THE VALUE OF A STORAGE FACILITY

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ABSTRACT

The paper derives the value of a storage facility that is too small to affect an exogenously defined price process of a storable good with seasonal and mean-reverting components. It provides an elegant new continuous-time model of storage under simple assumptions, which could be applied, for example, to natural gas. In the case without a seasonal component, closed form solutions are obtained as functions of the underlying price. A local time analysis provides an even simpler unconditional formula, which generalizes to the full model. The value of a storage facility under a model with a seasonal and a stochastic component is represented as a time integral which is easily evaluated numerically. The analysis provides a proper treatment of the true nature of the "option to store" and insights into the value of each component. An interesting feature of the model is that all transactions (whether to buy or to sell) are triggered by a single critical price. The analysis enables us to compare the profitability of storing under alternative price process assumptions.

KEY WORDS

real options, storage, option to store, local time.

1. Introduction

Although analyses of the theory of storage date back to Working [1], and many important models have been contributed since then, there are still some basic questions which remain unanswered within the literature. Such questions are of particular interest since the development of new markets for energy contracts.

The current paper is motivated by the following question: "For a commodity, such as natural gas, with both a seasonal component and a stochastic mean-reverting one, how does the value of storage depend on the magnitudes of these two effects?" For example, what is the difference between the economics of storing a commodity whose price is driven by a mean reverting random process (such as an Ornstein-Uhlenbeck process) and one for which there is a purely deterministic seasonal pattern?

The paper not only answers these questions, but provides an elegant new model of storage, which provides important new insights into the nature of "the option to store". Starting with the case where there is no seasonal component, it is shown that closed form solutions as functions of the underlying price can be obtained in the form of solutions to the Hermite differential equation. An analysis based on local time provides an even simpler unconditional formula. Solutions are obtained using both additive and multiplicative specifications.

The local time analysis provides a new and important characterisation of the value of storage facility when there is both a seasonal and a stochastic component. The value can be expressed as a time integral which is easily evaluated numerically. The formulation is related to that of a recent papers by Fackler and Livingston [2], and by de Jong and Walet [3], which provide numerical solutions to a similar problem, but are unable to make much progress analytically.

The paper has the following structure. Section 2 formulates the model and describes some of its general features. Section 3 solves a simple form of the model in which there is mean-reversion but no seasonal component. The analysis follows a traditional real options type of solution but with some novel features. Section 4 shows how a local time analysis can be used to obtain a simple closed form solution for the unconditional value of the storage facility. Section 5 extends this analysis to the situation with a seasonal component and mean reversion. Further numerical results are presented in Section 6. Section 7 concludes.

2. Formulation and General Features

We derive the value of a storage facility that is too small to affect the exogenously defined price process of a storable good.

The assumptions are:

1. A commodity price, P_t, follows one of the following mean-reverting processes[1]:
   (a) $P_t = \overline{P} + b\sin 2\pi t + u_t$, or
   (b) $P_t = \widehat{P}\exp\{b\sin 2\pi t + u_t\}$,
   under the risk neutral measure, unaffected by the actions of this facility, where
   $u_t = -\alpha\, u_t\, dt + \sigma\, dB_t$.
2. The storage facility has unit capacity and infinite life.
3. A storage cost is paid as a continuous cash flow at rate $c\times P$ per unit stored.
4. The risk free interest rate is r, (known and constant).
5. The facility is managed so as to maximise its market value.

For simplicity, the model wi1l be developed first in its additive specification (a), with the equations for the multiplicative version given as an Appendix.

This problem may be formulated as one of stochastic optimal control, as follows:

$$V(P_t, I_t; t) = \underset{s_t}{Max}\; E_t\left[\int_t^\infty e^{-r(s-t)}\left(-s_s - cI_s\right)P_s\, ds\right]$$

subject to $dI_t = s_t dt,\ 0 \le I_t \le 1,$ for all t,

where P_t follows the process as defined earlier, s_t is the rate at which the commodity is stored, and I_t is the level of inventory held.

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[1] In the additive form of the model, by specifying values for b and σ as percentages of $\overline{P}$, the two models will have very similar properties for the same input values.

However, it is unnecessary for us to solve within this complicated set-up. Examination of the first order condition shows that the optimal control is of the bang-bang type: the inventory is filled the instant $\frac{\partial V(P_t,0;t)}{\partial I} > P_t$, and it is emptied the instant $\frac{\partial V(P_t,1;t)}{\partial I} < P_t$.

3. Real Options Solution: No Seasonal Component

In this section we will solve the problem when there is no seasonal component (i.e. $b = 0$). In this case the time symmetry of the problem means that the value function is independent of time. We may therefore define and work with:

$$V_0(P_t) = \text{Value of empty storage facility when price is } P_t,$$
$$V_1(P_t) = \text{Value of full storage facility when price is } P_t.$$

It is no coincidence that the model has many similarities to the Brennan and Schwartz [4] paper on natural resource investments, and to the Dixit [5] model of entry and exit decisions. As in Dixit's model, we have two alternative value functions for which we can write ordinary differential equations. Similarly, there are boundary conditions which link the two functions. We will see, though, that this model also has a number of entirely different and distinctive features. First, since we assume that there are no fixed costs associated with filling or emptying the storage facility, there is just a single critical price at which the store is filled or emptied, and this price is very easily found. It also means that the two value functions always differ by exactly the commodity price. Second, since we have an Ornstein-Uhlenbeck process instead of Geometric Brownian Motion, the differential equation involved is not a simple homogeneous one, but instead is the rather more complicated Hermite differential equation, (see, for example, Andrews [6] for this).

There is a single critical price, P^*, in our model. It is optimal to buy stock as soon as P_t falls below P^*. It is also optimal to sell stock as soon as P_t rises above P^*. It is very easy to characterise P^*. Whenever the drift of P_t exceeds the cost of carrying inventory (interest rate, r plus storage cost, c), we should be holding stock. Conversely, whenever the drift of P_t is less than $r + c$, the store should be empty. The process for P_t is simply:

$$dP_t = -\alpha(P_t - \overline{P})\,dt + \sigma\,dB_t, \text{ with drift } \alpha(\overline{P} - P_t).$$

The critical value, P^*, is therefore given by:

$$(r+c)P^* = \alpha(\overline{P} - P^*), \text{ so}$$

$$P^* = \frac{\alpha}{\alpha + r + c}\overline{P}.$$

The value of the empty storage facility is characterised by the equations:

$$\tfrac{1}{2}\sigma^2 V_0'' + \alpha(\overline{P} - P)V_0' - rV_0 = 0, \text{ for } P > P^* \text{ and}$$
$$V_0 = V_1 - P, \qquad \text{for } P \le P^*.$$

The value of the full storage facility is characterised by the equations:

$$\tfrac{1}{2}\sigma^2 V_1'' + \alpha(\overline{P} - P)V_1' - rV_1 = cP, \text{ for } P < P^* \text{ and}$$
$$V_1 = V_0 + P, \qquad \text{for } P \ge P^*.$$

The two solutions differ from each other by P for all values of P. The first boundary condition is provided by:

$$V_1(P^*) = V_0(P^*) + P^*.$$

We also have the usual "smooth-pasting" condition[2], which takes the form:

$$V_1'(P^*) = V_0'(P^*) + 1.$$

Both of our ordinary differential equations are the Hermite equation: the one for $V_0(P)$ in homogeneous form, and that for $V_1(P)$ with an extra non-homogeneous term. If we make the substitutions:

$$x = \frac{\sqrt{\alpha}}{\sigma}\left(P - \bar{P}\right), \text{ so } P = \bar{P} + \frac{\sigma x}{\sqrt{\alpha}}, \text{ and}$$

$$y(x) = V(P(x)),$$

we obtain the equations in canonical form as:

$$y'' - 2xy' + 2\lambda y = 0, \text{ where } \lambda = -\frac{r}{\alpha}.$$

Note that under this (conventional) normalisation, x has a steady state Normal distribution with mean 0 and variance ½, so its law is that of the error function instead of the standard Normal distribution. In the most common applications of the Hermite equation λ is positive, and usually an integer. Here it is negative and unlikely to be a whole number. We will subsequently substitute $\eta = -\lambda = r/\alpha$ so that we are working with a positive variable. The Hermite equation has linearly independent solutions of the form[3]:

$$y_1(x) = {}_1F_1(-\tfrac{1}{2}\lambda, \tfrac{1}{2}; x^2) = {}_1F_1(\tfrac{1}{2}\eta, \tfrac{1}{2}; x^2),$$

$$y_2(x) = x\,{}_1F_1(-\tfrac{1}{2}\lambda + \tfrac{1}{2}, \tfrac{3}{2}; x^2) = x\,{}_1F_1(\tfrac{1}{2}\eta + \tfrac{1}{2}, \tfrac{3}{2}; x^2).$$

${}_1F_1(a, c; x)$ is the confluent hypergeometric function; it is defined (and readily calculated) according to the power series:

$${}_1F_1(a,c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \ldots$$

and its derivative with respect to x is given by:

$$\frac{d}{dx}\,{}_1F_1(a,c;x) \equiv \frac{a}{c}\,{}_1F_1(a+1, c+1; x).$$

We need to form the value functions $V_0(P)$ and $V_1(P)$ as linear combinations of these solutions. However, $y_1(x)$ and $y_2(x)$ are awkward to work with directly, as they both explode rapidly to ± infinity for large ± x. It is worth noting that $y_1(x)$ is an even function with $y_1(0) = 1$, $y'_1(0) = 0$, and $y_2(x)$ is an odd function with $y_2(0) = 0$, $y'_2(0) = 1$. We therefore take an alternative intermediate step.

It is apparent from the differential equation that, with η positive, there is an asymptotic solution for large x which converges to zero as $x^{-\eta}$. Solutions of this form will be much more convenient to work with. We can use the asymptotic behaviour of $y_1(x)$ and $y_2(x)$ to find their two combinations which take this form. Standard results give us that:

[2] See, for example, Dixit or Dixit and Pindyck ([7] and [8]) for further details of the solution method.

[3] Again, see Andrews[6] or other texts on special functions for details of the solution functions and their properties.

$$\lim_{x\to\infty} y_1(x)\,x^{1-\eta}\,e^{-x^2} = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}, \text{ and}$$

$$\lim_{x\to\infty} y_2(x)\,x^{1-\eta}\,e^{-x^2} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\eta}{2}\right)}.$$

We therefore form the rather better behaved functions:

$$g_0(x) = y_1(x) - k(\eta)y_2(x),$$

$$g_1(x) = g_0(-x), \text{ where}$$

$$k(\eta) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{\eta}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \eta\frac{\Gamma\left(\frac{1}{2}+\frac{\eta}{2}\right)}{\Gamma\left(1+\frac{\eta}{2}\right)}, \text{ (since } \Gamma(x+1) = \Gamma(x)).$$

The function $g_0(x)$ passes through $g_0(0) = 1$ with a negative slope and is asymptotic to 0 as x goes to infinity. It blows up as $\exp(x^2)$ as x goes to minus infinity. $g_1(x)$ passes through $g_1(0) = 1$ with a positive slope and is asymptotic to 0 as x goes to minus infinity and "blows up" for large positive x. The values and slopes of these two functions are easily calculated for any x using the formulae already given.

We can now write down solutions for $V_0(P)$, and $V_1(P)$ in terms of these functions. $V_0(P)$ is just a linear combination of the two $g_.(x)$ functions, $V_1(P)$ is the same, but with an extra term for the non-homogeneous part:

$$V_0(P) = a_{00}g_0(x) + a_{01}g_1(x),$$

$$V_1(P) = a_{10}g_0(x) + a_{11}g_1(x) - \frac{c}{\alpha+r}\left[P + \frac{\alpha}{r}\overline{P}\right], \text{ where}$$

$$x = \frac{\sqrt{\alpha}}{\sigma}\left(P - \overline{P}\right).$$

All that remains is to determine the values of the four constants, a_{00}, a_{01}, a_{10}, a_{11}. The economics of the problem tells us that $V_0(P)$ should go to zero (slowly) as P goes to infinity, and so a_{01} must be identically zero. Similarly that $V_1(P)$ should remain only linear in P when P becomes very small (in the limit to minus infinity). We therefore have a_{10} identically zero. The remaining constants, a_{00}, a_{11}, are found by solving the pair of simultaneous linear equations arising from the value matching and smooth pasting conditions at P^*:

$$V_1(P^*) - V_0(P^*) = a_{11}g_1(x^*) - a_{00}g_0(x^*) - \frac{c}{\alpha+r}\left[P^* + \frac{\alpha}{r}\overline{P}\right] = P^*,$$

$$V_1'(P^*) - V_0'(P^*) = a_{11}\frac{\sqrt{\alpha}}{\sigma}g_1'(x^*) - a_{00}\frac{\sqrt{\alpha}}{\sigma}g_0'(x^*) - \frac{c}{\alpha+r} = 1.$$

$$\text{where } x^* = \frac{\sqrt{\alpha}}{\sigma}\left(P^* - \overline{P}\right).$$

The following figures show results for:

$\overline{P} = \$100$, $\alpha = 2$, $\sigma = 10$ and $r = 0.05$. Figure 1 is for $c = 0$, and Figure 2 for $c = 0.03$.

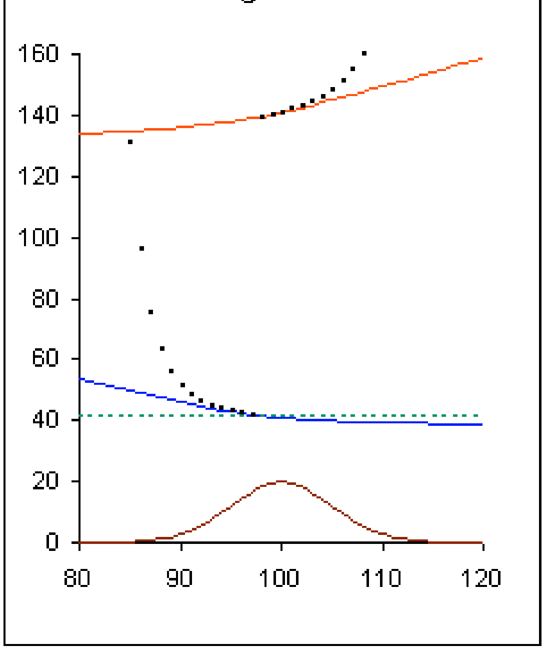


Figure 1

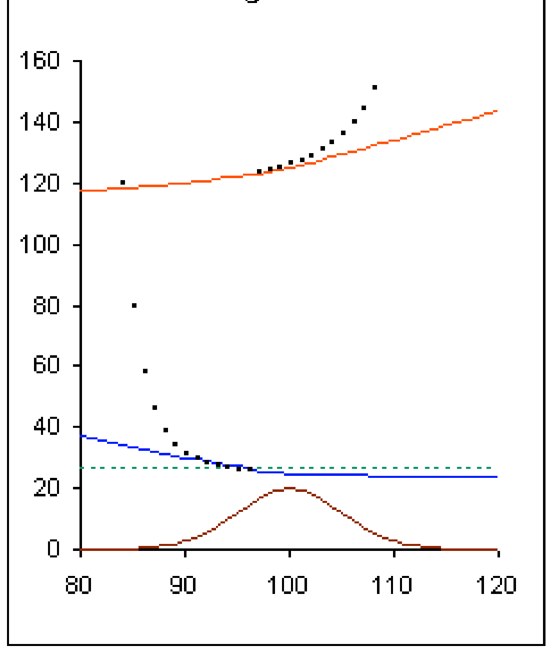


Figure 2

In each figure the top solid line shows how the value of the full store increases with the underlying price. The middle solid line shows how the value of the empty store decreases with the underlying price. The heavily dotted lines show the paths of the solution functions after the smooth pasting has occurred and they are no longer valid. Finally, the shape of the steady state density of the price is shown at the bottom of each figure. The results are essentially as expected.

4. Local Time Solution

All transactions take place at P^*. It is therefore possible to characterize the solution to our problem in terms of the local time spent by P at P^*. We will now do so. The role of local time in our model is very similar to its use by Carr and Jarrow [9] to characterize the value of a call option. The intuition and essence of their analysis is also contained in a slightly earlier paper by Seidenverg [10] which uses a binomial tree method.

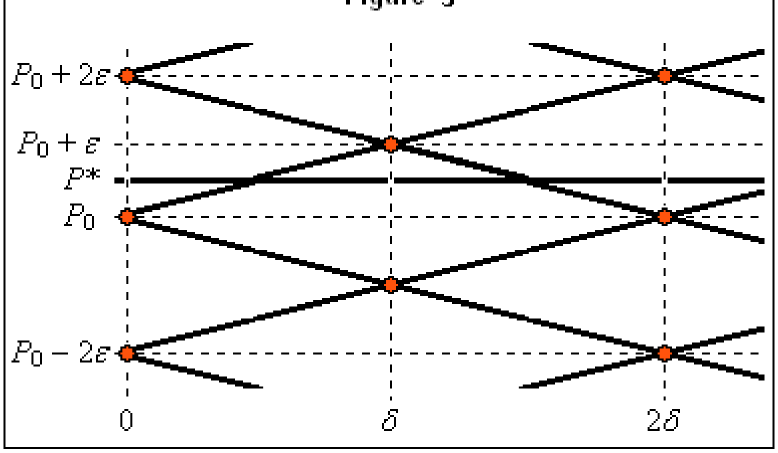


Figure 3

Since local time still has relatively few applications in finance, with first provide some intuition for it with a binomial tree version of our model. Figure 3 shows a lattice approximation to the price process as a discrete random walk. At the first date the middle price, P_0, is just below the critical price of P^*.

The time steps are at intervals of δ, and at each step the price goes up or down by $\varepsilon = \sigma\sqrt{\delta}$ with equal probability. Locally, since δ is small, we may imagine that the probability of reaching each node is the same and equals π, and we will also treat the drift as being zero.

Corresponding to our model, whenever the process is below P^* we are holding the commodity, and whenever it is above P^* we are not. On the lattice this means that whenever we go down from $P_0 + \varepsilon$ to P_0 we buy at P_0, and whenever we go up from P_0 to $P_0 + \varepsilon$ we sell at $P_0 + \varepsilon$. In a time period of length 2δ, the expected sales and purchases cancel out in quantity terms but leave us with a small expected profit amounting to:

$$E[\text{Profit}] = \pi[P_0 + \varepsilon - P_0]/2 = \pi\varepsilon/2.$$

If we divide this by 2δ we get the rate at which profit is expected to be earned, and we can write this in terms of the local probability density of P defined as $f(P) = \dfrac{\pi}{2\varepsilon}$. This gives:

$$E[\text{Profit Rate}] = \frac{\varepsilon\pi}{4\delta}$$

$$= \frac{\varepsilon^2}{2\delta}\frac{\pi}{2\varepsilon} = ½\sigma^2 f(P^*).$$

Note that this rate of profit does not depend on the choice of δ. The finer the grid, the more often we trade for less profit on each trade, but the total rate of profit remains the same.

More formally, the "doubled Brownian local time" of a path ω of stochastic process $X_t(\omega)$, at a level x over the time interval $[0, T]$, is usually defined as:

$$l(T, x, \omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^T 1_{[x-\varepsilon, x+\varepsilon]}\left(X_t(\omega)\right) dt$$

(although, exceptionally, Chung and Williams [11] refer to this simply as local time).
Note that $1_A(X)$ is an indicator function which takes the value 1 when the variable X_t belongs to the set A, and is zero otherwise. The profit we earn is a result of Tanaka's Theorem. The theorem states that if X_t follows a diffusion with diffusion coefficient σ, then:

$$|X_t| = |X_0| + \int_0^t sign(X_s)dX_s + \sigma^2 l(t, 0, \omega) \text{ on almost every path } \omega.$$

Carr and Jarrow used this to show that the outcomes from a "hedging through the cap" policy[4] fall short of replicating the call option by half the variance times the (doubled) local time. They showed how the Black-Scholes formula can be derived from this argument. In our analysis we apply Tanaka's Theorem to find that:

$$\int_0^T 1_{P_s < P^*} dP_t = Min(P_T - P^*) - Min(P_0 - P^*) + ½\sigma^2 l(T, P^*, \omega),$$

where σ the diffusion coefficient of P at P^*. In other words, we get a rent simply from trading backwards and forwards at P^* equal to the (doubled) local time at P^* for that path times $½\sigma^2$.

[4] The strategy holds one unit of the underlying when the (written) call is in-the-money and none when it is out-of-the-money.

We will complete the application of this to our storage problem. Although the theory applies along any path, or may be conditioned from any starting point, it is simplest to apply it to find the unconditional value of the empty storage facility. We will find that we obtain the simplest of analytic solutions, consistent with, but not hinted at by our previous analysis.

In the steady state, the expected local time of the process at P^* per unit time interval is simply its density:

$$f(P^*) = \frac{n(d^*)}{SD}, \text{ where } d^* = \frac{P^* - \overline{P}}{SD},$$

$$SD = \frac{\sigma}{\sqrt{2\alpha}} \text{ denotes the steady state standard deviation,}$$

and $n(.)$ denotes the standard Normal density.

The expected cash flow stream from trading is therefore $\frac{1}{2}\sigma^2 f(P^*)$ per unit time, as in our random walk approximation. Valuing this in perpetuity, we see that the unconditional value of the gross revenue stream is this divided by r. However, this is an average of cases, in some of which the storage facility is full and in some of which it is empty. We need to disentangle them. We also need to adjust for the explicit storage costs, incurred at rate at $c \times P$ whenever the store is full. Since the value of the full store is always the value of the empty one plus the price of the commodity, we have the following equation:

$$\frac{\sigma^2}{2r}\frac{n(d^*)}{SD} - \frac{c}{r}E[P.1_{P<P^*}] = E\left[\left(V_0(P)+P\right).1_{P<P^*} + V_0(P).1_{P\geq P^*}\right].$$

Note that $1_{P<P^*}$ is an indicator function (1 when $P < P^*$, 0 when $P > P^*$) for when the store is full. This gives:

$$E\left[V_0(P)\right] = \frac{\sigma^2}{2r}\frac{n(d^*)}{SD} - \left(1+\frac{c}{r}\right)E[P.1_{P<P^*}].$$

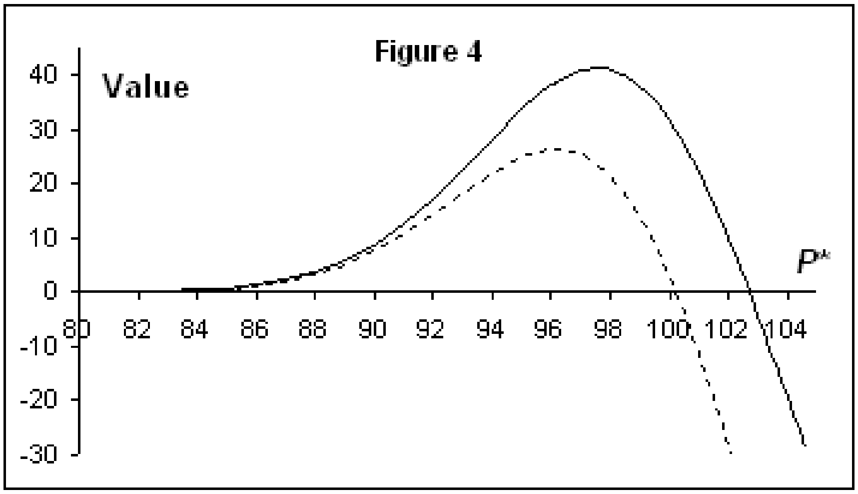
Finally, the last term is computed to provide the following simple solution for the unconditional value of the empty storage facility:

$$E[V_0(P)] = \frac{\sigma^2}{2r}\frac{n(d^*)}{SD} - P_C\left(1+\frac{c}{r}\right)N(d^*), \text{ where}$$

$$P_C = E[P \mid P < P^*] = \overline{P} - SD.\frac{n(d^*)}{N(d^*)}.$$

In the examples of Figures 1 and 2, we get values of \$41.43 and \$26.39 respectively, which are shown by the horizontal dashed lines.

What role has P^* (or d^* in the transformed coordinates) played in the above calculation? We know the value of P^* from the first order condition for storing or selling in the original stochastic control optimization. It is used as an argument in the expression above for the value of the storage facility, but this formula also provides the value for applying the same kind of bang-bang storage policy at any price level P^*, not necessarily the optimal one. Figure 4 shows the result of plotting our value formula against P^* for the two cases we considered before.



Note that the optimal values occur correctly at the P^* valued computed at the beginning. In fact, it is easily verified that the first order condition for the maximum of $E[V_0]$ (obtained by setting its derivative with respect to P^* equal to zero) does indeed provide the correct value of P^* . To the left of P^* we are waiting too long to buy and reducing value as a result. To the right we are buying too soon, when the drift back is not strong enough. It is interesting that the value is fairly insensitive to the precise choice of P^*, and, remarkably, it is possible to buy above the mean of 100 and still obtain some value.

5. Seasonal and Stochastic Components

Finally, we will see how the local time analysis can be extended to solve a problem with both seasonal and stochastic components. For the sake of continuity an additive model will again be used, though the solution for the multiplicative form (given in the Appendix) is no more complicated. Assume that at time t the price of the commodity is given as:

$$P_t = \overline{P} + b\sin 2\pi t + u_t, \text{ where}$$

$$du_t = -\alpha u_t\, dt + \sigma\, dB.$$

In other words, there is an additional annual seasonal component, $b \sin 2\pi t$, to P_t, where t is measured in years. The remaining assumptions are also as before, including the interest rate, r, and storage cost, c. The process for P_t is now:

$$dP_t = (2\pi b\cos 2\pi t - \alpha u_t)dt + \sigma\, dB.$$

The critical value, P_t^*, at which transactions take place is now time dependent, but is still determined by where the drift of P_t equals $(r + c)P_t$, and so it is given by:

$$(\overline{P} + b\sin 2\pi t + u_t^*)(r+c) = 2\pi b\cos 2\pi t - \alpha u_t^*, \text{ giving}$$

$$u_t^* = \frac{2\pi b\cos 2\pi t - (\overline{P} + b\sin 2\pi t)(r+c)}{r+c+\alpha}, \text{ and}$$

$$P_t^* = \overline{P} + b\sin 2\pi t + u_t^*.$$

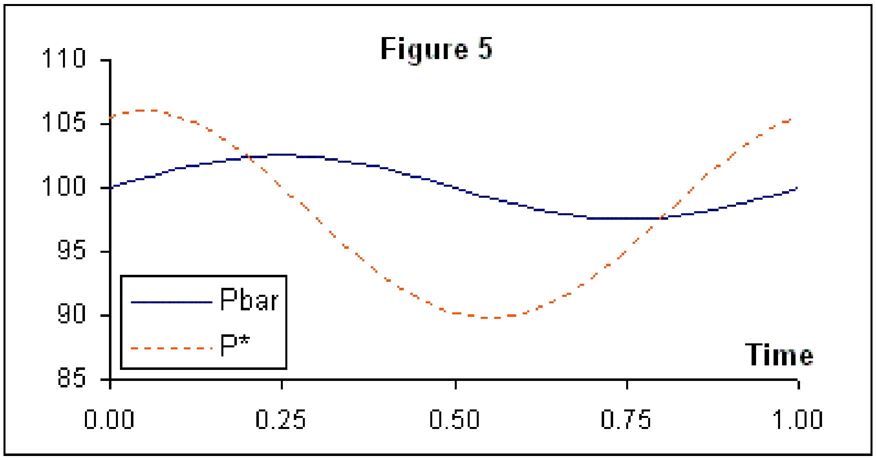


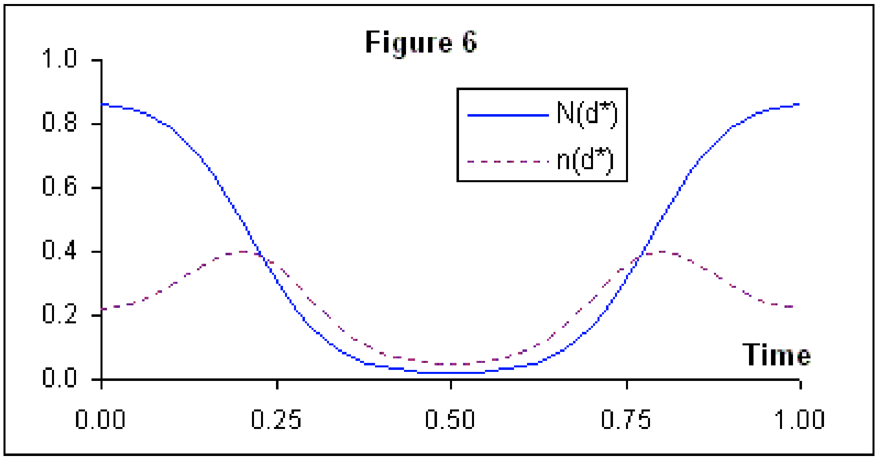
Figure 5 illustrates the expectation of P_t (solid line) and the position of P_t^* (dashed line) through a single year, where $b = 2.5$, and the other constants are as before, with $c = 0$.
Inventory will be held whenever the price is below the dotted line, which fluctuates considerably more than the price P_t.
Again it is possible to use the local time method to find the value of the storage facility. We will calculate the unconditional expected cash flows. There are now three components. In addition the component from the local time at P_t^*, and outflow from incurring a storage cost when P_t is below P_t^*, there is now a third component from the change in the probability of being below P_t^*, which generates net buy or sell transactions at P_t^*. The rate of cash flow is given by:

$$\begin{aligned} CF_t &= \frac{\sigma^2}{2}\frac{n(d_t^*)}{SD} - c\,E[P_t.1_{P_t<P_t^*}] - P_t^*\frac{\partial N(d_t^*)}{\partial t} \\ &= \frac{\sigma^2}{2}\frac{n(d_t^*)}{SD} - c\left[\overline{P}_t.N(d_t^*) - SD.n(d_t^*)\right] - 2\pi b P_t^*\frac{2\pi\sin 2\pi t + (r+c)\cos 2\pi t}{SD(r+c+\alpha)}.n(d_t^*). \end{aligned}$$

Finally, we integrate the discounted cash flow to get the value. This is easily done numerically and, since the expected cash flows repeat ever year, we need only to integrate over a single cycle of a year. We get:

$$\begin{aligned} \int_0^\infty e^{-rt} CF_t dt &= \frac{1}{1-e^{-r}}\int_0^1 e^{-rt}\, CF_t\, dt \\ &= E\left[\left(V_0(P_0)+P_0\right).1_{P_0<P_0^*} + V_0(P_0).1_{P_0\geq P_0^*}\right], \text{ so} \\ V_0(P_0) &= \frac{1}{1-e^{-r}}\int_0^1 e^{-rt}\, CF_t\, dt - \left[\overline{P}\,N(d_0^*) - SD\,n(d_0^*)\right]. \end{aligned}$$

Figure 6 shows the probability of holding stock through the annual cycle.



6. Further Numerical Results

Table 1 provides numerical results of the storage value for a range of values of b and σ. The other constants are as before:

$$\bar{P} = \$100,\ \ \alpha = 2,\ \ r = 0.05 \text{ and } c = 0.03.$$

The value increases very faster with b than it does with σ.

Table 1

b	σ: 0	5	10	15	20
0.0	0.00	2.79	26.39	60.49	98.09
2.5	33.27	40.18	59.54	86.93	119.42
5.0	126.52	129.38	138.40	154.42	176.78

Table 2 shows the corresponding results for a multiplicative model, where b and σ are the corresponding percentages. The results are very similar indeed.

Table 2

b	σ: 0%	5%	10%	15%	20%
0.0%	0.00	2.32	24.04	56.31	92.18
2.5%	33.29	39.67	57.77	83.55	114.35
5.0%	126.63	129.29	137.63	152.45	173.29

Figure 7 explores the comparative statics of varying the mean reversion rate alpha, while holding the steady state standard deviation constant. Two cases are shown, the upper curves have a 10% seasonal component, and the lower ones no seasonal component. Three sets are given for different levels of the standard deviation: SD = 2.5, 5, 10. Mean reversion by itself gives rise to a maximal expected cash flow of:

$$CF_{\max} = \frac{\sigma^2}{2}\frac{n(0)}{\text{SD}} = \frac{\text{SD}\times\alpha}{\sqrt{2\pi}}.$$

The dashed lines show this maximal cash flow capitalised by dividing by r. It is remarkable how close the functions approach to this level.

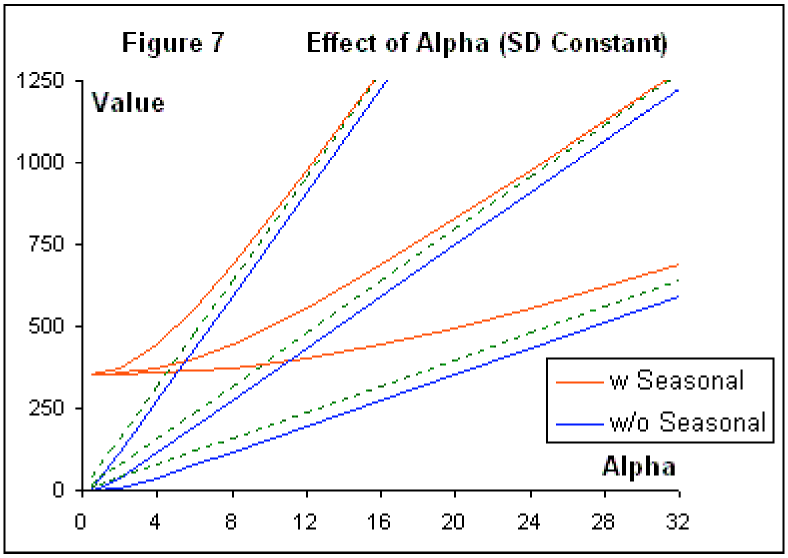
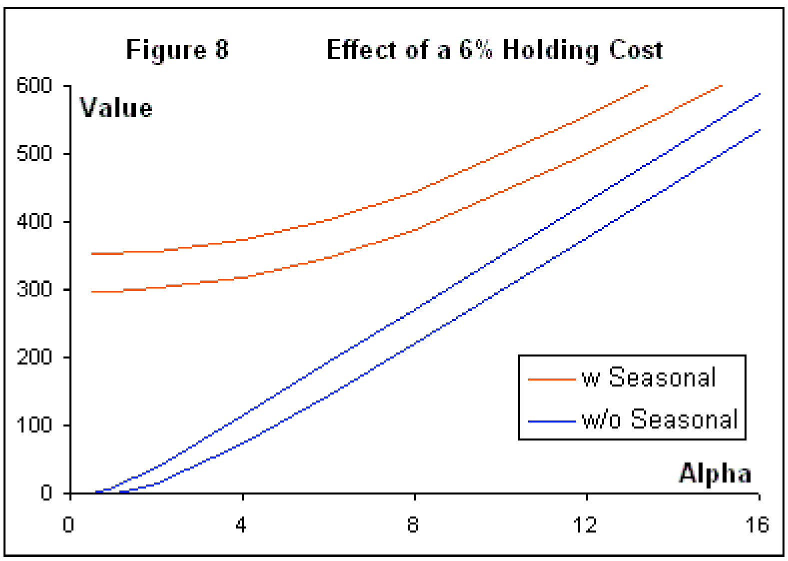
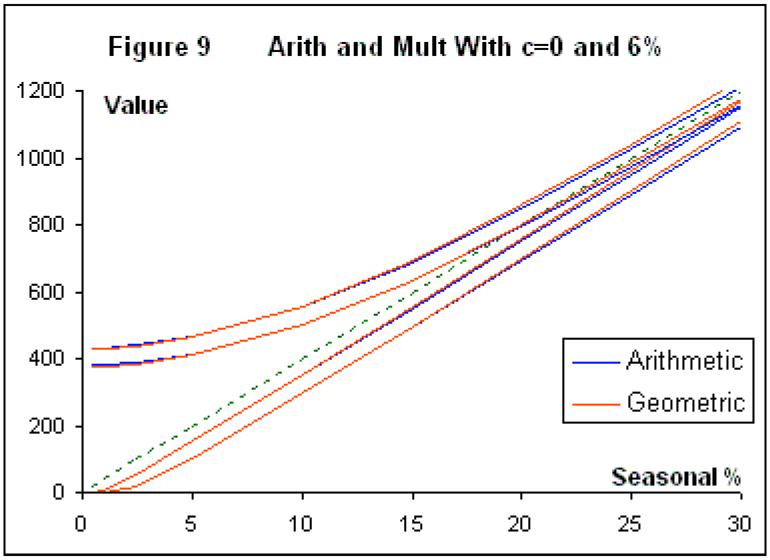


Figure 8 takes the two middle curves of Figure 7 and shows the effect of imposing a 6% holding cost. The curves drop in an almost parallel fashion, except close to the horizontal axis.



Finally, Figure 9 provides comparative statics for the magnitude of the seasonal component. The top curves show the case where there is our standard stochastic component as well, the two at the bottom show a pure seasonal component; in each case the lower of the pair of lines corresponds to a 6% holding cost. The dashed line shows the discounted value of the frictionless seasonal swing, 2*b*. Again, the actual functions approach remarkably closely to the simple formula. Both additive and multiplicative results are plotted in this figure and they are almost indiscernably close together..



We have seen that the expected cash flow from the pure Ornstein-Uhlenbeck process is approximately

$$\frac{\text{SD} \times \alpha}{\sqrt{2\pi}}$$

while that from the purely seasonal process is approximately $2b$. We may use this to determine how fast the mean reversion would have to be for a purely mean reverting process to generate as much storage value as a purely seasonal one. If we had a mean reverting process with SD = b, that is, the same steady state standard deviation as the amplitude of another seasonal process, then we would need

$$\alpha \geq 2\sqrt{2\pi} = 5.013$$

for the mean reversion to give as much profit as the seasonal commodity. This equates to a half life of 50½ days.

7. Conclusions

The paper has derived a simple model for the optimal storage of a storable commodity whose price mean reverts under an exogenously defined process, and which may or may not also possess a deterministic seasonal component. The model has been implemented in both arithmetic and geometric forms, though their results are extremely similar. The model assumes that there is a holding cost of storage proportional to the value of the inventory but that no incremental costs (other than the cash flow arising from buying or selling) are incurred when the inventory level is changed. Extending the model to incorporate adjustment costs of this kind would not be too difficult, but would involve a non-linear search for some parameters characterizing the solution.

The structure described has some particularly interesting features. Unsurprisingly, the solution to the optimal storage problem in continuous time involves switching between alternative solutions to an ordinary differential equation; each one corresponding to the value of either an empty or full storage facility at different price levels. Because there are no frictions in buying or selling stock, there is a single critical price which triggers all transactions: stock is purchased as soon as the price falls below this level and sold as soon as it rises above it. At first sight it seems surprising that any profit at all can be made under these circumstances. However, the critical price merely acts as a trigger. The storage activity results in a large number of transactions each making a tiny profit, but giving a sensible rate of profit in the limit. The strategy exactly corresponds to the Carr and Jarrow analysis of hedging a vanilla option "through the cap", except that in our case the optionality lies with the storer (i.e. the sign is reversed in our favour). This local time perspective clarifies the precise nature of the "option to store".

It is also possible to extend the analysis to look at situations where the buy trigger is strictly below the sell one. This is sub-optimal if there are no frictions, and the solution involves two value-matching conditions but no smooth-pasting one. Our numerical solutions can be obtained as the limit of this type of strategy, where the two price triggers are brought together at P^* If we had a cost of moving the commodity in or out of store, then the two critical prices would be endogenous.. In this case value-match and smooth-pasting conditions apply at each price, and it would be necessary to solve a rather more complex arrangement of four equations.

The local time analysis enables the solution of the general form of the model, with both a seasonal and stochastic component, to be expressed as a simple time integral. The formulation also provides useful insights into the nature of the optimal trading strategy, the value of storage and the determinants of the probability of storage at any date.

The local time analysis could easily be developed further to derive conditional expectations rather than unconditional ones, and also the distribution of realised cash flows from the optimal strategy. Our analysis has assumed a specific process under the risk neutral measure. It is worth noting that within the framework we have adopted, a single futures contract on the commodity would be sufficient to make our world dynamically complete, and for the values obtained to be hedged to become risk free.

The solution to the optimal storage problem may be regarded as a pre-requisite for the more difficult analysis of equilibrium storage and price behaviour. The equilibrium problem is a much more difficult one, which a number of papers have tackled numerically. At this point it seems unlikely that much progress can be made towards useful analytic solutions of equilibrium formulations.

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Appendix: Equations for the Multiplicative Case

In the multiplicative case we have

$$P_t = \exp\{z_t\}, \text{where}$$
$$z_t = \ln \widehat{P} + b \sin 2\pi t + u_t, \text{ and } u_t = -\alpha\, u_t\, dt + \sigma\, dB_t$$

under the risk neutral measure.

The process for P_t is:

$$dP_t = \left[2\pi b \cos 2\pi t - \alpha u_t + \tfrac{1}{2}\sigma^2 \right] P_t\, dt + \sigma\, dB_t.$$

The No-Seasonal Case

In the case $b = 0$, this gives:

$$dP_t = \left[\alpha(\ln \widehat{P} - \ln P_t) + \tfrac{1}{2}\sigma^2 \right] P_t\, dt + \sigma\, dB_t.$$

The trading price P^* is characterized by:

$$\alpha(\ln \widehat{P} - \ln P^*) + \tfrac{1}{2}\sigma^2 = r + c, \text{ giving}$$

$$z^* = \ln P^* = \ln \widehat{P} + \frac{\tfrac{1}{2}\sigma^2 - r - c}{\alpha}, \text{ and}$$

$$P^* = \widehat{P} \exp\left\{ \frac{\tfrac{1}{2}\sigma^2 - r - c}{\alpha} \right\}.$$

The value of the empty storage facility is characterised by the equations:

$$\tfrac{1}{2}\sigma^2 V_0'' + \alpha(\mu - z) V_0' - rV_0 = 0, \text{ for } z \geq z^* \text{ and}$$
$$V_0 = V_1 - e^z, \qquad \text{for } z < z^*.$$

The value of the full storage facility is characterised by the equations:

$$\tfrac{1}{2}\sigma^2 V_1'' + \alpha(\mu - z) V_1' - rV_1 = ce^z, \text{ for } z \leq z^* \text{ and}$$
$$V_1 = V_0 + e^z, \qquad \text{for } z > z^*.$$

The two solutions differ from each other by P for all values of P. The first boundary condition is provided by:

$$V_1(z^*) = V_0(z^*) + e^{z^*}.$$

We also have the usual "smooth-pasting" condition which takes the form:

$$V_1'(P^*) = V_0'(P^*) + e^{z^*}.$$

Both of our ordinary differential equations are the Hermite equation: the one for $V_0(z)$ in homogeneous form, and that for $V_1(z)$ with an extra non-homogeneous term. If we make the substitutions:

$$x = \frac{\sqrt{\alpha}}{\sigma}(z - \mu), \text{ so } z = \mu + \frac{\sigma x}{\sqrt{\alpha}}, \text{ and}$$
$$y(x) = V(z(x)),$$

we obtain the equations in canonical form as before, as:

$$y'' - 2xy' + 2\lambda y = 0, \text{ where } \lambda = -\frac{r}{\alpha}.$$

The solutions takes the form, as before, as:

$$V_0(P) = a_{00} g_0(x),$$
$$V_1(P) = a_{11} g_1(x) + C(x),$$

but where the particular solution $C(x)$ is given as:

$$C(x) = G(z(x)) = G(z) \text{ where}$$
$$G(z) = -\frac{c\overline{P}}{r} - c\overline{P}\int_0^{\infty} \exp\{-rt\}\left[\exp\left\{ue^{-\alpha t} - ke^{-2\alpha t}\right\} - 1\right]dt,$$
$$\text{where } u = z - \ln \widehat{P}, \text{ and } k = \frac{\sigma^2}{4\alpha}.$$

Its derivative is:

$$G'(z) = -c\overline{P}\int_0^{\infty} \exp\left\{-(r+\alpha)t + ue^{-\alpha t} - ke^{-2\alpha t}\right\}dt.$$

Local Time Solution

$$E[V_0(P)] = \frac{\sigma P^*}{r}\sqrt{\frac{\alpha}{2}}\, n(d^*) - \left(1 + \frac{c}{r}\right)E\left[P.1_{P<P^*}\right], \text{ (almost unchanged) but where}$$

$$E\left[P.1_{P<P^*}\right] = \overline{P}\, N\left(d^* - SD\right), \quad SD = \frac{\sigma}{\sqrt{2\alpha}}, \quad d^* = \frac{z^* - \ln \widehat{P}}{SD}.$$

Seasonal Solution

With seasonal and stochastic components, we have:

$$P_t = \widehat{P} e^{u_t} e^{b \sin 2\pi t} \text{ gives } dP_t = \left[2\pi b \cos 2\pi t - \alpha u_t + \tfrac{1}{2}\sigma^2\right] P_t\, dt + \sigma P_t\, dB_t.$$

Equating the drift to the cost of carry, we find:

$$u_t^* = \frac{2\pi b \cos 2\pi t + \frac{1}{2}\sigma^2 - r - c}{\alpha}, \quad P_t^* = \widehat{P}\, e^{u_t^*} e^{b \sin 2\pi t}.$$

The cash flow has three components: local time, storage cost and expected change in inventory:

$$CF_t = \sigma P_t^* \sqrt{\tfrac{\alpha}{2}}\, n(d_t^*) - c\overline{P}_t N(d_t^* - SD) + \frac{4\pi^2 b P_t^* \sin 2\pi t}{SD\,\alpha} n(d_t^*),$$

$$E[V_0(P_0)] = \frac{1}{1 - e^{-r}} \int_0^1 e^{-rt}\, CF_t\, dt - \overline{P}_0\, N(d_0^* - SD).$$