}{2}\frac{\partial^2\Sigma_t(\tau, m)}{\partial m^2}, \qquad (13)
\end{aligned}
$$

The risk neutral dynamics of quoted market implied volatilities are given by the following SDE:

$$
d\Sigma_t(\tau, m) = \alpha_t^*(\tau, m, \Sigma_t)dt + \beta_t(\tau, m, \Sigma_t)dB_t^{\Sigma}. \qquad (14)
$$

Remark that in the drift there appears the instantaneous (stochastic) volatility σ_t as well. So, in order to make expression (13) usable we should find the risk-neutral process for σ_t. We will discuss this important aspect in the next subsection.

We can notice that, the r.h.s. of (13) resembles the no-arbitrage restriction (3.7) obtained in Schönbucher (1999). The two expressions for the risk neutral drift of the stochastic market implied volatility differ one from another because of the different approaches followed to derive risk neutral implied volatilities. In fact, Schönbucher (1999) models the implied volatilities related to the absolute strike level and the time of maturity.

Instead, we impose directly the no-arbitrage condition by employing the process of the implied volatilities that are invariant to time to maturity and relative strike. The corrections to the derivation by Schönbucher (1999) arise from the relative parametrization introduced via the Itô-Venttsel formula. Our approach bears the dependence of the implied volatility dynamics on the current shape of the implied volatility surface and its evolution via the partial derivatives $\partial\Sigma_t/\partial\tau$, $\partial\Sigma_t/\partial m$ and $\partial^2\Sigma_t/\partial m^2$. In fact, as time goes by, the shocks on the implied volatility surface are reflected in changes of its term structure slope $\partial\Sigma_t/\partial\tau$ and in the smile slope $\partial\Sigma_t/\partial m$ and convexity $\partial^2\Sigma_t/\partial m^2$. The good news is that, owing to this result, the model is able to take into account the volatility information contained in option prices, information which is not available from the underlying security price. Unfortunately, the previous derivatives cannot be computed directly from a known formula, since there are no financial conditions to be imposed on the functional form of the implied volatility surface. The computational problem introduced by those derivatives will be faced in section (4.1).

Although our approach is fully consistent with Ledoit and Santa-Clara (1999), by modeling relative option prices $C(\tau, m, \Sigma_t)$ we emphasize directly the role played by relative implied volatilities on financial markets. Indeed, even Ledoit and Santa-Clara (1999) start with the actually quoted implied volatilities that are invariant to time to maturity and relative strike. However, in a second moment they change the dynamics of the relative implied volatilities invariant to time to maturity to the ones of the equivalent absolute implied volatilities invariant to time of maturity, reproducing the methodology already developed by Schönbucher (1999). In other words, they consider again the process of absolute implied volatilities because the arbitrage restriction they use is imposed on the price process for contracts with fixed absolute strike and fixed time of maturity, i.e. on $\hat{C}(T, X, \hat{\Sigma}_t)$.

One of the greatest advantages of modeling the real market implied volatilities, quoted according to the mechanism described in the second section, is that we can capture the floating nature of volatility smiles. In fact, since option traders quote implied volatilities with respect to relative futures levels, if the futures price changes from one day to another, the following day option traders will be quoting a new implied volatility relatively to a new futures price, determining a kind of migration of the previously quoted implied volatility with respect to absolute strike levels[6]. This migration of market implied volatilities is allowed in our model, because by replicating option traders' behavior we disregard the absolute level at which the futures price is trading. The migration of each market implied volatility along with the futures price is modeled through the correlation between the Brownian motions that drive the market implied volatility and the spot process respectively.

Also, we would like to notice that our model is consistent with the local volatility framework developed by Derman and Kani (1998). In fact, they start from the market implied volatilities and then from the correspondent option prices they find the local volatilities via the forward Black-Scholes equation. In this way, given any future local volatility, the risk neutral dynamics of which are given by Derman and Kani (1998), it is always possible to determine again via the forward Black-Scholes equation the futures-option prices and consequently the future implied volatilities. In our approach we model directly market implied volatilities with the special feature of considering implied volatilities invariant to time to maturity and relative strike.

3.2 At the money market implied volatility and instantaneous spot volatility

Given the risk neutral dynamics of each market implied volatility, we now focus on at the money market implied volatilities. In particular, by imposing $m = 1$ in equation (13), the risk neutral drifts for the set of at the money market implied volatilities spanned for different maturities are:

$$\begin{aligned}
\alpha_t^*(\tau, 1, \Sigma_t(\tau, 1)) &= \frac{1}{2\tau\Sigma_t(\tau,1)}\left((\Sigma_t(\tau,1))^2 - \sigma_t^2\right) \\
&+\rho_t(\tau,1,\Sigma_t)\beta_t(\tau,1,\Sigma_t)\sigma_t\left(\frac{\Sigma_t(\tau,1)}{2} - 1\right) \\
&-\frac{\Sigma_t(\tau,1)\,\tau\,(\beta_t(\tau,1,\Sigma_t))^2}{8} \\
&+\rho_t(\tau,1,\Sigma_t)\sigma_t \left.\frac{\partial\beta_t(\tau,m,\Sigma_t)}{\partial m}\right|_{m=1} + \frac{\partial\Sigma_t(\tau,1)}{\partial\tau} \\
&+\sigma_t\left[\frac{\sigma_t}{2} + \frac{\rho_t(\tau,1,\Sigma_t)\beta_t(\tau,1,\Sigma_t)(\Sigma_t(\tau,1))^2\tau}{4}\right]\left.\frac{\partial\Sigma_t(\tau,m)}{\partial m}\right|_{m=1} \\
&+\frac{\sigma_t^2}{8}\Sigma_t(\tau,1)\,\tau\left(\left.\frac{\partial\Sigma_t(\tau,m)}{\partial m}\right|_{m=1}\right)^2 + \frac{\sigma_t^2}{2}\left.\frac{\partial^2\Sigma_t(\tau,m)}{\partial m^2}\right|_{m=1}. \qquad (15)
\end{aligned}$$

We would like to stress the fact that if we set $\beta_t(\tau, 1, \Sigma_t) = 0$, we recover the dynamics for the square of the implied volatility consistently with a time dependent (but not stochastic) spot volatility framework, where the square of the at the money implied volatility is equal to the total future variance of the underlying spot price[7]. Indeed, in the

[6] For further details on the floating nature of volatility smiles see Rebonato (1999) in chapter 4.

[7] On this point see section 3.4.2 in Schonbucher (1999).

Black model, if σ_t is deterministic but time dependent, we have that Σ_t^2 is not stochastic, independent on m and equal to $\frac{1}{\tau}\int_t^{t+\tau}\sigma_u^2 du$. In such a case, if we write the differential equation for Σ_t^2, we find exactly expression (15), where $\beta_t(\tau, 1, \Sigma_t) = 0$.

It is interesting to see what happens to the stochastic at the money implied volatility if we let the time to maturity τ converge to zero.

As proven in Appendix C, the absence of maturity arbitrage requires the instantaneous market at the money implied volatility to be equal to the instantaneous spot volatility, i.e. we must have[8]:

$$\sigma_t = \lim_{\tau\to 0} \Sigma_t(\tau, 1).$$

The previous result, joint with expression (15), allows us to obtain the risk neutral dynamics of the instantaneous spot volatility. Owing to the fact that we are working on a subset of implied volatilities, i.e. the volatilities that refer to at the money options, we can just move along the time dimension to get the dynamics of the instantaneous volatility. Therefore, we can proceed analogously to Brace et al. (1997) when they find the dynamics for the short interest rate. Thus, the instantaneous spot volatility has dynamics given by:

$$d\sigma_t = d\Sigma_t(\tau, 1)|_{\tau=0}.$$

By means of expression (15), the drift of the istantaneous spot volatility can be easily specified, i.e.:

$$\begin{aligned}
\lim_{\tau\to 0^+} \alpha_t^*(\tau, 1, \Sigma_t(\tau, 1)) = & \left\{ \frac{1}{2\sigma_t} 2\Sigma_t(\tau, m) \frac{\partial \Sigma_t(\tau, m)}{\partial \tau} \right. \\
& + \rho_t(\tau, m, \sigma_t)\beta_t(\tau, m, \sigma_t)\left(\frac{\sigma_t}{2} - 1\right) \\
& + \rho_t(\tau, m, \sigma_t)\sigma_t \frac{\partial \beta_t(\tau, m, \sigma_t)}{\partial m} + \frac{\partial \Sigma_t(\tau, m)}{\partial \tau} \\
& \left. + \frac{\sigma_t^2}{2}\frac{\partial \Sigma_t(\tau, m)}{\partial m} + \frac{\sigma_t^2}{2}\frac{\partial^2 \Sigma_t(\tau, m)}{\partial m^2} \right\}\Bigg|_{\tau=0, m=1}.
\end{aligned} \tag{16}$$

Reordering terms, we obtain:

$$\begin{aligned}
d\sigma_t = & \left[-\frac{1}{2}\rho_t(0, 1, \sigma_t)\beta_t(0, 1, \sigma_t)\sigma_t^2 + \rho_t(0, 1, \sigma_t)\sigma_t \left.\frac{\partial \beta_t(0, m, \sigma_t)}{\partial m}\right|_{m=1} \right. \\
& \left. + 2\left.\frac{\partial \Sigma_t(\tau, 1)}{\partial \tau}\right|_{\tau=0} + \frac{\sigma_t^2}{2}\left(\left.\frac{\partial \Sigma_t(0, m)}{\partial m}\right|_{m=1} + \left.\frac{\partial^2 \Sigma_t(0, m)}{\partial m^2}\right|_{m=1}\right)\right] dt \\
& + \beta_t(0, 1, \sigma_t) dB_t^{\Sigma}.
\end{aligned} \tag{17}$$

As pointed out by Britten-Jones and Neuberger (2000), when $\beta_t(0, 1, \sigma_t) = 0$, the implied volatility has to be equal to the total realized volatility and we should find a flat smile, so $\partial\Sigma_t(0, 1)/\partial m = \partial^2\Sigma_t(0, 1)/\partial m^2 = 0$. Therefore, in a non-stochastic scenario the drift of the instantaneous spot volatility depends on the shape of the implied volatility surface only via the slope of the term structure of at the money implied volatilities.

The dynamics obtained for the instantaneous volatility play a key role in our model, giving a circular interpretation of financial market mechanisms which is not allowed by models conceived in a Black-Scholes environment. In fact, in classical financial models, the uncertainty propagates unilaterally from underlying security markets towards derivative

[8] By means of a different procedure based on asymptotic arguments, Ledoit and Santa-Clara (1999) have proven the same result.

markets. On the contrary, in our framework the stochastic dynamics of the instantaneous volatility affect the underlying asset dynamics following a behavior consistent with the evolution of the implied volatility surface it belongs to. In other words, we capture a feedback effect of option markets on underlying prices that cannot be measured even by well-known stochastic volatility models (e.g., Heston (1993)), where the instantaneous volatility is treated as a fundamental variable devoid of the linking feature we have just pointed out.

As Ledoit and Santa-Clara (1999) have suggested, the joint risk neutral diffusion of the underlying security price, described by the SDE (5), and of the instantaneous spot volatility, described by the SDE (17), can be used to price any exotic derivative on the spot price only, via Monte Carlo simulations.

4 Valuation of implied volatility derivatives

In this final section we are interested in pricing two kinds of derivative securities depending on the at the money market implied volatility: a futures contract on an at the money implied volatility and a forward starting at the money compound option. We would like to point out that we always refer to the forward price when we say at the money.

The first contract belongs directly to the implied volatility derivative family, i.e. a new class of exotic derivatives that reveals the option traders' interest on implied volatilities as market indices. With regard to this topic, Whaley (2000) states that the VIX, a synthetic index composed of implied volatilities relative to the S&P100, measures the "investor fear gauge. The index is set by investors and expresses their consensus view about expected future stock market volatility. The higher the fear, the higher the VIX." Therefore, one can write contracts on these indices to change the exposure to the future implied volatility risk.

Another important derivative instrument like the VIX is the VOLAX, we will show how to price. The VOLAX is a futures contract on the VDAX that has started being listed in Germany at the DTB. The VDAX is a weighted average of the implied volatilities of a basket of eight near the money options; it is a synthetic implied volatility corresponding to a hypothetical 45-calendar day at the money DAX option. The VDAX construction can be briefly explained. First of all, we need to select the eight options (four puts and four calls) whose implied volatilities constitute the VDAX. We denote the exercise price just below the current forward price as X_t^l and the exercise price just above the current forward price as X_t^h. Also, we define as $T_1 = t + \tau_1$ and $T_2 = t + \tau_2$ the two expiring dates of standard DAX options nearest to the remaining lifetime equal to 45 calendar days, i.e. such that:

$$\tau_1 < \tau^* < \tau_2.$$

The parameter τ^* terms the 45 calendar days characterizing the VOLAX contract. The VDAX is computed in two steps. First, two sub-indices, $V_t(\tau_i)$, $i = 1, 2$ are constructed by means of the implied volatilities of the four nearest the money options (two calls and two puts), i.e.:

$$\begin{aligned} V_t(\tau_i) &= \frac{\left[m_t^h - 1\right]\left[\Sigma_t^{Put}\left(\tau_i, m_t^h\right) + \Sigma_t^{Call}\left(\tau_i, m_t^h\right)\right]}{2\left(m_t^h - m_t^l\right)} \\ &+ \frac{\left[1 - m_t^l\right]\left[\Sigma_t^{Put}\left(\tau_i, m_t^l\right) + \Sigma_t^{Call}\left(\tau_i, m_t^l\right)\right]}{2\left(m_t^h - m_t^l\right)}, \quad i = 1, 2, \end{aligned} \tag{18}$$

where $m_t^h = X_t^h / F_t(\tau_i)$ and $m_t^l = X_t^l / F_t(\tau_i)$. The next step is to calculate a time-weighted average of the two sub-indices as:

$$VDAX_t = V_t(\tau_1)(1-\varepsilon) + V_t(\tau_2)\varepsilon, \tag{19}$$

where

$$\varepsilon = \frac{45 - \tau_1}{\tau_2 - \tau_1}.$$

It should be stressed that the VDAX is lifetime-independent, because it is related to a hypothetical option that does not expire: this feature eliminates the effect of strong fluctuations of volatilities that typically occurs close to option delivery dates. Moreover, the VDAX takes into account only at the money DAX options, and therefore it records only the price changes of the most liquid options. From expression (19), it is clear that the VDAX can be heuristically proxied as one of the at the money implied volatilities whose risk neutral drifts have been specified by expression (15) . Therefore, we assume the VOLAX contract expiring at time T has futures price at time t given by:

$$VOLAX_t(T) = \tilde{\mathbb{E}}_t\left[VDAX_T\right] = \tilde{\mathbb{E}}_t\left[\Sigma_T^{DAX}(\tau^*, 1)\right]. \tag{20}$$

On this point we would like to remark that the market model proposed by Schönbucher (1999) cannot be used immediately to price the VOLAX, because Schönbucher (1999) models the absolute at the money implied volatility. On the contrary, at the pricing date it is not known what forward level will be traded at the money at the VOLAX expiration. In other words, in the absolute framework it is not possible to forecast which options will be near the money in the future and will supply the implied volatilities weighted in the VDAX. In fact, one should keep track of all the possible implied volatilities that are likely to be at the money on the expiry date.

The other exotic derivative we want to price is a new popular instrument dealt in the OTC derivative markets, the forward starting at the money compound option. This is an option to deliver at time $T_1 > t$ an underlying at the money option with maturity at time $T_2 > T_1$ (again with time to maturity $\tau = T_2 - T_1$) at a given exercise price $h := H/F_{T_1}(\tau)$, expressed as a percentage, known at time t, of the forward price at time $T_1 > t$. The forward starting at the money compound option has price at time t given by:

$$\begin{aligned} CO_t(T_1, \tau, h) &= \tilde{\mathbb{E}}_t\left[e^{-r(T_1-t)}\left(C_{T_1}(\tau, 1, \Sigma_{T_1}(\tau, 1)) - H\right)^+\right] \\ &= \tilde{\mathbb{E}}_t\left[e^{-r(T_1-t)} F_{T_1}(\tau)\left(\Psi_{T_1}(\tau, 1, \Sigma_{T_1}(\tau, 1)) - h\right)^+\right]. \end{aligned} \tag{21}$$

Albeit this is not properly a volatility contract, it can be seen in the expression (21) that as an actual underlying of a forward starting at the money compound option we have the implied volatility that will be traded at the money at time T_1. The analogy with the VOLAX argument is straightforward: in an absolute context, we are not able to forecast which option will be at the money at time T_1 and, therefore, will be delivered to the compound option writer. Differently, our model overcomes this drawback focusing directly on relative implied volatilities.

In section (4.2), in order to price the contracts described above, we will run Monte Carlo simulations. Then, we will compare the prices of each of these two exotic instruments with the prices of the same derivatives obtained by assuming that the square of the at the money implied volatility multiplied by the time to maturity is equal to the total future variance of the underlying security price. This assumption, commonly held by practitioners, would be mathematically correct only if the instantaneous spot volatility were a deterministic (not necessarily constant) function of time. According to this view, if $\Sigma_t(T_1 - t, 1)$ and $\Sigma_t(T_2 - t, 1)$ are respectively the at the money implied volatilities for an option expiring at time $T_1 > t$ and $T_2 > T_1 > t$, then the future total variance that will be experienced by the underlying security price between T_1 and T_2 is given by the square of the following expression:

$$\Sigma_t(T_1, T_2; m = 1) := \sqrt{\frac{(T_2 - t)\left(\Sigma_t(T_2 - t, 1)\right)^2 - (T_1 - t)\left(\Sigma_t(T_1 - t, 1)\right)^2}{T_2 - T_1}}. \tag{22}$$

The l.h.s. of the previous expression is also known as forward implied volatility and must always be quoted as a positive quantity in order to prevent arbitrage opportunities consisting in at the money options maturing at time $T_1 > t$ more expensive than at the money options expiring at $T_2 > T_1$[9].

4.1 The interpolation technique

The practical implementation of the Monte Carlo simulation requires at each step the computation of the partial derivatives appearing in the risk neutral drift of the implied volatilities. In the following, a simple parametric approach is proposed. The aim is to reproduce the typical implied volatility surfaces observed in option markets with an interpolating function that extends the "smoothing" technique proposed by Shimko (1993) and captures the information embedded in the volatility term structure.

The implied volatility surface can be represented as:

$$\Sigma_t(\tau, m) = g(m, a_1, a_2, a_3)\, f(\tau, \lambda), \tag{23}$$

where

$$g(m, a_1, a_2, a_3) = a_1 m^2 + a_2 m + a_3, \tag{24}$$

$$f(\tau, \lambda) = \frac{1 - e^{-\lambda\tau}}{\lambda\tau}. \tag{25}$$

The parabola g is the interpolating function proposed by Shimko (1993); the parameters a_1, a_2 and a_3 are such to ensure the best least-squares fit of the smile curve drawn by implied volatilities with identical time to maturity. However, in a more realistic fashion the relationships between different smiles referred to the same volatility surface should be considered. Also, the floating nature of the volatility surface should be taken into account. The function f introduces the time dimension in the volatility surface, and the parameter λ can be used to fit the slope of the term structure. In particular, when choosing $\lambda > 0$ we obtain a decreasing implied volatility term structure, which is typically observed in equity option markets[10].

The implied volatility surface can be reproduced thanks to the interpolating function in expression (23) by solving a least-squares minimization problem with respect to the four parameters a_1, a_2, a_3 and λ:

$$\mathcal{P}(a_1, a_2, a_3, \lambda) = \min_{a_1, a_2, a_3, \lambda} \left\{ \begin{array}{l} \sum\limits_{j=1}^{N} \left[\Sigma_t(\tau_i, m_j) - g(m_j, a_1, a_2, a_3)\, f(\tau_i, \lambda)\right]^2 \\ + \sum\limits_{i=1}^{M} \left[\Sigma_t(\tau_i, m_j) - g(m_j, a_1, a_2, a_3)\, f(\tau_i, \lambda)\right]^2 \end{array} \right\}. \tag{26}$$

[9] On this point see Rebonato (1999) in chapter 1.

[10] However, different patterns are not totally unusual. The interpolating function can also reproduce increasing (with $\lambda < 0$) or flat term structures (with $\lambda = 0$), the latter being the Black-Scholes classical case. More realistic shapes of the term structure could be taken into account by means of other interpolating functions as the one proposed by Nelson and Siegel (1987).

We choose $N = 3$ in order to fit the volatility surface across the moneyness dimension, and $M = 2$ to fit the volatility term structure. Then, we take a point of the implied volatility surface as a fulcrum, for instance $\Sigma_t(\tau_1, m_2)$, which is the implied volatility we wish to model as an underlying. Since $\Sigma_t(\tau_1, m_2)$ appears in both sections of the implied volatility surface considered in $\mathcal{P}$, along with $\Sigma_t(\tau_1, m_2)$ we need to consider three other implied volatilities only. We choose the observed volatilities $\Sigma_t(\tau_1, m_1)$ and $\Sigma_t(\tau_1, m_3)$ that share the same time to maturity τ_1 as $\Sigma_t(\tau_1, m_2)$ and exhibit moneyness just below (i.e. m_1) and just above (i.e. m_3) the "central" level m_2. Also, we choose the observed implied volatility $\Sigma_t(\tau_2, m_2)$ that shares the same moneyness as the central volatility and exhibits the first time to maturity τ_2 longer than τ_1 (i.e.: $\tau_2 > \tau_1$). At each step of the Monte Carlo simulation the minimization problem (26) is solved by means of the following two-step *routine.*

First, we fit the smile the implied volatilities $\Sigma_t(\tau_1, m_1)$, $\Sigma_t(\tau_1, m_2)$ and $\Sigma_t(\tau_1, m_3)$ lie on, i.e. we solve the following least-squares problem w.r.t. a_1, a_2, and a_3 :

$$\mathcal{P}_1(a_1, a_2, a_3; \lambda_0) = \min_{a_1, a_2, a_3} \left\{ \sum_{j=1}^{3} \left[\Sigma_t(\tau_1, m_j) - g(m_j, a_1, a_2, a_3) f(\tau_1, \lambda_0)\right]^2 \right\}.$$

If we fix an arbitrary λ_0, $\mathcal{P}_1$ turns out to be a simple linear system:

$$\begin{aligned}
\frac{1-e^{-\lambda_0\tau_1}}{\lambda_0\tau_1}\left[a_1 m_1^2 + a_2 m_1 + a_3\right] &= \Sigma_t(\tau_1, m_1), \\
\frac{1-e^{-\lambda_0\tau_1}}{\lambda_0\tau_1}\left[a_1 m_2^2 + a_2 m_2 + a_3\right] &= \Sigma_t(\tau_1, m_2), \\
\frac{1-e^{-\lambda_0\tau_1}}{\lambda_0\tau_1}\left[a_1 m_3^2 + a_2 m_3 + a_3\right] &= \Sigma_t(\tau_1, m_3).
\end{aligned}$$

The previous linear system returns the vector $A^*(\lambda_0) = [a_1^*(\lambda_0),\ a_2^*(\lambda_0),\ a_3^*(\lambda_0)]$ containing the smile interpolating parameters as functions of a pre-defined λ_0.

The second step is required in order to fit also the volatility term structure. In fact, given the vector $A^*(\lambda_0)$, we solve another least-squares problem w.r.t. the time-parameter λ:

$$\mathcal{P}_2(\lambda; A^*(\lambda_0)) = \min_{\lambda} \left\{ \sum_{i=1}^{2} \left[\Sigma_t(\tau_i, m_2) - g(m_2, A^*(\lambda_0)) f(\tau_i, \lambda)\right]^2 \right\}.$$

The numerical solution of $\mathcal{P}_2$ is a certain value $\lambda = \lambda_1$ replacing the initial guess of the time-parameter in $\mathcal{P}_1$.

By repeating the first step, we obtain a new vector $A^*(\lambda_1)$ that, plugged in $\mathcal{P}_2$, returns another choice for λ. The procedure can be carried over until a pre-fixed tolerance bound is achieved[11]. Once the desired precision is reached, we obtain an analytic expression for the interpolating volatility surface, which is represented by expression (23), where the calibrated parameters a_1^*, a_2^*, a_3^* and λ^*, calculated by the described *routine,* are plugged.

At each simulated step, the interpolation *routine* is repeated in order to correct for changes in the shape of the volatility surface over time. The partial derivatives of the volatility surface that affect each drift can be easily calculated by differentiating the interpolating surface. In this way, the Monte Carlo simulation for the four relevant volatilities, according to their risk neutral drifts as defined in Theorem 1, is run.

[11] The tollerance level we have decided to fix is equal to 10^{-8}.

4.2 Numerical results

Following expression (20), the VOLAX has underlying $\Sigma_t^{DAX}(\tau^*, 1) := VDAX_t$, related to a theoretical DAX option which is always at the money and has constant time to maturity equal to $\tau^* = 45$ calendar days. In our model, this volatility has dynamics given by:

$$d\Sigma_t^{DAX}(\tau^*, 1) = \alpha_t^*\left(\tau^*, 1, \Sigma_t^{DAX}(\tau^*, 1)\right) dt + \beta_t\left(\tau^*, 1, \Sigma_t^{DAX}(\tau^*, 1)\right) dB_t^{\Sigma}. \tag{28}$$

We will simulate the evolution of the previous at the money volatility. Also, the implementation requires the simulation of three other volatilities: we choose $\Sigma_t^{DAX}(\tau^*, 0.95)$ and $\Sigma_t^{DAX}(\tau^*, 1.05)$, i.e. the volatilities narrowest to $\Sigma_t^{DAX}(\tau^*, 1)$ lying on its smile in Table 1, and $\Sigma_t^{DAX}(\tau^* + 45/360, 1)$, i.e. the volatility sharing the same term structure as $\Sigma_t^{DAX}(\tau^*, 1)$ with the closest time to maturity ahead of τ^*, again according to Table 1. In particular, following Table 1 the initial values for all the relevant volatilities are $\Sigma_t^{DAX}(\tau^*, 1) = 31.75\%$, $\Sigma_t^{DAX}(\tau^*, 0.95) = 34.00\%$, $\Sigma_t^{DAX}(\tau^*, 1.05) = 30.05\%$, and $\Sigma_t^{DAX}(\tau^* + 45/360, 1) = 30.50\%$. We do not need to simulate the instantaneous volatility $\Sigma_t^{DAX}(\tau, 1)\big|_{\tau=0}$, because it can be obtained by setting $m = 1$ and $\tau = 0$ in (23) at each step of the simulation, once we have interpolated the surface. Moreover, we assume that $\beta_t(\tau, m, \Sigma_t) = b\Sigma_t$ and $\rho_t(\tau, m, \Sigma_t) = \rho$, where b and ρ are constant and equal for all the volatilities[12]. In order to price the VOLAX contract we run 10,000 Monte Carlo simulations with antithetic variates each with 250 steps.

Table 2 shows the prices (expressed in percentage terms) of the VOLAX contract expiring in 3 months ($T - t = 0.25$ years) calculated for different choices of the volatility of volatility b and of the correlation ρ.

[INSERT HERE TABLE 2]

From the data in Table 2 we can explore how the parameters of the model affect the VOLAX price in the implied volatility scenario described by Table 1. It can be seen that, with highly negative or positive correlation levels, the lower b, the higher the VOLAX price. The effect of the volatility of the volatility on the VOLAX price is inverted when the correlation with the underlying movements takes values close to zero. In particular, in the range of low negative levels for the correlation parameter ρ the VOLAX price reaches its highest values by keeping b fixed.

Note that if we assume the implied volatility equal to the average total variance of the underlying security price, we get $VOLAX_t(T) = 29.19653$, which is just given by expression (22), employed to calculate the forward implied volatility.

The other instrument we are interested in pricing is the forward starting at the money compound option. As we have pointed out above, the value of such contract depends on the implied volatility $\Sigma_{T_1}(\tau, 1)$ of the option that will be delivered at the money at time T_1. Expression (21) states that the compound option price can be computed by simulating both the dynamics of the underlying at the money implied volatility and the dynamics of the futures price $F_t(\tau)$, which is affected by the stochastic instantaneous volatility obtained by setting $m = 1$ and $\tau = 0$ in (23) at each step of the simulation. We consider the option on option with 6 months to expiry (i.e.: $T_1 - t = 0.5$ years), and the underlying option with 1 year of residual life at the pricing date (i.e.: $\tau = T_2 - T_1 = 0.5$ years). If we assume that the DAX is again the reference market, following Table 1 $\Sigma_t^{DAX}(0.5, 1) = 30.00\%$ can be taken as the initial value of the underlying at the money implied volatility for the compound option. Moreover, in order to simulate the previuos implied volatility,

[12] Of course this hypothesis can be easily weakened, if we assume diffusion and correlation terms varying for different moneyness levels and maturities.

according to Table 1 we need $\Sigma_t^{DAX}(0.5, 0.95) = 33.00\%$, $\Sigma_t^{DAX}(0.5, 1.05) = 28.75\%$ and $\Sigma_t^{DAX}(1, 1) = 29.75\%$ as the initial conditions of the other simulated implied volatilities, required to apply the interpolation technique described in section (4.1). Also, we assume that $F_t(0.5) = 1$ and the risk-free interest rate is $r = 0.1$. As we have done with the VOLAX, in order to price the forward starting at the money compound option we run 10,000 Monte Carlo simulations with antithetic variates each with 250 steps.

Table 3 contains the compound option prices obtained for different choices of the volatility of volatility b, of the correlation ρ, and of the relative strike h.

[INSERT HERE TABLE 3]

Firstly, from Table 3 we can appreciate the impact of the volatility of the implied volatility on the price of the forward starting compound option, an effect that is not possible to capture by means of the total variance approach. It can be seen that, for every correlation value, the compound option price increases as the volatility of the volatility increases. This is what we should expect, because if the underlying option price becomes more volatile via its implied volatility, there is a greater probability that the compound option expires in the money. Secondly, for a given volatility of the volatility level, the compound option price reaches its highest values when the correlation level is close to zero. This result is consistent with the results we could observe in Table 2. In fact, when correlation is low, negative and close to zero the underlying at the money implied volatilty records its maxima and therefore the underlying option becomes more likely to be exercised[13].

We should notice that the compound option pricing formula given by Geske (1979) cannot be applied to our compound option because of the forward starting feature previously described.

Finally, if we assume that the implied volatility is equal to the average total variance of the underlying security price, we have $CO_t(0.5, 0.5, 0.1) = 0$, i.e. the compound option with strike $h = 0.10$ is worthless. This is the case since a six month call futures-option with forward price and strike equal to 1 and a forward implied volatility given by $\Sigma_t(0.5, 1; 1) = 29.49788\%$, calculated according to expression (22), is worth only 0.0830615, i.e. $\Psi_{0.5}(0.5, 1, 0.2949788) = 0.0830615$, below the compound option strike $h = 0.10$. In other words, if we assume, as many practitioners do, that the forward implied volatility is equal to the average future variance experienced by the underlying security price between T_1 and T_2, the compound option with strike $h = 0.10$ is out of the money and therefore worthless. Of course, this result is meaningless. In fact, if at time $T_1 = 0.5$ the six month implied volatility is such to make the underlying option be in the money, the compound option with strike $h = 0.10$ is worth a positive value. Therefore, in the case of the compound option, we could not even compare the prices obtained by following the methodology we propose with the ones obtained by the total variance approach, because the latter could produce meaningless results. Moreover, we would like to remark that even more paradoxical situations with more evident arbitrage scenarios could raise with strongly decreasing implied volatility term structures.

5 Conclusions

In this paper we have derived the risk neutral dynamics of the really quoted market implied volatilities. In particular, we have found the drift components of each single

[13] In particular, we have considered only negative levels of the correlation since this is what can be more commonly observed in practice.

implied volatility that are consistent with time to maturity and relative strike invariant futures-option prices. In fact, these prices are given by the formulas where the actually quoted market implied volatilities are plugged. This way of modeling is based on the assumption, fully confirmed by option traders' quoting practice, that the implied volatility is an input rather than an output of the Black-Scholes and Black option pricing formulas. In other words, the market implied volatility has the dignity of independent or exogenous variable. So, it results that option markets are incomplete not because the instantaneous spot volatility is unknown, but because the implied volatility has to be quoted by option traders in the same way as the underlying security price has to be quoted by spot or forward traders in order to price options.

By making the time to maturity approach to zero we have found the risk neutral dynamics of the instantaneous spot volatility which result automatically fitted to quoted option prices.

Finally, we have used the risk neutral dynamics of the market implied volatility to price futures contracts on the implied volatility like the VOLAX, listed at the DTB in Germany, and forward starting at the money compound options, dealt in the OTC exotic derivative markets. We have found noticeable differences with the prices of the same derivatives, when assuming not coherently with the absence of arbitrage that the at the money implied volatility, multiplied by the time to maturity, is equal to the future total variance of the underlying security price. We have noticed that, allowing for this view, forward starting compound options can be easily mispriced.

The paradoxical evidences found in this paper should cast serious doubts on the common market practice of using the at the money implied volatility as an average of the underlying security price variance.

As a future development of the methodology proposed in this paper, the risk neutral dynamics of the implied volatility surface with all the market implied volatilities, altogether and not individually considered, could be derived. In this way, the methodology introduced here could be successfully applied to price the family of all the forward starting derivative securities.

6 Appendix

A The Itô-Venttsel Formula

The following theorem, known as the Itô-Venttsel formula, can be found in Brace et al. (2001). This result allows us to find the SDE for $C_t(x_t)$ with x_t stochastic, given the SDE for $C_t(x)$ where x is a (fixed) parameter.

Theorem 2 *(Itô-Venttsel Formula) Let B_t^S be a Brownian motion. Suppose $C_t(x)$ is twice differentiable with respect to the parameter x and satisfies the following SDE:*

$$dC_t(x) = a(t,x)dt + b(t,x)dB_t.$$

If x_t satisfies the SDE:

$$dx_t = f(t,x_t)dt + g(t,x_t)dB_t,$$

then the SDE for $C_t(x_t)$ is:

$$\begin{aligned} dC_t(x_t) &= a(t,x_t)dt + b(t,x_t)dB_t \\ &+\frac{\partial C_t(x_t)}{\partial x}dx_t + \frac{1}{2}g^2(t,x_t)\frac{\partial^2 C_t(x_t)}{\partial x^2}dt \\ &+g(t,x_t)\frac{\partial b(t,x_t)}{\partial x}dt. \end{aligned} \tag{29}$$

Differently from the usual application of Itô's lemma, we have the appearance of the last term in (29) that comes from the covariation between the increment of order dB_t in the process for $C_t(x)$ and the increment of order dB_t in the process for x_t. An analogous result can be found in Ledoit and Santa-Clara (1999).

B Derivation of the drift risk-neutral restriction

The partial derivatives appearing in expression (10) have been computed by taking into account the fact that the absolute implied volatilities $\hat{\Sigma}_t$ are themselves functions of the strike X_t and the maturity T_t. The extensive notations for those partial derivatives are:

$$\begin{aligned}
\frac{\partial}{\partial T}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) &= \left(\frac{\partial \hat{C}_t}{\partial T} + \frac{\partial \hat{C}_t}{\partial \hat{\Sigma}}\frac{\partial \hat{\Sigma}_t}{\partial T}\right)(T_t, X_t, \hat{\Sigma}_t); \\
\frac{\partial}{\partial X}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) &= \left(\frac{\partial \hat{C}_t}{\partial X} + \frac{\partial \hat{C}_t}{\partial \hat{\Sigma}}\frac{\partial \hat{\Sigma}_t}{\partial X}\right)(T_t, X_t, \hat{\Sigma}_t);
\end{aligned}$$

$$\frac{\partial^2}{\partial X^2}\hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right) = \left(\begin{array}{c} \frac{\partial^2 \hat{C}_t}{\partial X^2} + 2\frac{\partial^2 \hat{C}_t}{\partial X \partial \hat{\Sigma}}\frac{\partial \hat{\Sigma}_t}{\partial X} + \\ \frac{\partial^2 \hat{C}_t}{\partial \hat{\Sigma}^2}\left(\frac{\partial \hat{\Sigma}_t}{\partial X}\right)^2 + \frac{\partial \hat{C}_t}{\partial \hat{\Sigma}}\frac{\partial^2 \hat{\Sigma}_t}{\partial X^2} \end{array}\right)(T_t, X_t, \hat{\Sigma}_t).$$

If the futures-option prices are computed by the common Black formula, their derivatives can be easily obtained as:

$$\begin{aligned}
\left.\frac{\partial \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial X}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= -N\left(d_2\right); \\
\left.\frac{\partial^2 \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial X^2}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= \frac{1}{mF_t\left(\tau\right)}\frac{n\left(d_2\right)}{\Sigma_t\left(\tau, m\right)\sqrt{\tau}}; \\
\left.\frac{\partial \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial T}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= \frac{\partial \hat{F}_t\left(T_t\right)}{\partial T}\Phi_t + \hat{F}_t\left(T_t\right)\left.\frac{\partial \Phi_t}{\partial T}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} = rC_t + F_t\left(\tau\right)\frac{n\left(d_1\right)\Sigma_t\left(\tau, m\right)}{2\sqrt{\tau}}; \\
\left.\frac{\partial \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial \hat{\Sigma}}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= \sqrt{\tau}F_t\left(\tau\right)n\left(d_1\right); \\
\left.\frac{\partial^2 \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial \hat{\Sigma}^2}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= \frac{d_1 d_2}{\Sigma_t\left(\tau, m\right)}\sqrt{\tau}F_t\left(\tau\right)n\left(d_1\right); \\
\left.\frac{\partial^2 \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial \hat{\Sigma}\partial X}\right|_{\substack{T_t=t+\tau \\ X_t=mF_t}} &= n\left(d_2\right)\frac{d_1}{\Sigma_t\left(\tau, m\right)}.
\end{aligned}$$

In order to obtain an homogeneous expression, also the derivatives of the implied volatility surface can be turned in terms of τ and m by using the following set of relations:

$$\begin{aligned}
\frac{\partial}{\partial T}\hat{\Sigma}_t(T_t, X_t) &= \frac{\partial}{\partial T}\Sigma_t\left(T_t - t, \frac{X_t}{\hat{F}_t(T_t)}\right) = \frac{\partial}{\partial \tau}\Sigma_t\left(\tau, m\right); \\
\frac{\partial}{\partial X}\hat{\Sigma}_t(T_t, X_t) &= \frac{\partial}{\partial X}\Sigma_t\left(T_t - t, \frac{X_t}{\hat{F}_t(T_t)}\right) = \frac{1}{\hat{F}_t(T_t)}\frac{\partial}{\partial m}\Sigma_t\left(\tau, m\right); \\
\frac{\partial^2}{\partial X^2}\hat{\Sigma}_t(T_t, X_t) &= \frac{\partial}{\partial X}\left(\frac{1}{\hat{F}_t(T_t)}\frac{\partial}{\partial m}\Sigma_t\left(\tau, m\right)\right) = \left(\frac{1}{\hat{F}_t(T_t)}\right)^2\frac{\partial}{\partial m}\Sigma_t\left(\tau, m\right); \\
\frac{\partial}{\partial X}\beta_t\left(T_t, X_t, \hat{\Sigma}_t\right) &= \frac{\partial}{\partial X}\beta_t\left(T_t - t, \frac{X_t}{\hat{F}_t(T_t)}, \hat{\Sigma}_t(T_t, X_t)\right) = \frac{1}{\hat{F}_t(T_t)}\frac{\partial}{\partial m}\beta_t\left(\tau, m, \Sigma_t\right).
\end{aligned}$$

Plugging the previuos results in expression (10), we get the dynamics for the futures-style option price whose expected value appears in expression (11), where the relation

$$\frac{\partial}{\partial X}\left[\frac{\partial \hat{C}_t\left(T_t, X_t, \hat{\Sigma}_t\right)}{\partial \hat{F}_t}\sigma_t \hat{F}_t\left(T_t\right)\right]\Bigg|_{\substack{T_t=t+\tau \\ X_t=mF_t(\tau)}} = -\frac{\sigma_t n\left(d_1\right)}{m\Sigma_t\left(\tau, m\right)\sqrt{\tau}}$$

has been exploited in order to take into account the term arising from the application of the Itô-Venttsel formula.

C The instantaneous volatility as a point of the implied volatility surface

Theorem 3 *Let $\{\Sigma_t(\tau, m) : \tau \in \Theta, m \in M\}$ be an implied volatility surface. If: (a) for each $t < \tau$ the function $\Sigma_t(.,.) \in C^{1,2}(\Theta \times M)$; (b) $(0,1) \in \Theta \times M$; (c) the functions $\beta_t(\tau, m, \Sigma_t)$ and $\rho_t(\tau, m, \Sigma_t)$ are continuous and bounded in $(\tau, m) = (0,1)$, then the following relationship holds:*

$$\sigma_t = \lim_{(\tau,m)\to(0,1)} \Sigma_t(\tau, m) = \lim_{\tau\to 0}\Sigma_t(\tau, 1). \tag{30}$$

Proof. *Let us consider the drift of a relative implied volatility $\Sigma_t(\tau, m)$ extracted from a futures-option with time to maturity τ and moneyness m. To ensure a unique bounded solution for the SDE that describes the evolution of the previous variable, we shall require that:*

$$\lim_{\tau\to 0^+} \alpha_t^*(\tau, m, \Sigma_t) < \infty, \qquad \forall(\tau, m) \in \Theta \times M.$$

The previous condition must hold for every implied volatility that belongs to the same surface, in order to ensure that all futures-option prices are arbitrage-free. We can rewrite this condition by using the expression (15) for α_t^*:

$$\begin{aligned}
&\left(\Sigma_t\left(\tau, m\right)\right)^2 - \sigma_t^2\Big) + 2\Sigma_t\left(\tau, m\right)\rho_t(\tau, m, \Sigma_t)\beta_t(\tau, m, \Sigma_t)\sigma_t\left[d_1\sqrt{\tau} - \tau\right] \\
&\quad -d_1 d_2 \tau\left(\beta_t(\tau, m, \Sigma_t)\right)^2 + 2\Sigma_t\left(\tau, m\right)\tau m \rho_t\left(\tau, m, \Sigma_t\right)\sigma_t \tfrac{\partial \beta_t(\tau, m, \Sigma_t)}{\partial m} \\
&\quad +2\Sigma_t\left(\tau, m\right)\tau \tfrac{\partial \Sigma_t(\tau, m)}{\partial \tau} \\
&\quad +2m\sigma_t\left[\sigma_t d_1\sqrt{\tau} + \Sigma_t\left(\tau, m\right)\rho_t(\tau, m, \Sigma_t)\beta_t(\tau, m, \Sigma_t)d_1 d_2 \tau\right]\tfrac{\partial \Sigma_t(\tau, m)}{\partial m} \\
&\quad +m^2\sigma_t^2 d_1 d_2 \tau\left(\tfrac{\partial \Sigma_t(\tau, m)}{\partial m}\right)^2 + m^2\sigma_t^2\Sigma_t\left(\tau, m\right)\tau\tfrac{\partial^2 \Sigma_t(\tau, m)}{\partial m^2} \quad = o\left(\tau\right).
\end{aligned} \tag{31}$$

Noting that:

$$\lim_{\tau\to 0} d_1\sqrt{\tau} = \lim_{\tau\to 0} d_1\sqrt{\tau} = -\lim_{\tau\to 0}\left(\frac{\ln m}{\Sigma_t(\tau,m)}\right),$$

we can take the limit for the drift. By hypotheses (a) and (c), from expression (31) we get the relation between the instantaneous volatility and every component of the implied volatility smile containing the instantaneous volatility itself that ensures the no-explosion condition:

$$\begin{aligned}(\Sigma_t(0,m))^2 - \sigma_t^2 &= 2\rho_t(0,m,\Sigma_t)\,\beta_t(0,m,\Sigma_t)\sigma_t \ln m - \left(\frac{\beta_t(0,m,\Sigma_t)\ln m}{\Sigma_t(0,m)}\right)^2 \\ &\quad + \frac{2m\ln m\sigma_t}{\Sigma_t(0,m)}\left[\rho_t(0,m,\Sigma_t)\,\beta_t(0,m,\Sigma_t)\ln m - \sigma_t\right]\frac{\partial \Sigma_t(0,m)}{\partial m} \\ &\quad + \left(m\ln m\frac{\sigma_t}{\Sigma_t(0,m)}\frac{\partial\Sigma_t(0,m)}{\partial m}\right)^2. \qquad (32)\end{aligned}$$

In particular, for $m = 1$, equation (32) is verified if and only if the thesis is true. ■

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Table 1: Market implied volatility matrix (in % terms).

Moneyness→ Maturity↓	-20%	-10%	-5%	0%	5%	10%	20%
1 Month	54.75	46.75	39.75	32.75	32.25	29.25	27.75
1.5 Months	47.75	42.25	34.00	31.75	30.05	27.60	27.20
3 Months	44.00	39.50	33.50	30.50	29.00	27.25	26.25
6 Months	40.00	36.50	33.00	30.00	28.75	27.25	26.00
1 Year	37.25	34.25	32.00	29.75	28.25	27.00	26.00
2 Years	34.75	32.50	30.75	29.25	27.90	26.80	25.90
3 Years	33.55	31.65	30.15	28.75	27.60	26.80	26.10
4 Years	32.60	31.10	29.75	28.50	27.45	26.65	25.85
5 Years	32.10	30.80	29.60	28.50	27.60	26.70	25.90
6 Years	31.70	30.50	29.50	28.50	27.50	26.70	25.90
7 Years	31.50	30.50	29.50	28.50	27.50	26.70	25.90

Table 2: VOLAX prices for different levels of **b** and ρ. The standard errors are in brackets.

ρ	b		
	0.2	0.3	0.5
0.5	28.23781 (0.021661)	27.54664 (0.029650)	27.32072 (0.057386)
0	28.69184 (0.023028)	28.68171 (0.044566)	29.60727 (0.078548)
−0.1	28.91208 (0.031963)	28.98764 (0.048963)	29.15614 (0.07442)
−0.3	28.94043 (0.031412)	28.69037 (0.049570)	28.51257 (0.083749)
−0.5	28.29713 (0.028090)	28.20270 (0.039009)	27.68667 (0.058578)

Table 3: Forward starting compound option prices for different levels of $\mathbf{b}$ and $\boldsymbol{\rho}$. The standard errors are in brackets.

	$h = 0.06$	$h = 0.08$	$h = 0.10$
b		$\rho = 0$	
0.2	0.125403 (0.000685)	0.104529 (0.000636)	0.084391 (0.000580)
0.3	0.160019 (0.00114)	0.141450 (0.001108)	0.123306 (0.001073)
0.5	0.246351 (0.003429)	0.228990 (0.003332)	0.211252 (0.00323)
b		$\rho = -0.3$	
0.2	0.074747 (0.000288)	0.055826 (0.000269)	0.037152 (0.000252)
0.3	0.087027 (0.000476)	0.067902 (0.000456)	0.049680 (0.000431)
0.5	0.188163 (0.002704)	0.171894 (0.002634)	0.156723 (0.002560)
b		$\rho = -0.5$	
0.2	0.095992 (0.000392)	0.076769 (0.000376)	0.058228 (0.000351)
0.3	0.1517565 (0.000879)	0.134117 (0.000866)	0.116674 (0.000852)
0.5	0.171454 (0.001964)	0.155152 (0.001921)	0.139774 (0.001873)