Financial Options Research Centre, 12 November 2003

IMPLIED KERNEL MODELS

PETER WEIGEL
Financial Options Research Centre, Warwick Business School,
University of Warwick, Coventry CV4 7AL, Great Britain

We develop a class of models within the pricing kernel framework. I.e., we model the pricing kernel directly, and not a particular interest rate or a set of rates. The construction of the kernel is explicitly linked to the calibrating set of instruments. Thus, once the kernel is constructed it will price correctly the chosen set of instruments, and have a low-dimensional Markov structure. We test our model on yield, at-the-money cap, caplet implied volatility surface, and swaption data. The quality of fit is very good.

Keywords: Term structure of interest rates, derivative pricing, pricing kernel, radial and ridge functions.

1. Introduction

The market models developed by Miltersen *et al* (1997) and Brace *et al* (1997) are very popular among practitioners, as they allow almost instantaneous calibration to liquid market prices. The main disadvantage of market models turned out to be their high-dimensional Markov structure. This inhibits the use of these models for pricing exotic products, where the short rate approach is still the preferred choice. Furthermore, extensions of the market models to account for volatility smiles and skews are quite complex.

At this point we ask what we might expect from a good term structure model. We want it to produce explicit solutions for interest rate derivatives. Most of the interest rate derivatives in a portfolio will have quite distinct structures. To obtain explicit solutions for even a small subset of derivatives we will need to make very strong modelling assumptions at the expense of the other derivatives. It is questionable whether we want to sacrifice prudent modelling assumptions to achieve this.

Do we want exact calibration to the prices of all traded derivatives? All derivatives are traded with a bid-ask spread. So the price is not uniquely defined. Furthermore, the prices are not updated continuously and simultaneously. Calibrating the model exactly to the quoted prices will mean calibrating the model to old and improperly synchronised information. So, it is again questionable whether it is sensible to calibrate the model exactly. However, one might expect that the actual prices will not be too far from the observed ones, so the prices should be reasonably well approximated by the model.

Do we want the model to produce simple numerical solutions? The answer is yes. This is an important property. In a trading environment one needs the price information very fast. So, one expects a model to deliver a price within seconds, even for exotic-type products.

Thus there is a need for finding a class of models that enjoys the same ease of calibration to liquid market prices as the market models, and at the same time has a low-dimensional Markov structure as in the case of short rate models. It should be flexible in fitting volatility smiles and skews. Furthermore, it should produce simple numerical solutions for most of the interest rate derivatives.

In this paper we work within the pricing kernel framework, i.e. framework in which the bond price can be represented as

$$B(t,T) = \mathbb{E}^{\mathbb{P}^*}\left[\frac{K_T}{K_t}|\mathcal{F}_t\right],$$

where K_t is positive pricing kernel and $\mathbb{P}^*$ some probability measure equivalent to the objective $\mathbb{P}$. The pricing kernel framework has been studied far less than the HJM and short rate frameworks. Only a few authors have exploited this framework directly. The idea of using this framework in arbitrage pricing was pioneered by Constantinides (1992). It was then pursued by Flesaker and Hughston (1996a), Rogers (1997b), Balland and Hughston (2000), and Hunt *et al* (2000). We describe their ideas in Sec. 2.

Though we model the pricing kernel, we link its construction explicitly with the calibrating set of instruments. Thus, once the kernel is constructed it prices correctly the chosen set of instruments and has a low-dimensional Markov structure. In particular, we assume that a pricing kernel K_t can be approximated by a series of functionals of some underlying Markov process, i.e.,

$$K_t \equiv \sum_i a_i(t) f_i(t, X_t).$$

The coefficients of this approximation are then implied from the set of liquid market instruments, such as bond, caps, swaptions, etc. We call models of this type Implied Pricing Kernel (IPK) models.

The outline of this paper is as follows. In Sec. 2, we motivate our modelling framework and describe its relation to other pricing kernel models. We also briefly describe the literature on modelling non-flat implied volatilities. In Sec. 3, we introduce several families of approximating functions that can be used to approximate the pricing kernel. We also discuss a general Gaussian diffusion process that we use in the implementation of IPK models. Furthermore, we present pricing formulae for several fixed income instruments within the pricing kernel approach. We discuss implementation issues and conduct several calibrating studies in Sec. 4. We conclude in Sec. 5.

2. Related Research

In this section we motivate our class of models and describe its relation with the existing pricing kernel literature. As our approach can also handle non-flat volatilities, we also discuss literature relating to this topic.

2.1. *Kernel Models*

To define a pricing kernel model one needs two ingredients: a model for the underlying noise in the economy and a functional form relating that noise to the pricing kernel. The underlying noise in the economy is usually modelled by a simple stochastic process, such as the Ornstein-Uhlenbeck process. Then the pricing kernel is defined to be some strictly positive function of the process.

For example, Constantinides (1992) chooses the kernel,

$$K_t = \exp\left(-\alpha t + \sum_{i=1}^{N}(X_{i,t} - \alpha_i)^2\right)$$

where the stochastic process is a sum of squares of displaced Ornstein-Uhlenbeck processes $X_{i,t}$, and the function relating the process to the kernel is simply the exponential. Das and Foresi (1996) choose as the driving process the sum of two components, the Ornstein-Uhlenbeck process X_t with stochastic mean, and a pure jump process $y(t)$. The function is again exponential,

$$K(t) = \exp(-y_t - X_t).$$

Flesaker and Hughston (1996a) in their *rational log-normal* model use a kernel that can be be written

$$K_t = f(t) + \exp(c_t + X_t),$$

where f_t and c_t are deterministic functions. In this kernel the underlying process is a simple Gaussian diffusion and the kernel is again defined as the exponential of this diffusion plus some deterministic functions. Rogers (1997b) considers kernels of the form

$$K_t = \frac{e^{-\alpha t} R_\alpha g(X_t)}{R_\alpha g(X_0)},$$

which is a positive supermartingale, i.e. he defines a positive supermartingale of the resolvent of some process X_t with function $\exp(-\alpha t)$. As examples of his framework, among others, he considers the exponential-linear kernel

$$K_t = \exp(-\alpha t + a \cdot X_t),$$

the exponential-quadratic

$$K_t = \exp(-\alpha t + \frac{1}{2}(X_t - c)^T Q(X_t - c)),$$

and the quadratic kernel,

$$K_t = \gamma + \frac{1}{2}(X_t - c)^T Q(X_t - c).$$

In all above examples the functional form was a simple positive function such as an exponential or quadratic. An opposite point of view has been taken by Hunt[a] *et al* (2000). They develop a class of pricing kernel models, the Markov-functional interest rate models (MFM), in which the positive functional is constructed from the market price information. As the underlying noise in the economy they choose a simple Gaussian diffusion.

For example, in their LIBOR Markov-functional model they choose the reciprocal of a bond of fixed maturity as a pricing kernel, $K_t \equiv 1/B(t, T_{n+1})$. The market price information, in this case, is given by a set of cap prices. More precisely, they take as given the set of caplet prices for maturities T_i, $i = 1, \ldots, n$, and for all possible strikes. They choose as the underlying Markov process Gaussian diffusion of the form

$$dX_t = \sigma_t^n dW_t,$$

where $\sigma_t^n = \sigma \exp(at)$. To specify the model completely they use a backward induction. To start the induction they assumed that the last forward LIBOR rate follows a log-normal diffusion[b] $dL_t^n = \sigma_t^n L_t^n dW_t$, under the forward measure $\mathbb{P}_{n+1}$, corresponding to the numéraire $B(t, T_{n+1})$. Thus, the distribution of $B(T_n, T_{n+1})$ can be recovered from its dependence on $L_{T_n}^n$. To determine the model completely one needs to find functional forms of the kernel $K(X_{T_i})$ for the dates T_i, $i = 1, \ldots, n-1$. These functional forms can be found by inverting the digital caplet prices into the bond price $B(T_i, T_{n+1})$.

To summarise, the main idea of Markov functional models is that the distribution of a pricing kernel can be recovered from the prices of digital options via an inversion procedure. Thus, by construction, the LIBOR Markov-functional model prices exactly the digital caplets for a given set of maturities and a continuum of strikes, and consequently the initial term structure. However, the main disadvantage of this method is that it imposes a rather rigid distribution on the kernel which is needed to price the digital caplets correctly. Furthermore, the method is rather difficult to generalise to higher dimensions. We have implemented elsewhere a multidimensional version of the Markov functional model and have found that the results are quite similar to the one dimensional case. Thus, it is not clear if this method provides any more generality when working in higher dimensions.

[a]See, also, Hunt *et al* (1996) and Hunt and Kennedy(1998b).
[b]This assumption is not necessary; we only need to assume that the functional form of the forward LIBOR rate with respect to some underlying Markov process is explicitly known (and is monotone function of the process).

It is worth mentioning a parallel paper by Balland and Hughston (2000). They develop a lattice type model based on exactly the same idea as the MFM framework. However, the implementation of this lattice model is rather cumbersome. They make use of the change of numéraire technique in every time step, which makes the method very tedious to implement. At the heart of the method, as with MFM, is the relation of the digital option prices with the functional form of the kernel.

The pricing kernel models described above can be thought of as belonging to two classes: parametric and non-parametric. Models belonging to the parametric class are pricing kernel which are defined as simple positive functional forms, such as an exponential or quadratic, of some Markov process. In the models belonging to the non-parametric class the functional form of the kernel is implied non-parametrically from the data.

In this paper we take a "middle way" point of view. We assume that the pricing kernels can be approximated by a series of functions of some Markov process. The actual coefficients and parameters of the series are then implied from the market prices, such as bonds, caps, swaptions, etc. We refer to this class of models as Implied Pricing Kernel models (IPK). This class of models presents several advantages over previous pricing kernel models. It has low-dimensional Markov structure. It is just as simple in a one- as in a multi-dimensional framework. It is flexible in fitting volatility skews. Furthermore, it does not require fitting continuum of price information to construct its functional form. Moreover, this method allows the analytical pricing of certain instruments, as in the case of parametric kernel models.

2.2. *Modelling Non-Flat Volatilities*

When plotting implied volatilities of the caplet prices observed in the interest rate market[c] against strikes and maturities, one usually observes non-flat surfaces. This means that the distribution of the forward LIBOR rates is not log-normal as implied by the Black formula. To account for this phenomenon several modelling approaches have been suggested. They range from simple parametric to fully non-parametric methods. We briefly describe them in this section.

The first approach is based on an extension of the standard log-normal LIBOR market model by assuming alternative dynamics for the forward-rate process that lead to volatility smiles or skews. For example, Andersen and Andreasen (2000), used the Cox (1975) CEV process for the forward rate. Zühldorf (2000) considered affine and quadratic volatility functions for the instantaneous volatility function of the forward rate diffusions. All these extensions exhibit quite flexible volatility skews. However, these extensions do not allow for the calibration of the whole volatility surface.

The class of models for the dynamics of forward LIBOR rates introduced recently by Brigo and Mercurio (2000), and Brigo *et al* (2002), can be considered as a semi-parametric approach. It is based on the assumption that the forward rate density is

[c]The same phenomenon occurs in the equity and foreign-exchange markets as well.

given by a mixture of known basic densities. Simple log-normal dynamics as well as mixture dynamics can be generalised by a displace-diffusion technique. This involves shifting a process by a constant. Both, the mixture-diffusion and displace-diffusion techniques enable better, albeit not exact, fit to the implied volatilities.

Yet another way of modelling non-flat volatilities has recently been considered by Joshi and Rebonato (2001). They consider a stochastic volatility extension of the standard LIBOR market model. In particular, they assume a certain parametric form for the instantaneous forward volatility function. The parameters of this volatility function are assumed to follow certain stochastic processes. By a clever choice of implementation procedures they achieve very good results both in fitting the implied volatilities (though not exact), and in rapid pricing of path-dependent options.

Finally, the approach based on the assumption of continuum of traded strikes can be considered as a fully parametric. This goes back to Breeden and Litzenberger (1978). This method[d] is commonly referred to as implied pricing. The main problems of this method are numerical instability and the need for interpolation between option prices[e] between consecutive strikes. The models described above by Hunt *et al* (2000) and Balland and Hughston (2000) can be considered as examples of this approach as well.

In this section we have described several approaches to model non-flat volatilities. They range from parametric to fully non-parametric methods. The characteristic feature of parametric methods is that they are rather limited in their flexibility to fit shapes of the implied volatility structures whereas non-parametric methods fit any implied volatility exactly. As we have noted in the introduction the exact fit is probably not desirable, although a reasonably good fit may be expected from a good term structure model. In this light the semi-parametric model yield the best result. In the next section we describe our implied pricing kernel class of models which, can be considered as a semi-parametric approach.

3. Modelling Framework

In this section we describe two classes of functions which can be used to approximate a pricing kernel. The member functions of both classes are strictly positive. Thus, if we choose only positive coefficients in the approximation then our model will be arbitrage-free. Note, both classes of functions are suitable for multivariate approximation. This allows a simple extension of the IPK models from a one-factor to a multi-factor framework. We also describe the underlying process we used in the implementation of the model.

We start by assuming that the unknown kernel is of the form

$$K_t = f(t, X_t), \tag{3.1}$$

[d]See also Amin and Ng (1997), and Coutant *et al* (2001).
[e]Alternatively, the interpolation can be performed on Arrow-Debreu prices, the density function, or local drift and volatility functions.

where X_t is some Markov process and f_t is a strictly positive continuous function on $\mathbb{R}^d$ for every t. Furthermore, we take the view that the pricing kernel (3.1) can be approximated by a kernel of the form,

$$K_t = \sum_i a_i(t) f_i(t, X_t), \tag{3.2}$$

where X_t is some Markov process and f_i a family of strictly positive continuous functions.

The first class of functions is that of positive functions which have radial symmetry. A real-valued function F on an inner-product space is said to be *radial* if $F(x) = F(y)$ whenever $\|x\| = \|y\|$. If this property is present, the value of $F(x)$ depends only on $\|x\|$, and consequently there exists another function $f : [0, \infty) \to \mathbb{R}$ such that

$$F(x) = f(\|x\|^2).$$

This type of function is referred to in Approximation Theory as a *radial basis function.*

Certain classes of radial basis functions have nice interpolating and approximating properties. Specifically, it can be shown that for, say, a radial function f, for each compact Q in $\mathbb{R}^n$, the set

$$\{x \to f(x - y) : y \in Q\}$$

is fundamental in $C(Q)$, where $C(Q)$ is the set of all continuous functions. Put in another way, for each $g \in C(Q)$ and for each $\epsilon > 0$, there is a linear combination $\sum_{y \in Q} c_j f(x - y)$, so that $\|g(x) - \sum_{y \in Q} c_j f(x - y)\| < \epsilon$. (The sum is finite.)

For example, if f is completely monotone[f] but not constant on $[0, \infty)$, and Q is a compact subset in $\mathbb{R}^n$, then the set of functions

$$\{x \to f(\|x - y\|) : y \in Q\}$$

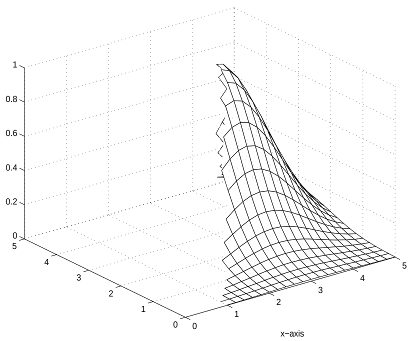
is fundamental in $C(Q)$.

This result provides a rich source of functions that are suitable for approximation of data in the Euclidean spaces $\mathbb{R}^1, \mathbb{R}^2, \ldots$. The following functions satisfy the specified conditions

(a) $f(t) = (t + 1)^{-1}$,
(b) $f(t) = e^{-t}$,
(c) $f(t) = (t + 1)^{-1/2}$.

In Fig. 1 and Fig. 2 we plot two examples of functions of this class. Note the typical bell-shaped form. These functions can be used to interpolate or approximate arbitrary data by functions of the form,

[f]This class of function is due to Schoenberg (1938). In particular, he showed, that the function $x \to f(\|x\|^2)$ is a radial strictly positive definite function on any inner-product space. I.e., for any n distinct points $x_1, x_2, \ldots, x_n$ in such a space the matrix $A_{ij} = f(\|x_i - x_j\|^2)$ is positive definite.



1
0.8
0.6
0.4
0.2
0
5
4
3
2
1
0
0
1
2
3
4
5
x−axis

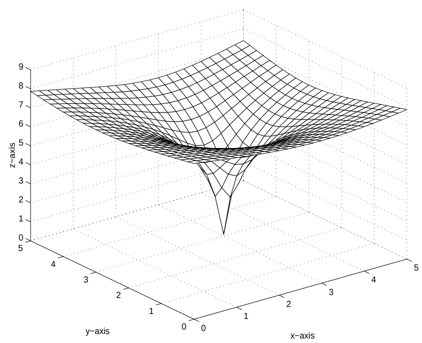


Fig. 3. A radial basis function of the form $\log(1.1 + -a\|x-b\|)$, with $a = 200$ and $b = (3,3)$.

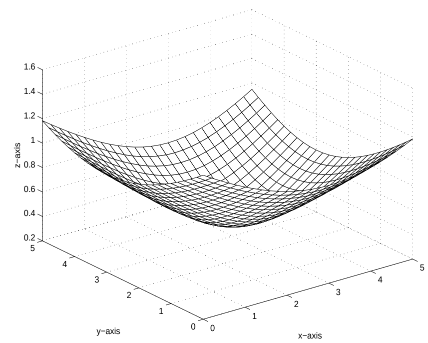


Fig. 4. A radial basis function of the form $\sqrt{0.1 + a\|x-b\|}$, with $a = 0.1$ and $b = (3,3)$.

on X. A function $f : X \to \mathbb{R}$ is called *ridge function* if it can be represented in the form $f = g \circ \phi$, where $g : \mathbb{R} \to \mathbb{R}$ and $\phi \in X^*$. Every continuous linear functional on $\mathbb{R}^s$ is of the form

$$\phi(x) = \alpha_1\zeta_1 + \alpha_2\zeta_2 + \cdots + \alpha_s\zeta_s.$$

A ridge function on $\mathbb{R}^s$ is then a function of the form

$$f(x) = g(\alpha_1\zeta_1 + \alpha_2\zeta_2 + \cdots + \alpha_s\zeta_s),$$

where $x = \zeta_1, \zeta_2, \cdots, \zeta_s$.

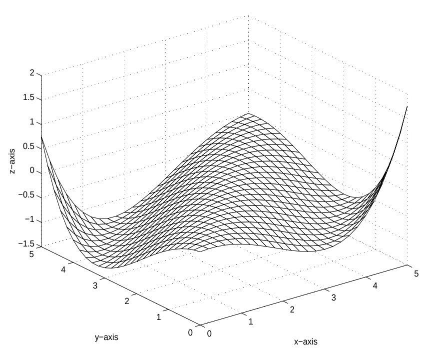


Fig. 5. A typical ridge function.

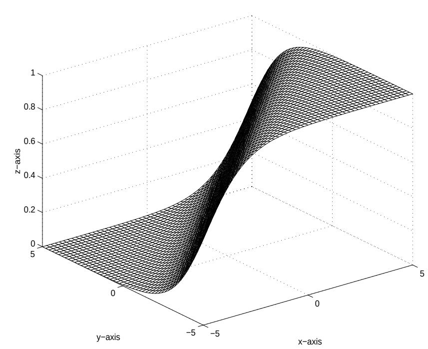


Fig. 6. A sigmoid ridge function.

Fig. 5 shows the ridge function $z = p(x-y)$, where p is a polynomial of degree 4. A single ridge function is very limited in its capacity to approximate an arbitrary continuous function on X. In particular, the graph of a ridge function is a ruled surface. For approximation purposes one considers a linear combination of ridge functions of the form

$$f = \sum_{i=1}^{m} c_i g_i \circ \phi_i \qquad (g_i \in C(\mathbb{R}), \phi_i \in X^*).$$

The coefficients c_i are unnecessary in this representation, since they can be absorbed by the functions g_i. It is important to note that not all continuous functions on X are linear combinations of ridge functions. But every continuous function on X can be well approximated by such linear combinations.

For some applications, such as neural networks[h], it is very desirable to employ a single function g in the ridge functions, which is usually called a *sigmoid function* and denoted σ, and has the property that

$$\lim_{t\to\infty} \sigma(t) = 1 \text{ and } \lim_{t\to-\infty} \sigma(t) = 0.$$

The following functions are examples of sigmoids:

(a) The logistic sigmoid $f(x) = \frac{1}{1+\exp(-ax)}$.
(b) The Heaviside function $f(x) = 1$ if $x \geq 0$, $f(x) = 0$ if $x < 0$.

A typical example of ridge function is plotted in Fig. 5, and sigmoid ridge functions in two dimensions are shown in Fig. 6.

For numerical implementation of the models of the form (3.2) we choose a d-dimensional linear stochastic differential equation of the form

$$dx_t = (A(t)x_t + a(t))dt + B(t)dW_t, \qquad x_{t_0} = c \tag{3.3}$$

where $A(t)$ is a $d \times d$-matrix valued function, $a(t)$ is $\mathbb{R}^d$-valued function, and $B(t)$ is $d \times m$-matrix valued. This process is very tractable and has been a popular choice in interest rate modelling, especially within the short rate framework. Examples are Vasicek (1977), Langetieg (1980), El Karoui and Lacoste (1995).

The solution of the linear SDE (3.3) is a *Gaussian* stochastic process x_t, with mean value

$$m_t = Ex_t = \Phi(t)\left(c + \int_{t_0}^{t} \Phi(u)^{-1} a(s)ds\right) \tag{3.4}$$

and covariance matrix

$$\begin{aligned} K(s,t) &= \mathbb{E}(x_s - \mathbb{E}x_s)(x_t - \mathbb{E}x_t)' \\ &= \Phi(s)\left(\int_{t_0}^{\min(s,t)} \Phi(u)^{-1}B(u)B(u)'(\Phi(u)^{-1})'du\right)\Phi(t)', \end{aligned} \tag{3.5}$$

[h]This type of function have been used by Bansal and Viswanathan (1993), and Bansal *et al* (1993) used the logistic sigmoid, to approximate the pricing kernel within the asset pricing framework.

where the matrix $\Phi(t) = \Phi(t, t_0)$ is the matrix of solutions of the homogeneous equation[i]

$$\dot{x}_t = A(t)x_t,$$

with the unit vectors $c = e_i$ in the x_i-direction as initial value If, for example, $A(t) \equiv A$ is independent of t, then

$$\Phi(t) = \exp(A(t - t_0)) = \sum_0^{\infty} A^n (t - t_0)^n / n!.$$

Furthermore, if we assume that the matrices A and B, and the vector a, are independent of t, to obtain the moments of (3.3), we need to evaluate the integral of the matrix exponentials in (3.4) and (3.5). This can be done efficiently by the method of diagonal Padé approximation with scaling and squaring as described in Van Loan (1978). Most of the integration in the numerical implementation of the model in this paper has been done with *Gauss-Hermite* quadrature. This type of quadrature is particularly useful when we consider stochastic processes with Gaussian distributions, as they approximate integrals of the type $\int_{-\infty}^{\infty} F(x)e^{-x^2}dx$, and the Gaussian density can be used as the weight function. This method can be generalised to integrals in higher dimensions.

4. Formulae for Pricing Zero-Coupon Bonds, Caps, and Swaptions

In this section we derive expressions for pricing formulae for zero-coupon bonds, caplets and swaptions. These expressions will be used in the calibration studies of the model in the next section. Assuming that the pricing formula has the form (3.2), the price of a zero coupon bond can be expressed as

$$P(t, T) = \frac{\mathbb{E}[K_T|\mathcal{F}_t]}{K_t} = \frac{\sum_i^N a_i(T)\mathbb{E}[f_i(T, X_T)|\mathcal{F}_t]}{\sum_i a_i(t) f_i(t, X_t)},$$

where $\mathbb{E}[\cdot|\mathcal{F}_t]$, expectation operator conditional on the sigma algebra $\mathcal{F}_t$, taken with respect to a reference measure, the equivalent to the risk neutral measure. The value of a caplet at time t with maturity date T_{n-1} and payoff at the date T_n,follows from observing that $P(T_{n-1}, T_n) = (1 + L(T_{n-1})\delta)^{-1}$, where $L(T_{n-1})$ is the δ-period LIBOR rate,

$$\begin{aligned}
\mathrm{Cpl}(t, T_{n-1}, T_n) &= \mathbb{E}\left[\frac{K_{T_{n-1}}}{K_t} P(T_{n-1}, T_n)\delta(L(T_{n-1}) - K)^+ |\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{K_{T_{n-1}}}{K_t}(1 - \tilde{\delta}P(T_{n-1}, T_n))^+ |\mathcal{F}_t\right]
\end{aligned} \qquad (4.6)$$

where $\tilde{\delta} = 1 + \delta K$, with K the strike of the caplet and δ the accrual period of the underlying LIBOR rate.

[i]Further details on SDE's of the form (3.3) can be found in Arnold (1974).

Similarly, the price of a (payer) swaption at time t with maturity date T_0 and last payment at time T_N, with the length of the accrual period δ, is given by

$$\mathrm{Swp}(t, T_0, T_N) = \mathbb{E}\left[\frac{K_{T_0}}{K_t} D(T_0, T_N)(\kappa(T_0, T_N) - K)^+ | \mathcal{F}_t\right],$$

where K is the strike, κ is the swap rate defined as

$$\kappa(T_0, T_N) = \frac{(1 - P(T_0, T_N))}{\delta \sum_1^N P(T_0, T_i)},$$

and $D(T_0, T_N) = \delta \sum_{i=1}^N P(T_0, T_i)$.

For our numerical study of the model in the next section we are going to use one particular family of radial functions.

$$F(t) = \sum_{j=1}^n a_j(t) e^{-\|x - b_j(t)\|^2}$$

This kernel allows analytical solution for the zero-coupon bond price,

$$P(t, T) = \frac{\sum_{j=1}^n a_j(T) e^{-(m_{Tt} - b_j(T))^T (I + 2V_{Tt})^{-1} (m_{Tt} - b_j(T))}}{\sqrt{\det(I + 2V_{Tt})} \sum_{j=1}^n a_j(t) e^{-(X_t - b_j(t))^T (X_t - b_j(t))}},$$

where[j] m_{Tt} and V_{Tt} are conditional mean and variance of the process X_T conditioned on the process at time t.

5. Implementation and Numerical Study of the Model

In this section we calibrate the implied kernel model to the data and study its properties. We choose the model based on the Gaussian radial functions and calibrate it to three sets of data: the yield curve and at-the-money caps for the GB pound on February 3, 1995, the yield curve and the caplet black implied volatility surface for the GB pound on August 4, 2000, and the yield curve and at-the-money black implied swaption matrix for 4th August, 2000. For all implementations we use only three basis functions.

5.1. *Yield Curve and ATM Caps*

In the first study of the model, we investigate whether our implied pricing kernel model was able to fit zero-coupon yield curve and ATM caps. To calibrate the models we used data from the UK market on February, 3, 1995, which we obtained from Brace *et al* (1997). We used the weighted least-squares spline with a judiciously chosen set of knots to construct whole term structure. The plots of the resulting forward rate and yield curves are in Fig. 7.

[j]This expression for the bond price can be easily confirmed by completing the square in the Gaussian density.

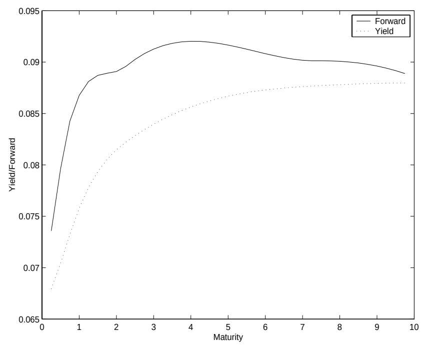


Fig. 7. Yield and forward rate curves. Date: 03-Feb-95.

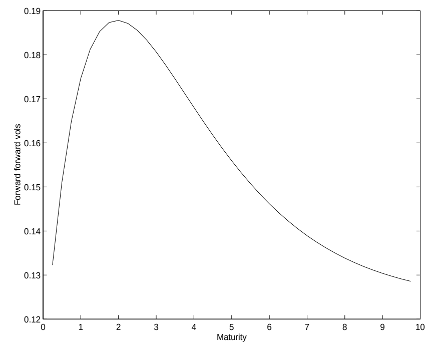


Fig. 8. Forward forward volatility curve. Date: 03-Feb-95.

To calibrate the model to the ATM cap prices we used a similar algorithm to the one used in calibration of the market models. I.e., we recovered the black implied forward-forward volatilities. This gave us prices of individual caplets comprising the caps. We plotted the forward forward volatilities in Fig. 8. As we see, it has the typical hump shape around 2 years to maturity and is monotonic elsewhere. We then calibrated our model to these caplet prices and subsequently to the quoted ATM cap prices.

Table 1. Results from the model calibration to the ATM caps and zero coupon bond prices. Date: 03-Feb-1995.

Length	A-T-M Strike (%)	Black Vol (%)	Market Price (bp)	Model Price (bp)	Error (%)	Error (bp)
1	7.91	15.50	27	27	0.04	0.01
2	8.41	17.75	99	99	0.32	0.32
3	8.60	18.00	183	186	1.57	2.92
4	8.74	17.75	272	273	0.40	1.09
5	8.82	17.75	356	356	-0.07	-0.25
7	8.88	16.50	508	507	-0.25	-1.25
10	8.90	15.50	701	699	-0.24	-1.71

The calibration results of our model are presented in Table 1. The fit to the yield curve is exact. The fit to cap prices is quite good. The largest error is just below 3bp. Measured in percentage terms most of the errors are less than 0.5 percent with the exception of the 3-year cap, which has an error of 1.5 percent. For comparison reasons we also calculated the swaption prices implied by our calibrated model. Then we compared the results with the swaption market prices. We presented the results in Table 2. Note that we did not try to calibrate the model to these swaption prices. As can be expected the errors are quite large.

Table 2. Model and market swaption prices, resulting from calibration to the ATM caps and zero coupon bond prices. Date: 03-Feb-1995.

Maturity × Swap Length	A-T-M Strike (%)	Black Vol (%)	Market Price (bp)	Model Price (bp)	Error (%)	Error (bp)
0.25 × 2	8.56	16.75	51	44	-13.05	-6.6
0.25 × 3	8.74	16.50	73	73	-0.44	-0.3
0.25 × 5	8.93	15.00	105	121	15.58	16.3
0.25 × 7	8.99	13.75	124	158	26.70	33.2
1 × 4	9.13	15.50	172	193	11.99	20.6
1 × 9	9.13	13.25	270	316	16.99	46.0
2 × 8	9.17	12.75	312	352	12.62	39.4

Most of the swaption prices are higher than the market prices. This is in line with other findings in the literature. For example, de Jong *et al* (2001) analyse the LIBOR market models on the US caplet and swaption data. They find that swaptions are overpriced with the average absolute pricing error around 1 volatility point. Similar results have been found by Driessen *et al* (2000) who investigate the performance of HJM multi-factor models on the US data.

Table 3: Implied black swaption volatility, resulting from calibration to the ATM caps and zero coupon bond prices. Date: 03-Feb-1995.

	1	2	3	4	5	6	7	8
2	16.43	14.67	16.92	16.15	15.04	9.87	5.66	9.54
3	17.06	15.63	16.76	15.88	11.94	8.94	7.91	
4	17.39	15.84	16.53	13.52	10.91	9.92		
5	17.40	15.96	14.66	12.52	11.45			
6	17.39	14.62	13.75	12.80				
7	16.14	13.92	13.89					
8	15.45	14.08						
9	15.52							

We also calculated the implied black swaption volatility matrix resulting from the model's calibration to the ATM caps. This matrix can be seen in Table 3. In most of the cases the contracts change smoothly from maturity to maturity, and between different underlying swap lengths.

5.2. *The Yield Curve and Caplet Volatility Surface*

In this study we tested whether the implied pricing kernel model could fit the yield curve and caplet volatility surface. The dataset was the yield curve and the caplet black implied volatility surface for the GB pound on August 4, 2000. In particular, we fitted the yield curve with maturities from 2 to 10 years with quarterly steps, and caplet prices with the same maturities and strikes ranging from 2% to 10%, with 0.01% step size. We have plotted the smoothed yield and forward curves in Fig. 9.

The fit to the yield curve is exact. In Table 4, we present the results of the fit to the caplet prices. The first line in each block presents the market prices, and the second the model prices. The third and forth lines present errors expressed in percentage terms and basis points.

The quality of fit of the caplet surface is very good for the longer maturities. In most cases it is less than 1%. However, for maturities less than 5 years and high strikes the quality decreases significantly. The reason for, we believe, lies in the low precision of the implementation programme. We plotted the market caplet prices in Fig. 11, and the model caplet prices in Fig. 12. We have also plotted market and model caplet prices together for selected maturities of 2.5, 5, 7.5, 10 years to maturity in Fig. 13.

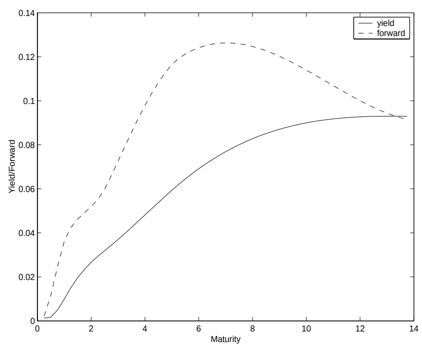


Fig. 9. Yield and forward rate curves. Date: 04-Aug-00.

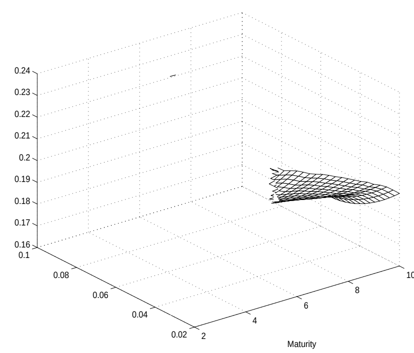


Table 4. Calibrated model and market caplets prices for a range of strikes and maturities, together with errors expressed in basis points and percentage. Date: 04-Aug-00.

	0.02%	0.03%	0.04%	0.05%	0.06%	0.07%	0.08%	0.09%	0.1%
2	75.09	51.79	30.37	14.51	5.95	2.06	0.61	0.15	0.03
M	74.60	51.48	30.42	14.69	5.84	1.77	0.48	0.13	0.04
bp	-0.49	-0.30	0.05	0.18	-0.10	-0.29	-0.13	-0.02	0.01
%	-0.65	-0.59	0.16	1.20	-1.79	-16.18	-26.78	-15.97	24.54
3	114.73	93.01	72.08	52.97	36.62	23.29	14.16	8.31	4.11
M	115.55	93.70	72.45	52.85	36.11	22.93	14.06	8.43	5.07
bp	0.83	0.69	0.37	-0.12	-0.51	-0.37	-0.09	0.13	0.96
%	0.71	0.73	0.51	-0.23	-1.41	-1.60	-0.67	1.53	18.94
4	156.05	136.02	116.27	97.28	79.21	62.76	48.63	36.05	26.61
M	156.70	136.61	116.71	97.29	78.81	62.00	47.98	36.63	27.82
bp	0.64	0.59	0.44	0.01	-0.40	-0.76	-0.65	0.57	1.21
%	0.41	0.43	0.38	0.01	-0.50	-1.23	-1.35	1.57	4.35
5	172.70	154.79	137.11	119.90	103.18	87.53	73.04	59.31	48.43
M	174.06	156.03	138.09	120.38	103.10	86.73	72.14	59.46	48.77
bp	1.36	1.24	0.99	0.48	-0.09	-0.80	-0.90	0.15	0.34
%	0.78	0.79	0.71	0.40	-0.09	-0.92	-1.25	0.26	0.70
6	165.99	150.03	134.27	118.65	103.95	89.30	76.71	64.13	53.54
M	166.74	150.75	134.86	119.17	103.85	89.30	76.13	64.45	54.36
bp	0.75	0.73	0.59	0.52	-0.10	0.00	-0.58	0.32	0.82
%	0.45	0.48	0.44	0.43	-0.10	0.00	-0.77	0.49	1.50
7	150.13	136.14	122.35	108.64	95.78	82.91	71.42	60.56	49.80
M	150.45	136.34	122.35	108.59	95.26	82.69	71.28	61.10	52.22
bp	0.32	0.20	-0.00	-0.06	-0.52	-0.21	-0.14	0.54	2.43
%	0.21	0.14	-0.00	-0.05	-0.54	-0.26	-0.19	0.89	4.65
8	130.67	118.36	106.21	94.33	83.07	71.82	62.37	53.02	44.24
M	130.92	118.48	106.18	94.18	82.69	72.01	62.30	53.64	46.08
bp	0.25	0.12	-0.03	-0.15	-0.38	0.19	-0.07	0.62	1.84
%	0.19	0.10	-0.02	-0.16	-0.46	0.26	-0.12	1.16	3.99
9	111.21	100.11	89.25	79.01	69.14	60.36	52.29	44.23	38.17
M	111.15	100.15	89.35	78.92	69.13	60.16	52.00	44.75	38.44
bp	-0.06	0.04	0.10	-0.09	-0.01	-0.19	-0.29	0.52	0.28
%	-0.05	0.04	0.11	-0.11	-0.02	-0.32	-0.57	1.17	0.72
10	92.96	83.11	73.54	64.62	55.92	48.73	41.75	35.12	30.51
M	92.91	83.13	73.59	64.48	56.08	48.55	41.68	35.63	30.42
bp	-0.05	0.02	0.05	-0.14	0.17	-0.17	-0.07	0.52	-0.09
%	-0.06	0.03	0.07	-0.21	0.29	-0.36	-0.17	1.45	-0.29

5.3. *The Yield Curve and ATM Swaptions*

In this exercise we tested whether the implied pricing kernel model could fit the yield curve and ATM swaption prices. The dataset was again GB pounds on August 4, 2000, comprising the zero coupon bond prices and ATM swaptions. The model bond prices fit the market prices almost exactly. The Table 5 presents the market and model swaption prices for several maturities and lengths of the underlying swap contracts together with percentage errors of the fit. In many cases the fit is quite good, but in some cases the error is quite large. This is not unexpected, as our model is only one-factor. We would need higher-factor models to achieve a better fit.

Table 5. Swaptions contracts expressed as maturity $\times$ length. Model fit to the ATM swaption prices in basis point units together with market prices and the percentage errors. Date: 04-Aug-00.

	2	3	4	5	6	7	8	9	10
0.5	24	41	55	62	64	64	58	53	47
Mk	24	41	55	62	64	64	58	53	47
%	0.07	-0.00	0.00	0.01	-0.00	-0.00	0.00	0.00	-0.00
1	43	74	113	91	121	117	113	79	80
Mk	51	87	113	124	123	117	106	95	84
%	15.36	15.63	-0.00	26.08	1.99	-0.01	-5.97	17.47	5.38
2	102	174	197	201	225	182	175	157	150
Mk	119	179	220	232	228	212	191	171	150
%	14.50	2.91	10.51	13.33	1.40	14.11	8.75	7.99	0.00
3	184	247	298	302	295	258	240	227	216
Mk	195	269	310	318	307	284	255	227	202
%	5.33	8.05	3.88	4.95	4.14	8.98	5.94	-0.00	-7.12
4	244	339	381	364	369	332	305	291	
Mk	265	346	384	385	369	340	305	272	
%	7.86	2.03	0.61	5.42	-0.00	2.24	0.00	-6.98	
5	323	410	444	435	440	399	368		
Mk	329	410	444	435	415	381	344		
%	1.95	0.00	0.01	0.00	-6.05	-4.77	-6.85		
6	387	467	502	502	505	461			
Mk	385	465	496	484	461	427			
%	-0.45	-0.42	-1.26	-3.65	-9.41	-7.99			
7	428	520	560	563	564				
Mk	428	510	538	523	499				
%	0.00	-1.92	-4.13	-7.68	-13.01				
8	473	571	616	619					
Mk	470	545	569	556					
%	-0.77	-4.81	-8.14	-11.33					
9	515	619	666						
Mk	492	569	597						
%	-4.59	-8.69	-11.62						
10	554	665							
Mk	513	593							
%	-8.06	-12.03							

5.4. *Properties of the Model*

We also investigated distributional properties of the implied kernel model calibrated to the yield curve and the caplet black implied volatility surface for GB pound on August 4, 2000. We simulated the 3-month LIBOR rates implied by our model using Monte-Carlo technique.

Table 6. Mean, standard deviation, skewness and kurtosis for LIBOR rates, resulting from the calibration of the model to the zero-coupon bond and caplets. Date: 04-Aug-00.

Maturity	Mean	St.Dev.	Skewness	Kurtosis
2.00	0.054	0.014	0.686	0.646
2.25	0.058	0.016	0.715	0.678
2.50	0.063	0.019	0.533	0.407
2.75	0.070	0.023	0.482	0.332
3.00	0.078	0.025	0.492	0.354
3.25	0.085	0.028	0.491	0.354
3.50	0.093	0.032	0.740	0.733
3.75	0.100	0.036	0.508	0.373
4.00	0.107	0.038	0.557	0.443
4.25	0.114	0.041	0.540	0.422
4.50	0.120	0.044	0.552	0.444
4.75	0.126	0.047	0.556	0.452
5.00	0.131	0.050	0.516	0.398
5.25	0.136	0.053	0.635	0.587
5.50	0.139	0.055	0.620	0.562
5.75	0.143	0.058	0.682	0.677
6.00	0.145	0.059	0.699	0.713
6.25	0.148	0.061	0.739	0.797
6.50	0.149	0.058	0.633	0.694
6.75	0.154	0.067	0.880	1.055
7.00	0.153	0.061	0.627	0.699
7.25	0.156	0.068	0.822	0.975
7.50	0.158	0.071	0.873	1.083
7.75	0.159	0.071	0.862	1.100
8.00	0.158	0.070	0.811	1.086
8.25	0.163	0.080	0.983	1.364
8.50	0.164	0.081	0.999	1.414
8.75	0.164	0.082	1.043	1.619
9.00	0.164	0.082	0.977	1.371
9.25	0.165	0.085	1.035	1.579
9.50	0.166	0.088	1.092	1.734
9.75	0.166	0.090	1.171	2.077
10.00	0.165	0.089	1.153	2.006

In Table 6 we present the descriptive statistics for the LIBOR rates for maturities between 2 and 10 years, resulting from our simulation, including the mean, standard deviation, skewness and kurtosis. Observe the presence of skewness and kurtosis which grows for longer maturities.

We have also plotted in Fig. 14, the histograms of the LIBOR rates with maturities of 2.5, 5, 7.5, and 10 years. In each plot we have also added the density functions of normal and log-normal distributions with the same moments as the simulated distribution of the LIBOR rates. Note the IPK model produced positive

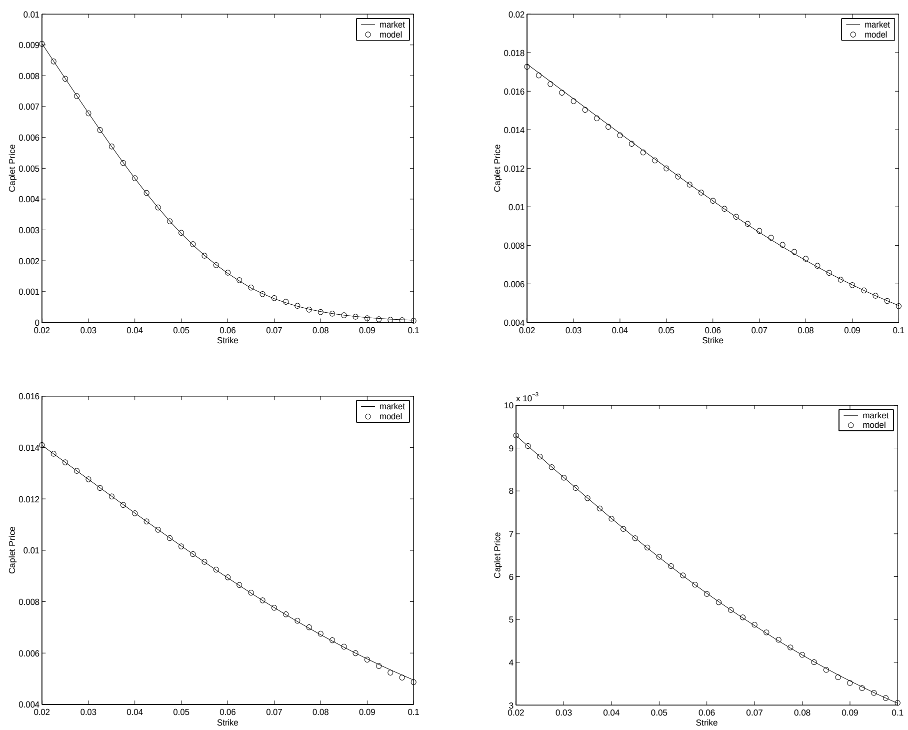
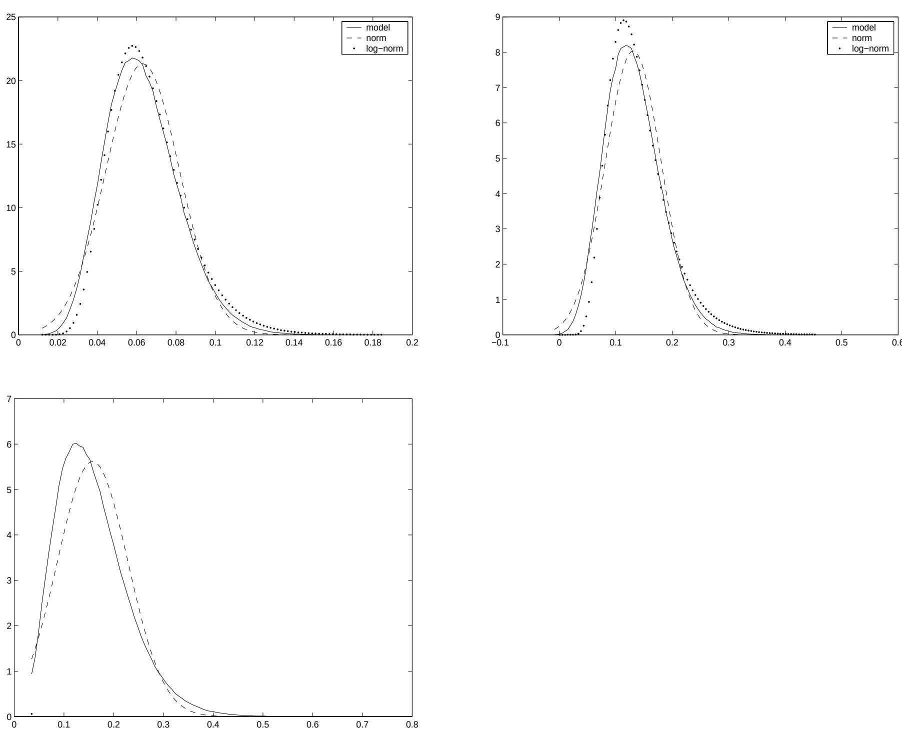


Fig. 13. Model and caplet prices with 2.5, 5, 7.5 and 10 years to maturity. Date: 04-Aug-00.



interest rates though, it was not constrained to do so.

6. Conclusions

In this paper we have introduced a class of pricing kernel models, the Implied Pricing Kernel (IPK) models, for the term structure of interest rates. This class is mainly intended for pricing interest rate derivatives. The main motivation for its development has been the goal of finding an alternative to the market models. This alternative should combine the advantages of the market models such as the ease of calibration to liquid market prices, with the advantages of the short rate models, such as low-dimensional Markov structure. We also wanted this class to be flexible in fitting other price information such as skews etc.

Unlike previous literature which has utilised a pricing kernel approach, we have approximated the pricing kernel by a functional series. Thus, implicitly, we have acknowledged that we do not know its correct functional form. However, we believe that the pricing kernel can be approximated sufficiently closely by a judiciously chosen functional series and the underlying process. We have suggested two types of approximating series: the first is based on the radial basis functions, and the second on the ridge functions. The basis functions of both types are strictly positive. Thus, by choosing only non-negative coefficients in the series we have achieved a strictly positive pricing kernel which guarantees absence of arbitrage in the model. Furthermore, the coefficients and parameters of the series have been chosen so that the model fits the market price information, such as zero coupon bond prices, caps, caplets, and swaptions. As the underlying noise in the economy we chose a simple multi-factor Gaussian diffusion. Our approach is just as simple in a one-factor as in a multi-factor setting. In a summary, the class of implied kernel models is Markov by construction, calibration for a small number of factors is relatively easy, it can deal with American type options, and is flexible in fitting to skews.

The IPK class models have an advantage over the models with a fixed pricing kernel designed by Constantinides (1992), or Flesaker and Hughston (1996a). They are more flexible in fitting the market price information. It also has an advantage over the non-parametric type of kernel models designed by Hunt *et al* (2000), and Balland and Hughston (2000). This non-parametric approach relies on inferring the kernel from the digital caplets or swaption prices. For a given maturity such kernels can only fit one type of instrument only, e.g. a cap price or possibly a set of caplets with different strikes. However, they are incapable of dealing with say a set of caplets and swaptions at the same time, or swaptions with different lengths of underlying swaps. The possibility of extending that type of model to a multi-factor setting is questionable. The IPK models can fit simultaneously to any market price information available. The quality of the fit can be improved by adding more basis functions in the approximating series. Moreover, the extension of the underlying noise to a multi-factor structure is trivial.

We conducted several model calibration studies for one-factor IPK models. This comprised of calibrations to the yield curve and at the money cap prices, the yield curve and caplet implied volatility surface, and the yield curve and swaption data. Overall, we achieved a reasonably good quality of fit. We also studied numerically the distributional properties of the forward LIBOR rates implied by a calibrated IPK model. These studies are, of course, only preliminary. More extensive tests of this class of models is needed to assess its full advantages and shortcomings.

There are several questions that can be addressed by further research. We have suggested two classes of strictly positive approximating functions. It is not clear which class of functions and which members of these classes are better suited for approximation of the pricing kernel. We have assumed a Gaussian diffusion as the underlying noise structure. Other processes may be more appropriate in this case. On the implementation side, better evaluation and calibrating techniques may be needed, especially for multi-factor IPK models.

Acknowledgment

I would like to thank Prof. Stewart D. Hodges for helpful discussions. Thanks to participants of finance workshop at Birckbeck College, London, and finance seminar series at Warwick Business School. Any errors are my own.

Appendix

Lemma A.1 *Let $X \in \mathbb{R}^d$ be a random variable such that its push forward measure $\mathbb{P}_X$ is a measure with density g relative to the λ^d-Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}^d)$. Furthermore, let $\{f_n\}$, be a sequence of real, measurable function on $\mathbb{R}$ converging point-wise to a measurable function f. Then the sequence of random variables $f_n(X)$ converges to $f(X)$ in distribution.*

To prove convergence in distribution we need to show only that the characteristic function of $f_n(X)$, $\hat{f}_n(t)$, converges point-wise to the characteristic function of $f(X)$, $\hat{f}(X)$.

$$\lim_{n\to\infty} \hat{f}_n(t) = \lim_{n\to\infty} \int e^{-ity} \mathbb{P}_{f_n(X)}(dy) = \lim_{n\to\infty} \int e^{-itf_n(x)} \mathbb{P}_X(dx),$$

From the point-wise convergence of f_n to f follows point-wise convergence of e^{-itf_n} to e^{-itf}. Furthermore, from

$$|e^{-itf_n} g(x)| \leq g(x)$$

it follows that the integrand is dominated by a function $g \in L^1(\mathcal{B}^\lceil, \lambda^\lceil)$. The interchange of limit and integration is justified by Lebesgue's dominated convergence theorem. (See Rudin (1976), Theorem 11.32.) □

References

[1] K.I. Amin and V.K. Ng, *Inferring Future Volatility from the Information in Implied Volatility in Eurodollar Options: A New Approach*, The Review of Financial Studies **10** (1997), no. 2, 333–367.

[2] L. Andersen and J. Andreasen, *Volatility Skews and Extensions of the Libor Market Model*, Applied Mathematical Finance **7** (2000), 1–32.

[3] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, John Willey and Sons, 1974.

[4] P. Balland and L.P. Hughston, *Markov Market Model Consistent With Cap Smile*, International Journal of Theoretical and Applied Finance **3** (2000), no. 2, 161–181.

[5] R. Bansal, D.A. Hsieh, and S. Viswanathan, *A New Approach to International Arbitrage Pricing*, The Journal of Finance **XLVIII** (1993), no. 5, 1719–1747.

[6] R. Bansal and S. Viswanathan, *No Arbitrage and Arbitrage Pricing: A New Approach*, The Journal of Finance **XLVIII** (1993), no. 4, 1231–1261.

[7] A. Brace, D. Gatarek, and M. Musiela, *The Market Model of Interest Rate Dynamics*, Mathematical Finance **7** (1997), 127–154.

[8] D.T. Breeden and R.H. Litzenberger, *Prices of State-Contingent Claims Implicit in Option Pricess*, Journal of Business (1978).

[9] D. Brigo and F Mercurio, *A Mixed-up Smile*, Risk (2000).

[10] D. Brigo and F. Mercurio, *Lognormal-Mixture Dynamics and Calibration to Market Volatility Smiles*, International Journal of Theoretical and Applied Finance **5** (2002), no. 4, 427–446.

[11] D. Brigo, F. Mercurio, and F. Rapisarda, *Pricing the Smile in a forward LIBOR Market Model*, Working paper, Banca IMI, 2001.

[12] G.M. Constantinides, *A Theory of the Nominal Term Structure of Interest Rates*, The Review of Financial Studies **5** (1992), no. 4, 531–552.

[13] S. Coutant, E. Jondeau, and M. Rockinger, *Reading PIBOR futures options smiles: The 1997 snap election*, Journal of Banking and Finance **25** (2001), 1957–1987.

[14] J.C. Cox, *Notes on Option Pricing I: Constant Elasticity of Variance Diffusions*, Tech. report, Stantford University, 1975.

[15] S.R. Das and S. Foresi, *Exact Solutions for Bond and Option Prices with Systematic Jump Risk*, Review of Derivatives Research **1** (1996), 1–24.

[16] F. de Jong, J. Driessen, and A. Pelsser, *Libor Market Models versus Swap Market Models for Pricing Interest Rate Derivatives: An Empirical Analysis*, European Finance Review **5** (2001), 201–237.

[17] J. Driessen, P. Klaassen, and B. Melenberg, *The Performance of Multi-Factor Term Structure Models for Pricing and Hedging Caps and Swaptions*, Working paper, Tilburg University, 2000.

[18] N. El Karoui and V. Lacoste, *Multifactor Models of the Term Structure of Interest Rates*, Working Paper DR95010, Ecole Supérieure des Sciences Economiques et Commersiales, 1995.

[19] B. Flesaker and L. Hughston, *Positive Interest*, Risk **9** (1996a), no. 1, 1–4.

[20] P.J. Hunt and J.E. Kennedy, *Implied Interest Rate Pricing Models*, Finance and Stochastics **2** (1998b), 275–293.

[21] P.J. Hunt, J.E. Kennedy, and A. Pelsser, *Markov-Functional Interest Rate Models*, Finance and Stochastics **4** (2000), 391–408.

[22] P.J. Hunt, J.E. Kennedy, and E.M. Scott, *Terminal Swap-Rate Models*, Working paper, University of Warwick, 1996.

[23] M. Joshi and R. Rebonato, *A Stochastic-Volatility, Displaced-Diffusion Extension*

of the LIBOR Market Model, Tech. report, Royal Bank of Scotland Quantitative Research Centre, 2001.

[24] T.C. Langetieg, *A Multivariate Model of the Term Structure*, The Journal of Finance **35** (1980), no. 1, 71–97.

[25] C.A. Micchelli, *Interpolation of Scattered data: Distance Matrices and Conditionally Positive Definite Functions*, Constructive Approximation **2** (1986), 11–22.

[26] K. Miltersen, K. Sandmann, and D. Sondermann, *Closed form solutions for the Term Structure Derivatives with log-normal interest rates*, Journal of Finance **52** (1997), 409–430.

[27] L.C.G. Rogers, *The Potential Approach to the Term Structure of Interest Rates and Foreign Exchange Rates*, Mathematical Finance **7** (1997b), no. 2, 157–176.

[28] W Rudin, *Principals of Mathematical Analysis*, McGraw-Hill, 1976.

[29] I.J. Schoenberg, *Metric Spaces and Completely Monotone Functions*, Ann.Math. **39** (1938), 811–841.

[30] C.F. Van Loan, *Computing Integrals Involving the Matrix Exponential*, IEEE Transactions on Automatic Control **33** (1978), 395–404.

[31] O. Vasicek, *An Equilibrium Characterization of the Term Structure*, Journal of Financial Economics **5** (1977), 177–188.

[32] C. Zühldorf, *Extended Market Models with Affine and Quadratic Volatility*, Working paper, University of Bonn, 2000.