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A Note on Optimal Calibration of the Libor Market Model to the Correlations

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Abstract

We develop a simple, fast, non-parametric method for calibrating Libor market models to historical or implied correlation matrices. For a given symmetric matrix, the method utilises alternating projections to find the nearest correlation matrix of a lower rank.

Key words: LIBOR market model, non-parametric calibration, matrix nearness problem, correlation matrix, alternating projections method.
JEL Classification: C51, C61, G13
Mathematics Subject Classification: 65F30

1 Introduction

This short paper addresses the problem of calibrating the market models developed by Miltersen *et al* (1997) and Brace *et al* (1997) to correlation matrices. Correlation matrices can be either historical or implied by the option markets. These correlation matrices usually exhibit high-factor structure which hinders efficient use of market models. We describe a simple algorithm that finds the closest correlation matrix for any number of factors, i.e. our algorithm provides an efficient calibration of low-factor market models to the observed correlation matrices.

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We will not describe here the details of the Libor market models [1]. Instead we proceed directly to specification of the problem. The correlation matrices in Libor market models appear in two ways: first, as the historical correlation of instantaneous changes in the Libor rates; second, as the correlation of instantaneous changes of the Libor rates implied by the derivative data, usually the swaption prices. We concentrate here on the historical correlations; we will comment on implied correlations later in the paper. For given n Libor rates the associated instantaneous sample correlation matrix usually has rank n. To reproduce this correlation matrix the implementation of the Libor market model will require n Brownian motions. To reduce the dimensionality of the problem one could implement the model using only k Brownian motions, with $k < n$. In this case, however, one need to choose rank k, $n \times n$ correlation matrix, which is closest in some sense to the original rank n, $n \times n$ correlation matrix.

We define the sets

$$\begin{aligned} S &= \{Y = Y^T \in \mathbb{R}^{n \times n} : Y \geq 0\}, \\ U &= \{Y = Y^T \in \mathbb{R}^{n \times n} : y_{ii} = 1, i = 1 : n\}, \\ S_k &= \{Y = Y^T \in \mathbb{R}^{n \times n} : \text{Rank}(Y) \leq k, k \in \mathbb{N}, k \leq n\}. \end{aligned}$$

Here, for a symmetric Y, the notation $Y \geq 0$ means Y is positive definite.

Then, the problem we consider here can be stated as follows. For arbitrary symmetric $A \in \mathbb{R}^{n \times n}$, find a matrix $X \in S \cap U \cap S_k$. I.e. find a correlation matrix X of rank less or equal k minimizing the distance

$$\gamma(A) = \min\{\|A - X\|_F\}. \tag{1}$$

The norm is the Frobenius [2] norm, $\|A\|_F^2 := \sum_{i,j} a_{ij}^2$.

This problem has been addressed by Rebonato (1999). He suggested a procedure for constructing a correlation matrix for the calibration of Libor market models. In particular, he approximated the implied correlation matrix $\hat{C}$ by a matrix of the form BB^T, where the matrix B is constructed via the following

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[1] We refer the interested reader to several recent manuscripts which describe this modelling framework in detail, including Rebonato (2002) and Brigo and Mercurio (2001a).

[2] Other norms can be considered as well. For example the 2-norm, $\|A\| = \rho\sqrt{A^TA}$, where the spectral radius $\rho(B) = \max\{|\lambda| : \det(B - \lambda I) = 0\}$. However, Halmos (1972) showed that positive approximants of the Frobenius and 2- norms are the same when A is symmetric.

algorithm,

$$b_{ij} = \cos\theta_{ij} \prod_{k=1}^{j-1} \sin\theta_{ik} \quad : \quad j = 1, \ldots, k-1,$$
$$b_{ij} = \prod_{k=1}^{j-1} \sin\theta_{ik} \quad : \quad j = k,$$

for an arbitrary set of angles θ_{ij}. The matrix BB^T is a correlation matrix of rank k by construction. To find the closest correlation matrix of this form one needs to minimize the distance function $\gamma(A)$ in (1) over the set of $n \times k$ angles θ_{ij}. The computational time needed for calibration for a low number n of Libor rates may be acceptable. However, for a large number of Libor rates this nonlinear minimization becomes easily unmanageable.

Recently, Zhang and Wu (2003) have approached our problem using Lagrange multiplier[3]. Furthermore, they have rigorously justified the convergence of the Lagrange multiplier method for the low-rank approximation of correlation matrices.

Contrary to Zhang and Wu (2003), our approach is very simple to understand and implement. The iterative algorithm only involves projections on the sets S, K, and R_k, with the optimal correlation matrix of rank k as its limit. The implementation of the algorithm can be done in only a few lines of code. In the next section we describe in detail the method of alternating projections and derive the projections on the sets S, K, and R_k. We present numerical results in Section 3 and conclude in Section 4.

2 Alternating Projections

Our method is based on a procedure which is generally known as the *Method of Alternating Projections.* This is an iterative scheme for finding the best approximation to any given point in a Hilbert space from the intersection of a finite collection of closed subspaces. This result was originally proved by von Neumann (1950), and independently by Wiener (1955) for the case of two subspaces. It was further generalized to the case of several subspaces by Halperin (1962). In our case, the sets under considerations are not subspaces, so there is no guarantee that the alternating projection method will converge.

To illustrate this point we borrow the following example from Shih-Ping Han (1988). Consider the space $\mathbb{R}^2$ equipped with the Euclidean inner product. Let $M_1 := \{(\zeta_1, \zeta_2) | \zeta_2 \leq 0\}$ and $M_2 := \{(\zeta_1, \zeta_2) | \zeta_1 + \zeta_2 \leq 0\}$. The standard projection method does not work for any vector x outside M_1 and M_2. This is demonstrated in Figure 1. The figure represents application of the standard

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[3] Wu (2003) applied this method to the calibration of the Libor market model.

projection method to the vector x. The vector $x_{12} = P_{M_2}P_{M_1}$ is clearly sub-optimal. The iterative projection terminates at x_{12} since the vector is in the intersection of M_1 and M_2.

A generalization of the method to convex closed sets[4] has been given by Dykstra (1983) and Boyle and Dykstra (1985). In particular, their algorithm converges strongly to the nearest point in the intersection. Dykstra's idea was to introduce a correction procedure for pathological cases such as the above example. It consists of subtracting a correction term before the projection. One obtains this term from the previous run as a difference between the starting value and the projection on the first set. The second diagram in Figure 2 represents the effect of correction by I_{11}. The vector x_{12} is effectively taken out of the intersection $M_1 \cap M_2$ and the iterative procedure will proceed with the limit the projection of x onto $M_1 \cap M_2$.

Next, we identify the projections of a symmetric matrix A, $P_S(A)$, $P_{S_k}(A)$ and $P_U(A)$, on the sets S, S_k and U respectively. The projection of a symmetric matrix A, on the set S of symmetric positive semidefinite matrices is

$$P_S(A) = \Pi \text{diag}(\max(\lambda_i, 0))\Pi^T, \tag{2}$$

where $A = \Pi\Lambda\Pi^T$ is a spectral decomposition, with Π an orthogonal matrix of eigenvectors and $\Lambda = \text{diag}(\lambda_i)$ a diagonal matrix with eigenvalues on the diagonal. This result was derived by Higham's (1988).

The projection of a general matrix A on the set of matrices with a lower rank can be found using singular value decomposition[5]. When the matrix A is symmetric, the projection P_{S_k} is achieved by setting the eigenvalues in the spectral decomposition of A to zero, with the exception of the first k. I.e.

$$P_{R_k}(A) = \Pi \text{diag}(\max(1_{k<i}\lambda_i, 0))\Pi^T. \tag{3}$$

where $A = \Pi\Lambda\Pi^T$ is again a spectral decomposition, and $1_{k<i}$ is the standard indicator function.

The projection $P_U(A)$ of a symmetric matrix A on the set of symmetric matrices with unit diagonal U, is simply

$$P_U(A) = \begin{cases} a_{ij} & i \neq j, \\ 1, & i = j. \end{cases}$$

I.e., we set the diagonal entries of A to 1.

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4 Bregman (1965) also considered such a generalization. His algorithm, however, converges only weakly to a point in the intersection, but not necessarily to the best approximation.

5 See, for example, Horn and Johnson (1985), Chapter 7.4.

We summarize the proceeding discussions in the algorithm:

$\Delta I_0 = 0,\ Z_0 = A$
for $n = 1, 2, \ldots$
  $R_n = Z_{n-1} - \Delta I_{n-1}$, where ΔI_{n-1} is Dykstra's correction.
  $X_n = P_S(R_n)$
  $\Delta I_n = X_n - R_n$
  $Y_n = P_{S_k}(X_n)$
  $Z_n = P_U(Y_n)$
end

One could use the relative changes in one of the projections as a stopping criterion.

3 Numerical Results

For the numerical analysis of the method we use the same examples as Zhang and Wu (2003). The first example is based on the correlation matrix C, generated by the correlation function as suggested by Rebonato (1999). The entries of the matrix C are defined as,

$$c_{ij} = \alpha + (1 - \alpha)\exp(\beta |t_i - t_j|),$$
$$\beta = d_1 - d_2 \max(t_i, t_j),$$

with parameters[6] $\alpha = 0.3$, $d_1 = -0.12$, and $d_2 = 0.005$. We present the values of the correlation matrix C resulting from this choice of parameters in Table 1. We plot the corresponding distance function $\gamma(C)$ in Figure 3. We also plot the approximating correlation matrices for several ranks together with the model correlation matrix C in Figures 7 to 12. We see that approximations take on the shape of the model correlation quite quickly. The approximation at the diagonal, however, becomes reasonable only for a high number of factors, which is unavoidable due to smoothness of the first principal components. We plot the eigenvectors of the rank three approximation together with the first three eigenvectors of the model correlation in Figure 5.

We take the correlation matrix for the second example in Table 2 from Brace *et al* (1997)), who give the market forward rate correlations for GBP in 1995. As in the previous example, we plot the distance function γ for an increasing number of factors in the approximating correlation matrix in Figure 4. We plot the approximating correlation matrices for several ranks together with

[6] For some choices of the parameters this function fails to be positive definite. This, however, is irrelevant, as the correlation approximant will always be positive semi-definite.

the market correlation matrix in Figures 13 to 18. The shape of the market correlation matrix is not as well-behaved as in the previous example. To capture its basic shape one needs an approximation of at least rank six. We also plot the eigenvectors of the rank three approximation matrix together with the first three eigenvectors of the market correlation matrix in Figure 6. Even for a non-optimal implementation the computation time for the correlation approximations is fraction of a second.

4 Conclusions

In this paper we have developed a simple, fast, non-parametric method for calibrating Libor market models to historical or implied correlation matrices. The method is based on alternating projections of the solution on the set of all positive semi-definite matrices, the set of matrices of rank k, and the set of matrices with diagonal one. This iterative algorithm converges to a correlation matrix of rank k in the intersection of these three sets. The implementation of the algorithm is trivial and requires only a few lines of code. We have presented the tests of the method on a model correlation matrix and a historical market correlation matrix. In both cases convergence is achieved in a fraction of a second.

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	1	2	3	4	5	6	7	8	9	10	11
2	0.958										
3	0.916	0.957									
4	0.876	0.915	0.956								
5	0.837	0.874	0.913	0.955							
6	0.799	0.834	0.872	0.912	0.954						
7	0.763	0.796	0.832	0.870	0.910	0.953					
8	0.729	0.760	0.793	0.829	0.867	0.909	0.953				
9	0.696	0.725	0.756	0.790	0.826	0.865	0.907	0.952			
10	0.665	0.692	0.721	0.753	0.787	0.824	0.863	0.906	0.951		
11	0.635	0.660	0.688	0.718	0.750	0.784	0.821	0.861	0.904	0.950	
12	0.607	0.631	0.656	0.684	0.714	0.746	0.781	0.819	0.859	0.902	0.949

Table 1
Model correlation matrix resulting from the function $\alpha + (1-\alpha)\exp(\beta|t_i - t_j|)$.

	0.25	0.5	1	1.5	2	2.5	3	4	5	7
0.5	0.842									
1	0.625	0.790								
1.5	0.623	0.784	0.997							
2	0.533	0.732	0.811	0.815						
2.5	0.428	0.635	0.724	0.729	0.976					
3	0.327	0.452	0.543	0.538	0.568	0.546				
4	0.446	0.581	0.612	0.617	0.686	0.658	0.594			
5	0.244	0.344	0.443	0.446	0.497	0.492	0.608	0.485		
7	0.333	0.453	0.519	0.523	0.573	0.551	0.675	0.645	0.602	
9	0.263	0.366	0.425	0.430	0.477	0.458	0.602	0.567	0.520	0.989

Table 2
Market forward rate correlations for GBP.

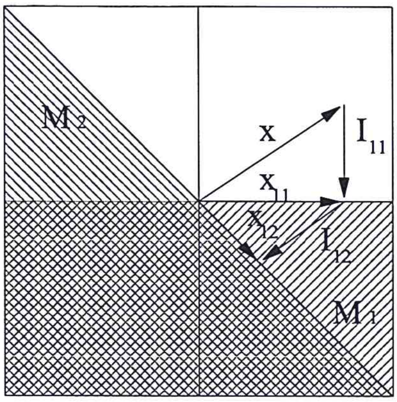

Fig. 1. Alternating Projection on closed subsets of $\mathbb{R}^2$.

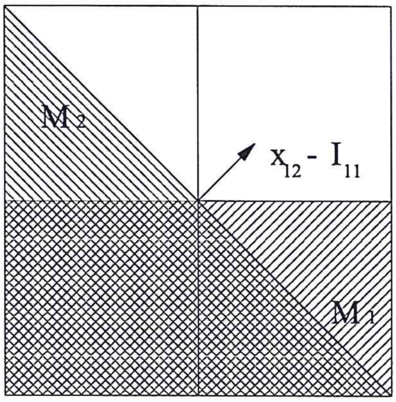

Fig. 2. The effect of Dykstra's correction.

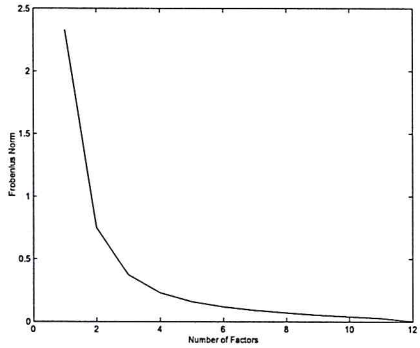

Fig. 3. Convergence to the model correlation matrix measured in Frobenius norm.

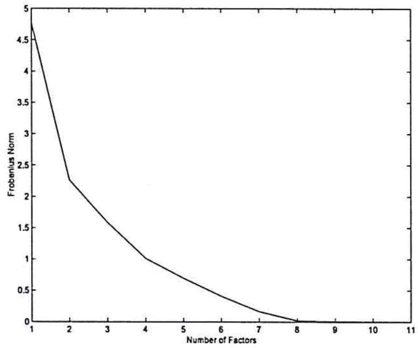

Fig. 4. Convergence to the market correlation matrix measured in Frobenius norm.

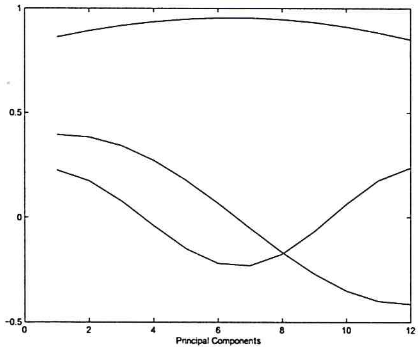

Fig. 5. The first three principal components of the rank three approximation and the model correlation.

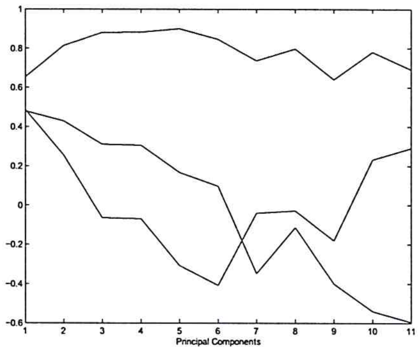

Fig. 6. The first three principal components of the rank three approximation and the market correlation.

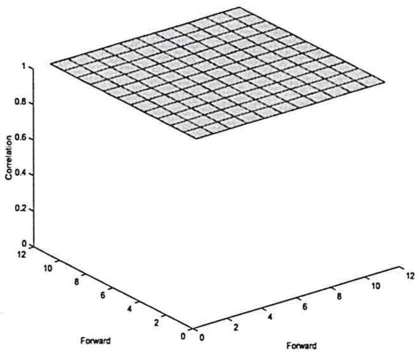

Fig. 7. Rank one approximation.

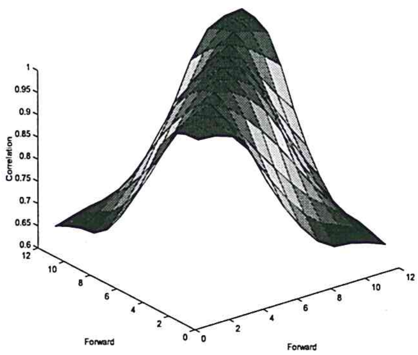

Fig. 8. Rank three approximation.

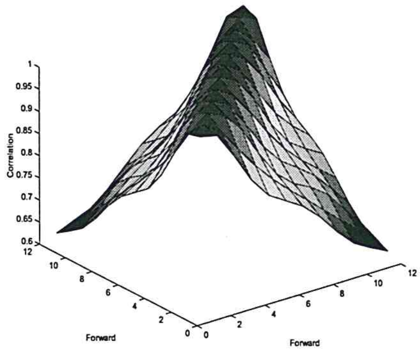

Fig. 9. Rank five approximation.

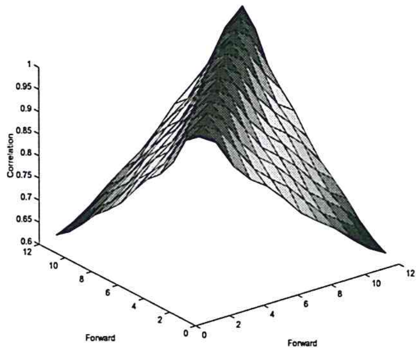

Fig. 10. Rank seven approximation.

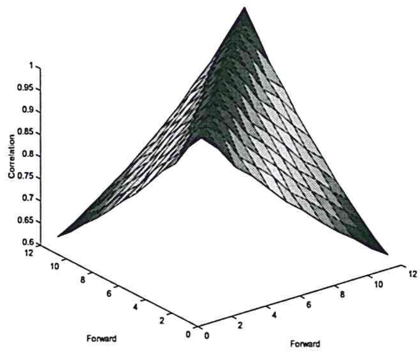

Fig. 11. Rank nine approximation.

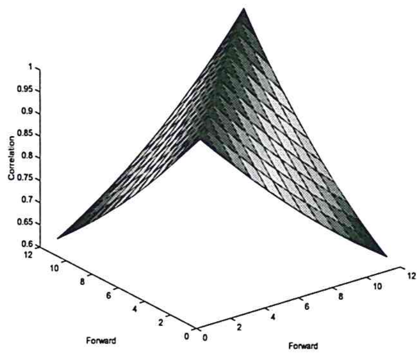

Fig. 12. Model correlation surface.

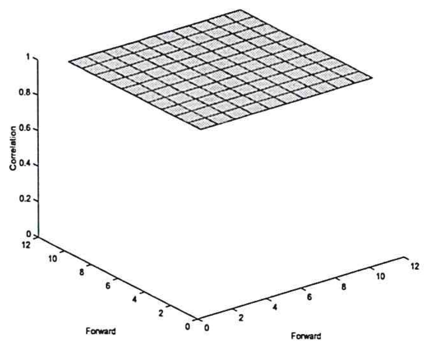

Fig. 13. Rank one approximation.

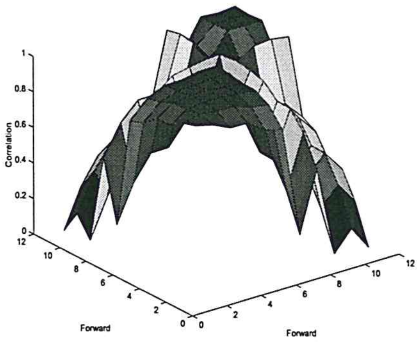

Fig. 14. Rank two approximation.

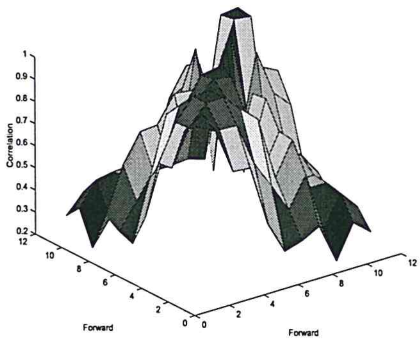

Fig. 15. Rank four approximation.

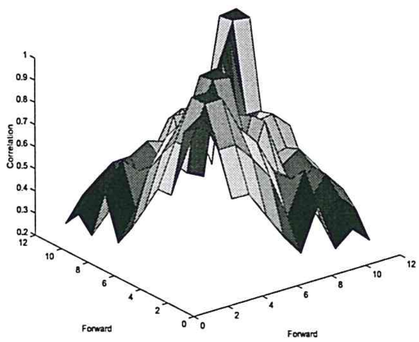

Fig. 16. Rank six approximation.

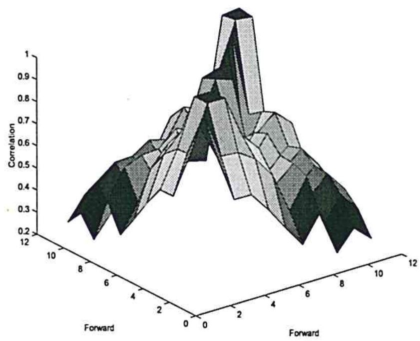

Fig. 17. Rank nine approximation.

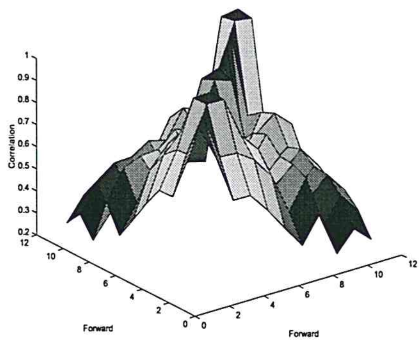

Fig. 18. Market correlation surface.