ES3C9

FLUID MECHANICS FOR MECHANICAL ENGINEERS

Answers: Example Sheet 3 - Pipe Flows

(1) Plane Poiseuille Flow: Consider the flow between two infinitely wide plates, driven by a pressure gradient, as shown in Fig. 1. 'b' is the width of the element of fluid consider in the 'z' direction.

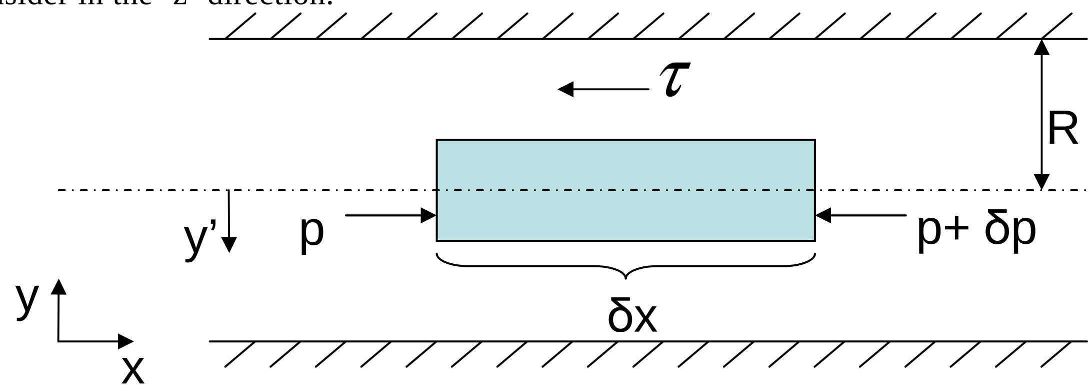


Fig. 1

(a) Use the suggested fluid element to show that the velocity profile for laminar flow is given by :

$$u(y) = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left(2Ry - y^2\right) \qquad (1.1)$$

Force balance : $p2y' = 2\tau b\delta x + (p+\delta p)2y'b$ Re-arrange : $\tau = \left(-\frac{dp}{dx}\right)y'$ with y'=R–y Newton law for the viscosity : $\tau = \mu\frac{du}{dy}$ So : $\frac{du}{dy} = \frac{1}{\mu}\left(-\frac{dp}{dx}\right)(R-y)$ After integration (y): $u = \frac{1}{\mu}\left(-\frac{dp}{dx}\right)\left(Ry - \frac{y^2}{2}\right) + C$ The boundary conditions give : u=0 at y=0 and at y=2R So C=0 Then : $u(y) = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left(2Ry - y^2\right)$

Write down the maximum velocity u_{max} in the pipe.

The maximum velocity occurs when du/dy change sign, ie y=R So : $u_{max} = \frac{R^2}{2\mu}\left(-\frac{dp}{dx}\right)$

(b) By first evaluating the volumetric flow rate in the duct, show that the mean velocity is given by $u_m = 2u_{max}/3$. Compare the present results for u_m and u_{max} with those found for Poiseuille flow in a pipe with circular cross-section, ie. (from the notes):

$$u_m = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) \qquad u_{\max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)$$

Volumetric flow rate : $Q = \int_0^{2R} u(y)\, b dy$

$$= \frac{b}{2\mu}\left(-\frac{dp}{dx}\right)\int_0^{2R}\left(2Ry - y^2\right)dy$$

$$= \frac{b}{2\mu}\left(-\frac{dp}{dx}\right)\left[Ry^2 - \frac{y^3}{3}\right]_0^{2R}$$

$$= \frac{2bR^3}{3\mu}\left(-\frac{dp}{dx}\right)$$

Mean velocity : $Q = 2Rbu_m$

So : $u_m = \frac{R^2}{3\mu}\left(-\frac{dp}{dx}\right)$

And then using u_{max} for question (1a) : $u_m = 2u_{max}/3$.

Is the difference reasonable ?

The effect of viscosity is greater in the circular-pipe case yielding lower speeds

(c) Compute (i) the wall shear stress (Use y' instead of y).

The wall shear follows from the definition of a Newtonian fluid :

$$\tau_w = \tau_{xy\ \text{wall}} = \mu\left(\frac{\partial u}{\partial y'} + \frac{\partial \mathrm{v}}{\partial x}\right)\Bigg|_{y'=\pm R} = \mu\frac{\partial}{\partial y'}\left[\left(\frac{\partial p}{\partial x}\right)\left(\frac{R^2}{2\mu}\right)\left(1 - \frac{y'^2}{R^2}\right)\right]\Bigg|_{y'=\pm R}$$

$$= \pm\frac{dp}{dx}R = \pm\frac{2\mu u_{\max}}{h}$$

The wall shear has the same magnitude at each wall, but by our sign convention, the upper wall has negative shear stress.

(ii) the stream function

Since the flow is plane, steady, and incompressible a stream function exists:

$$u = \frac{\partial \Psi}{\partial y} = u_{\max}\left(1 - \frac{y'^2}{R^2}\right) \qquad \mathrm{v} = -\frac{\partial \Psi}{\partial x} = 0$$

Integrating and setting ψ=0 at the centerline for convenience, we obtain

$$\Psi = u_{\max}\left(y - \frac{y'^3}{3R^2}\right)$$

At the walls, y' = ± R and ψ= $\pm 2u_{max}R/3$ respectively.

(iii) the vorticity

In plane flow, there is only a single nonzero vorticity component:

$$\zeta_z = (\mathrm{curl}\ V)_z = \frac{\delta v}{\delta x} - \frac{\delta u}{\delta y'} = \frac{2u_{\max}}{R^2} y'$$

The vorticity is highest at the wall and is positive (counterclockwise) in the upper half and negative (clockwise) in the lower half of the fluid. Viscous flows are typically full of vorticity and are not at all irrotational.

(iv) the velocity potential

From part (iii), the vorticity is finite. Therefore the flow is not irrotational, and the velocity potential does not exist.

(v) the average velocity

The average velocity is defined as V_{av} = Q/A, where $Q = \int \mathrm{u}\, dA$ over the cross section.

For our particular distribution u(y) from Eq. (1), we obtain :

$$V_{av} = \frac{1}{A}\int \mathrm{u}\, dA = \frac{1}{2Rb}\int_{-R}^{R} u_{\max}\left(1 - \frac{y'^2}{R^2}\right) b\,dy' = \frac{2}{3}u_{\max}$$

In plane Poiseuille flow between parallel, plates, the average velocity is two-thirds of the maximum (or centerline) value. This result could also have been obtained from the. stream function derived in part (ii). From Eq. (4.95) :

$$Q_{channel} = \Psi_{upper} - \Psi_{lower} = \frac{2u_{\max}R}{3} - \left(-\frac{2u_{\max}R}{3}\right) = \frac{4}{3}u_{\max}R \quad \text{per unit width}$$

whence $V_{av} = q / A_{b=1} = (4Ru_{\max}/3)/2R = 2u_{\max}/3$, the same result.

This example illustrates a statement made earlier: Knowledge of the velocity vector V [as in Eq. 1] is essentially the solution to a fluid-mechanics problem, since all other flow properties can then be calculated.

(2) Laminar flow exists in the inclined pipe shown in Fig. 2a. The flow is "downhill" being driven by both the gravitational force and an applied pressure gradient, dp/dx, along the length of the pipe. z is an upward coordinate whilst x is a coordinate along the pipe centerline.

(a) By considering the suggested fluid element in Fig. 2a, show that the velocity profile is given by:

$$u(r) = \frac{1}{4\mu}\left(-\frac{d}{dx}(p+\rho gz)\right)\left(R^2 - r^2\right)$$

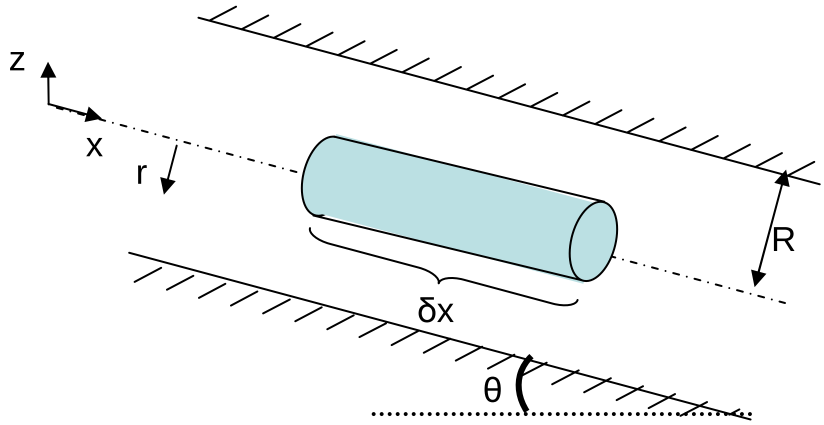


Fig. 2a

Force balance in x-direction : $\pi r^2 p + mg\sin\theta = \tau 2\pi r \delta x + (p+\delta p)\pi r^2$

Or m, the mass of the element is : $m = \rho\pi r^2 \delta x$

Also $\sin\theta = -\frac{dz}{dx}$ ('downhill' => negative gradient)

And $\partial p = \frac{dp}{dx}\partial x$

After substitution in the force balance equation :

$$\tau = -\frac{r}{2}\left(\rho g\frac{dz}{dx} + \frac{dp}{dx}\right) = \left(-\frac{d}{dx}(\rho gz + p)\right)\frac{r}{2}$$

Newton law for the viscosity : $\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$ (r is in the opposite direction to y)

So : $\frac{du}{dr} = -\left(-\frac{d}{dx}(\rho gz + p)\right)\frac{r}{2\mu}$

After integration and with the boundary conditions u=0 at r=R :

$$u(r) = \frac{1}{4\mu}\left(-\frac{d}{dx}(\rho gz + p)\right)\left(R^2 - r^2\right) \qquad (1.2)$$

(b) $\rho = 917 kg/m^3$) feeds the flow in an inclined pipe as shown in Fig. 2b. Given that h = 3.0 m, z_0=0.5m, x_o = 1.0m and R = 0.01m, find the maximum flow velocity U_{max} in the pipe.

[ANS. 2.71 m/s]

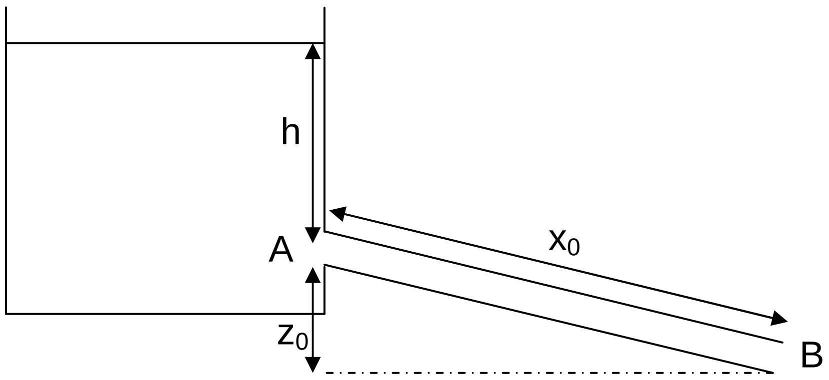


Fig. 2b

Pressure in A : $p_A = p_{Atm} + \rho g h$

Pressure in B : $p_B = p_{Atm} = p_A + \rho g z_0 + \frac{dp}{dx} x_0$

So the pressure gradient is : $\left(-\frac{dp}{dx} \right) = \frac{\rho g h + \rho g z_0}{x_0}$

Also if $p^* = p + \rho g z$ then $\frac{dp}{dx} = \frac{dp^*}{dx}$

Then, knowing that the maximum of velocity is at r =0 and using equation (1.2) :

$$u_{\max} = u(0) = \frac{1}{4\mu} \left(\frac{\rho g h + \rho g z_0}{x_0} \right) R^2 = 2.711 \text{ m/s}$$

(c) Show that the volumetric flow rate through the pipe, Q, is given by:

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{d}{dx} (p + \rho g z) \right)$$

$$Q = \int_0^R u(r) 2\pi r \partial r$$

Then using equation (1.2) : $Q = \frac{\pi}{2\mu} \left(-\frac{dp^*}{dx} \right) \int_0^R R^2 r - r^3 dr$

$$= \frac{\pi}{2\mu} \left(-\frac{dp^*}{dx} \right) \left[\frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R$$

$$= \frac{\pi R^4}{8\mu} \left(-\frac{d}{dx} (p + \rho g z) \right)$$

Hence determine the mean velocity and confirm that the flow is, indeed, laminar

[ANS. 1.36 m/s]

Mean velocity : $Q = \pi R^2 u_m$

Thus : $u_m = \dfrac{\pi R^2}{8\mu}\left(-\dfrac{d}{dx}(p+\rho g z)\right) = \dfrac{1}{2}u_{\max} = 1.356$ m/s

And then : $\mathrm{Re}_D = \dfrac{\rho u_m D}{\mu} = 86 \ll 2000$ so the flow is laminar.

(3) Kerosine (μ=0.00192Ns/m^2, ρ=804kg/m^3) flows in a horizontal pipe of diameter 2.0cm and length 5.0m. The volumetric flow rate, Q is 0.754 liters/s

(a) Calculate the Reynolds number of the flow.

Mean velocity : as $Q = \dfrac{\pi D^2 u_m}{4}$, so $u_m = 2.4$ m/s

Reynolds number : $\mathrm{Re}_D = \dfrac{\rho u_m D}{\mu} = 20000$ the flow is turbulent

(b) Using the index laws for velocity profile, calculate the flow speed at the pipe centerline.

From table in notes, choose n=6.6 for this Reynolds number.

$u_{\max} = \dfrac{(n+1)(2n+1)}{2n^2}u_m = 2.97$ m/s

(c) Use the Blasius equation to calculate the coefficient of friction and hence the shear stress at the wall for this flow.

The Blasius equation is applicable at this Reynolds number:

$C_f = \dfrac{\tau_w}{\rho u_m^2} = \dfrac{0.0395}{\mathrm{Re}_D^{1/4}} = 3.322\times10^{-3}$

So : $\tau_w = 15.38$ Pa

(d) Calculate the pressure head loss for the 5 m length of pipe.

To find the pressure loss, first find the pressure gradient: $\tau = \left(-\dfrac{dp}{dx}\right)\dfrac{r}{2}$

Also at the wall $r = D/2$: $\tau = \tau_w$

So : $\left(-\dfrac{dp}{dx}\right) = \dfrac{4\tau_w}{D} = 3076$ Pa/m

This is the pressure loss per unit length, ie $\left(-\dfrac{dp}{dx}\right) = -\dfrac{\Delta p}{L}$

Pressure loss for L : $\Delta p = L \times 3076 = 15380$ Pa

Equivalent head loss : $\Delta p = \rho g \Delta h$ then $\Delta h = \Delta p/\rho g = 1.95$ m

(e) Calculate the (output) power of a pump needed to drive this flow.

[ANS. (a) 20,000, (b) 2.97m/s, (c) 3.322x10^{-3}, (d) 1.95m, (e) 11.6W]

$P = Q\Delta p = 11.6$ W since $P = Q\rho g\Delta h = \dot{m}g\Delta h = \frac{d}{dt}(mg\Delta h)$

(4) For the same flow as described in Question (3), use the logarithmic law to calculate the coefficient of friction.

Logarithmic law for C_f: $\frac{1}{\sqrt{C_f}} = 5.7\log\left(\mathrm{Re}\sqrt{C_f}\right) + 0.3$

Or $C_f = \frac{1}{\left(5.7\log\left(\mathrm{Re}\sqrt{C_f}\right)+0.3\right)^2}$ and Re=20000 from question 4.

The above equation is difficult to solve (for C_f) because we can't write C_f explicitly. So use a iterative (numerical) method : Call the 'present' estimate C_{f_n} . Then an improve estimate will be $C_{f_{n+1}}$, where: $C_{f_{n+1}} = \frac{1}{\left(5.7\log\left(\mathrm{Re}\sqrt{C_{f_n}}\right)+0.3\right)^2}$

To get the method going, assume that $C_{f_{n+1}} \approx \frac{1}{(0+0.3)^2}$ and then :

n	C_{f_n} (*10^{-3})	$C_{f_{n+1}}$ (*10^{-3})
1	/	11110
2	11110	1.294
3	1.294	3.635
4	3.635	3.134
5	3.134	3.199
6	3.199	3.190
7	3.190	3.191
8	3.191	3.191

To this degree of accuracy, we can expect no further improvement with further value of n (i.e the result has converged to the solution of the equation) : $C_f = 3.191*10^{-3}$

Compare this with the answer for c_f found using the index law in question (3c).

Acceptable correlation with Blasius method

Calculate the friction velocity V* for the flow and hence the flow speed at the centerline using the logarithmic law for the velocity profile. Compare this with the answer to (3b).

[ANS. V*=2.841m/s, U_{max}=2.936m/s]

As $V^* = \sqrt{\dfrac{\tau_w}{\rho}}$ and $C_f = \dfrac{\tau_w}{\rho u_m^2}$, so : $C_f = \dfrac{V^{*2}}{u_m^2}$

Then : $V^* = \sqrt{C_f} .u_m$

From question (3a), $u_m = 2.4$ m/s so $V^* = 0.138$ m/s

Logarithmic law for velocity profile: $$\frac{u}{V^*} = 5.5\log\left(\frac{V^* y}{\nu}\right) + 5.4 \qquad (1.3)$$

At the centerline (y=D/2), the velocity is maximum. Then using V* from above and ν from question (3), we obtain : $u_{\max} = 2.841$ m/s

The index law of question (3b) gave $u_{\max} = 2.97$ m/s. We know that the logarithmic law underestimates the velocity at the pipe centre. In the notes it was suggested that near the pipe center, the multiplicative constant 5.5 is replaced by 5.75. If this is done in equation (1.3) then we find that $u_{\max} = 2.936$ m/s. This last result is in better agreement with the index-law result.

(5) The pipe used in Questions (3) and (4) is now discovered to have rough walls. The equivalent sand-grain roughness size, k_s, is 1.0 mm.

(a) Assuming the pipe wall to be completely rough calculate the new coefficient of friction.

$$C_f = \frac{1}{\left(5.7\log\left(R/k_s\right) + 4.75\right)^2} = 9.157\times10^{-3} \qquad \text{with R=D/2}$$

(b) By first calculating the friction velocity, V*, verify that we are justified in making the completely rough assumption.

$V^* = \sqrt{C_f} .u_m = 0.230$ m/s

Then $\dfrac{k_s V^*}{\nu} = 96.1$

The completely rough assumption require $\dfrac{k_s V^*}{\nu} > 100$. This is ‘almost’ completely rough !

(c) Using the value for the coefficient of friction found in part (5a), calculate the pressure head loss for the 5m length of pipe. Compare this answer with that of Question (3d) pertaining to the hydraulically smooth pipe assumption.

[ANS. (a) 9.157×10^{-3}, (b) V*=0.230m/s, (c) 5.38m]

We know that : $\tau = \left(-\frac{dp}{dx}\right)\frac{r}{2}$

Also at the wall $r = D/2$: $\tau = \tau_w$

$$\left(-\frac{dp}{dx}\right) = \frac{4\tau_w}{D} = \frac{4C_f \rho u_m^2}{D} = 8481.3 \text{ Pa/m}$$

This is the pressure loss per unit length, ie $\left(-\frac{dp}{dx}\right) = \frac{\Delta p}{L}$

Pressure loss for L : $\Delta p = L \times 8481.3 = 42407$ Pa with L=5 m

Equivalent head loss : $\Delta p = \rho g \Delta h$ then $\Delta h = \Delta p / \rho g = 5.38$ m

(6) Make use of the Moody chart (Fig. 7.11 in the notes) to determine the head loss due to friction when water flows through 300m of 150mm diameter galvanized steel pipe at 50 liters/s. Take the kinematic viscosity to be 1.14mm^2/s.

[ANS. 16.7m]

Mean velocity : as $Q = \frac{\pi D^2 u_m}{4}$, so $u_m = 2.83$ m/s

$$\text{Re}_D = \frac{u_m D}{\nu} = \frac{2.83 \times 0.15}{1.14 \times 10^{-6}} = 3.72 \times 10^5$$

Properties of the galvanized steel is founded in the Moody chart – roughness table : $k_s = 0.15$ mm so $k_s/D = 0.001$

Then $f = \dfrac{\Delta h}{\frac{L}{D} \cdot \frac{u_m^2}{2g}} = 0.0205$

So the head loss is : $\Delta h = 0.0205 \times \frac{300}{0.15} \times \frac{2.83^2}{2 \times 9.81} = 16.7$ m

(7) Using the Moody chart, determine the diameter of galvanized steel pipe needed to carry water ($\mu = 10^{-3}$Ns/m^2, $\rho = 10^3$kg/m^3) a distance of 180m at 85liter/s with a head loss of 9m.

[ANS. 0.2 m]

First $Q = \frac{\pi D^2 u_m}{4}$ so $u_m = \frac{0.108}{D^2}$

Then : (i) $\text{Re}_D = \frac{u_m D}{\nu} = \frac{0.108 \times 10^6}{D}$ (ii) $f = \dfrac{\Delta h}{\frac{L}{D} \cdot \frac{u_m^2}{2g}} = 84.1 D^5$ (iii) $\frac{k_s}{D} = \frac{0.00015}{D}$

We have to find D so that these three equations are verified (Moody chart). We use a trial error method to solve these equations.

Guess for D	Re_D (*10^6)	k_s/D	f (Moody)	f/D^5	Check solution
0.1	1.08	0.0015	0.0215	2150	D is too small
0.2	0.54	0.00075	0.0185	57.8	D is too large
0.18	0.6	0.00833	0.0192	101.6	D is too small
We can deduce that D=0.19 m from the statistic above. In pratice we would chose larger available size, i.e. D=0.2 m.					

(8) A simple hydraulic system is illustrated in Fig. 8a. It consists of a large reservoir and a horizontal pipeline 200m long discharging to atmosphere through a valve. The first 120m of pipe connected to the reservoir has a diameter of 200 mm. The remainder of the pipeline has a diameter of 250mm. Galvanized steel is used throughout as the pipe material. The pipeline is connected to the reservoir at a point 8m below the water surface, Formulae for head losses due to sudden changes in pipe diameter are given in Fig. 8b.

If the valve is set to discharge 0.1 m^3/s of Water ($\mu=10^{-3} Ns/m^2$, $\rho=10^3 kg/m^3$) plot, to scale, the hydraulic and energy gradelines (ie. the variations of piezometric and total heads) for the pipe-line.

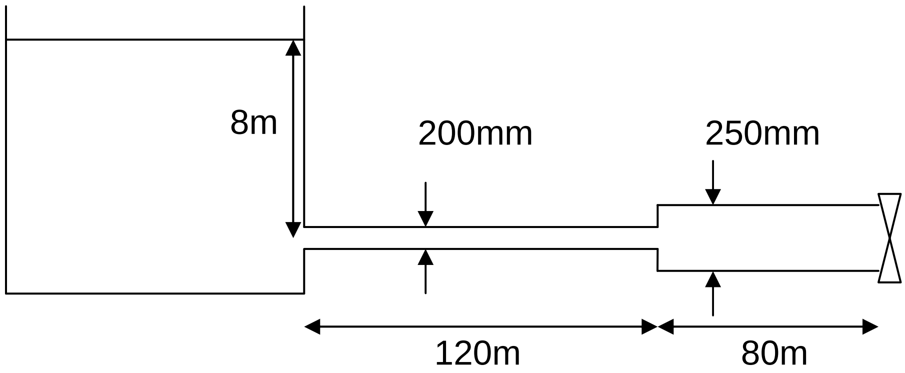
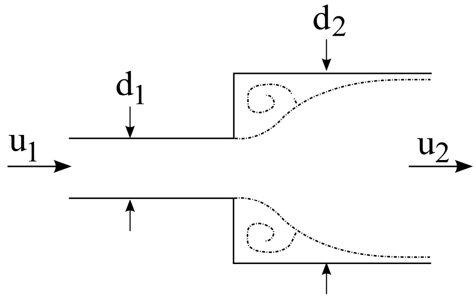


Fig. 8a

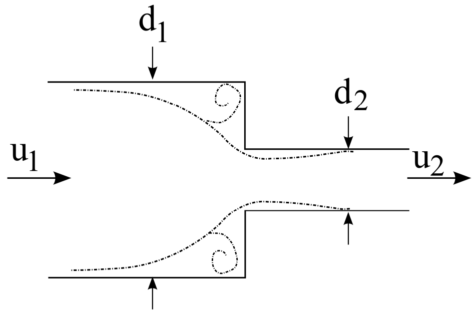
Fig. 8b (i) Head loss due to sudden enlargement : $\Delta h = \frac{u_1^2}{2g}\left(1-\left(\frac{d_1}{d_2}\right)^2\right)$

(ii) Head loss due to sudden contraction : $\Delta h = \frac{k u_2^2}{2g}$ where k verified :

d_2/d_1	0	0.2	0.4	0.6	0.8	1
k	0.5	0.45	0.38	0.28	0.14	0



(i)



(ii)

Work from pipe entry A toward valve V:

- Losses in the entry of pipe A:

$$\frac{d_2}{d_1} = \frac{0.2}{\infty} = 0 \text{ so k=0.5}$$

$$u_A = \frac{4Q}{\pi D_A^2} = 3.18 \text{ m/s then } \Delta h = \frac{k_s u_2^2}{2g} = \frac{u_A^2}{4g} = 0.258 \text{ m}$$

- Losses in the pipe A:

$$\mathrm{Re}_{D_A} = \frac{u_A D_A}{\nu} = 636000$$

Then $k_s = 0.00015$ m so $\frac{k_s}{D_A} = 0.00075$

Then in Moody chart we find : $f = \dfrac{\Delta h}{\frac{L}{D_A}.\frac{u_A^2}{2g}} = 0.019$ so $\Delta h = 5.88$ m

The dynamic head is $\frac{u_A^2}{2g} = 0.515$ m

- Losses in the entry of pipe B:

$$\Delta h = \frac{u_A^2}{2g}\left(1 - \left(\frac{D_A}{D_B}\right)^2\right) = 0.186 \text{ m}$$

- Losses in the pipe B:

$$u_B = \frac{4Q}{\pi D_B^2} = 2.04 \text{ m/s so } \mathrm{Re}_{D_B} = \frac{u_B D_B}{\nu} = 509000 \text{ and } \frac{k_s}{D_B} = 0.0006$$

Then in Moody chart we find : $f = \dfrac{\Delta h}{\frac{L}{D_B}.\frac{u_B^2}{2g}} = 0.0182$ so $\Delta h = 1.235$ m

The dynamic head is $\frac{u_B^2}{2g} = 0.212$ m

(9) A single uniform joins two reservoirs of fluid as shown in Fig. 9a. Calculate the percentage increase of flow rate obtainable if, from the mid-point of this pipe, another of the same diameter is added in parallel to it as shown in Fig 9b. Neglect all losses except pipe friction and assume a constant and equal f for both pipes.

[ANS. 26.5 %]

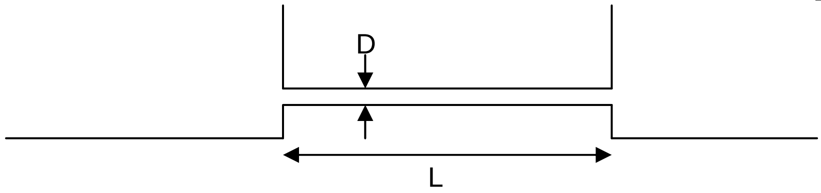


Fig. 9a

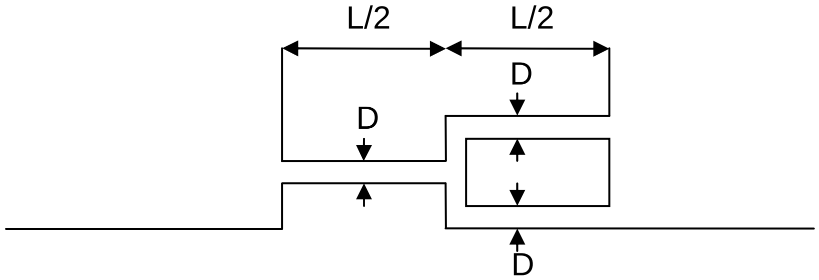


Fig. 9b

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u_m^2}$$

The balance force on the fluid element gives: $\tau_w = \left(-\frac{dp}{dx}\right)\frac{D}{4} = \frac{\Delta p}{L}\frac{D}{4}$

Combining above equations gives that: $u_m^2 = \frac{D}{2\rho C_f}\frac{\Delta p}{L}$

Cases of figure 9a:

$u_m = u_a$, $\Delta p = \Delta p_A$, L=L so $u_A^2 = \frac{D}{2\rho C_f}\frac{\Delta p_A}{L}$ and $\Delta p_A = \frac{L\rho C_f}{D}2u_A^2$

Cases of figure 9b:

1st half with a single pipe : $u_m = u_B^{'}$, $\Delta p = \Delta p_B^{'}$, L=L/2

so $u_B^{'2} = \frac{D}{\rho C_f}\frac{\Delta p_B^{'}}{L}$ and $\Delta p_{B'} = \frac{L\rho C_f}{D}u_{B'}^2$

2nd half with double pipe : $u_m = u_B^{"} = \frac{1}{2}u_B^{'}$, $\Delta p = \Delta p_B^{"}$, L=L/2

so $u_B^{"2} = \frac{D}{\rho C_f}\frac{\Delta p_B^{"}}{L}$ and $\Delta p_B^{"} = \frac{L\rho C_f}{D}u_B^{"2}$

But the pressure charge is the same for both cases (a) and (b):

$$\Delta p_A = \Delta p_B^{'} + \Delta p_B^{"} \Leftrightarrow \frac{L\rho C_f}{D}2u_A^2 = \frac{L\rho C_f}{D}\left(u_B^{'2} + \frac{1}{4}u_B^{'2}\right) \Leftrightarrow 2u_A^2 = \frac{5}{4}u_B^{'2} \Leftrightarrow u_B^{'} = \sqrt{\frac{8}{5}}u_A$$

Also $Q_A = \frac{\pi D^2 u_A}{4}$ and $Q_B^{'} = \frac{\pi D^2 u_B^{'}}{4}$ then $Q_B^{'} = \sqrt{\frac{8}{5}}Q_A = 1.265Q_A$

Moody diagram :