On the informational structure in optimal dynamic stochastic control[1]

Matija Vidmar (joint work with S. D. Jacka (Warwick))

VeSFiM seminar, UL FMF

March, 2015

[1] {arXiv:1503.02375} The support of the Slovene Human Resources Development and Scholarship Fund (contract number 11010-543/2011) is acknowledged.

Outline of talk

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- At the level of random (stopping) times, and in the context of (completed) natural filtrations of processes, this 'obvious' requirement becomes surprisingly non-trivial (at least in continuous time).

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  2. With respect to the second question, a generalization of (a part of) Galmarino's test to a non-canonical space setting is proved, although full generality could not be achieved. Several corollaries and related findings are given, which in particular shed light on the theme of 'informational consistency' (at random /stopping/ times).

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A collection of random times $\mathcal{S} = (\mathcal{S}^c)_{c\in\mathbf{C}}$ is called a **controlled time**, if $\mathcal{S}^c$ is a {defined up to P^c-a.s. equality} stopping time of $\mathcal{G}^c$ for every $c \in \mathbf{C}$.

and also a family $(\mathcal{D}(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ of subsets of $\mathbf{C}$ for which:

(1) $c \in \mathcal{D}(c,\mathcal{S})$ for all $(c,\mathcal{S}) \in \mathbf{C}\times\mathbf{G}$.

(2) For all $\mathcal{S} \in \mathbf{G}$ and $\{c,d\} \subset \mathbf{C}$, $d \in \mathcal{D}(c,\mathcal{S})$ implies $\mathcal{S}^c = \mathcal{S}^d$ {P^c & P^d-a.s}.

(3) If $\{\mathcal{S},\mathcal{T}\} \subset \mathbf{G}$, $c \in \mathbf{C}$ and $\mathcal{S}^c = \mathcal{T}^c$ {P^c-a.s}, then $\mathcal{D}(c,\mathcal{S}) = \mathcal{D}(c,\mathcal{T})$.

(4) If $\{\mathcal{S},\mathcal{T}\} \subset \mathbf{G}$ and $c \in \mathbf{C}$ for which $\mathcal{S}^d \leq \mathcal{T}^d$ {P^d-a.s.} for $d \in \mathcal{D}(c,\mathcal{T})$, then $\mathcal{D}(c,\mathcal{T}) \subset \mathcal{D}(c,\mathcal{S})$.

(5) For each $\mathcal{S} \in \mathbf{G}$, $\{\mathcal{D}(c,\mathcal{S}) : c \in \mathbf{C}\}$ is a partition of $\mathbf{C}$ ($\rightarrow$ denote the induced equivalence relation by $\sim_{\mathcal{S}}$).

(6) For all $(c,\mathcal{S}) \in \mathbf{C}\times\mathbf{G}$: $\mathcal{D}(c,\mathcal{S}) = \{c\}$ (resp. $\mathcal{D}(c,\mathcal{S}) = \mathbf{C}$), if $\mathcal{S}^c$ is identically {or P^c-a.s.} equal to ∞ (resp. 0).

. . . temporal consisteny and optimality

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Assumption (Temporal consistency)

For all $\{c, d\} \subset \mathbf{C}$ *and* $\mathcal{S} \in \mathbf{G}$ *satisfying* $c \sim_{\mathcal{S}} d$, *we have* $\mathcal{G}^c_{\mathcal{S}^c} = \mathcal{G}^d_{\mathcal{S}^d}$ *and* $\mathsf{P}^c|_{\mathcal{G}^c_{\mathcal{S}^c}} = \mathsf{P}^d|_{\mathcal{G}^d_{\mathcal{S}^d}}$.

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Definition (Optimal expected payoff)

We define $v := \sup_{c \in \mathbf{C}} \mathsf{E}^{\mathsf{P}^c} J(c)$ $(\sup \emptyset := -\infty)$, the **optimal expected payoff**. Next, $c \in \mathbf{C}$ is said to be **optimal** if $\mathsf{E}^{\mathsf{P}^c} J(c) = v$. Finally, a $\mathbf{C}$-valued net is said to be **optimizing** if its limit is v.

The conditional payoff and the Bellman system

The conditional payoff and the Bellman system

Definition (Conditional payoff & Bellman system)

We define for $c \in \mathbf{C}$ and $\mathcal{S} \in \mathbf{G}$:

$$J(c,\mathcal{S}) := \mathsf{E}^{\mathsf{P}^c}[J(c)|\mathcal{G}^c_{\mathcal{S}^c}],$$

and then

$$V(c,\mathcal{S}) := \mathsf{P}^c|_{\mathcal{G}^c_{\mathcal{S}^c}}\text{-esssup}_{d\in\mathcal{D}(c,\mathcal{S})} J(d,\mathcal{S});$$

and say $c \in \mathbf{C}$ is **conditionally optimal** at $\mathcal{S} \in \mathbf{G}$, if $V(c,\mathcal{S}) = J(c,\mathcal{S})$ P^c-a.s. $(J(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ is called the **conditional payoff system** and $(V(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ the **Bellman system**.

A 'technical' condition

A 'technical' condition

Proposition

Let $c \in \mathbf{C}$, $\mathcal{S} \in \mathbf{G}$ *and* $\epsilon \in [0, \infty)$, $M \in (0, \infty]$. *Then* (1)⇒(2)⇒(3).

(1) *(i) For all* $d \in \mathcal{D}(c, \mathcal{S})$, $\mathsf{P}^d = \mathsf{P}^c$. *AND*
(ii) For all $\{d, d'\} \subset \mathcal{D}(c, \mathcal{S})$ *and* $G \in \mathcal{G}^c_{\mathcal{S}^c}$, *there is a* $d'' \in \mathcal{D}(c, \mathcal{S})$ *such that* $J(d'') \geq M \wedge [\mathbb{1}_G J(d) + \mathbb{1}_{\Omega \backslash G} J(d')] - \epsilon$ P^c*-a.s.*

(2) *For all* $\{d, d'\} \subset \mathcal{D}(c, \mathcal{S})$ *and* $G \in \mathcal{G}^c_{\mathcal{S}^c}$, *there is a* $d'' \in \mathcal{D}(c, \mathcal{S})$ *such that* $J(d'', \mathcal{S}) \geq M \wedge [\mathbb{1}_G J(d, \mathcal{S}) + \mathbb{1}_{\Omega \backslash G} J(d', \mathcal{S})] - \epsilon$ P^c*-a.s.*

(3) $(J(d, \mathcal{S}))_{d \in \mathcal{D}(c, \mathcal{S})}$ *has the "*(ϵ, M)*-upwards lattice property":*

For all $\{d, d'\} \subset \mathcal{D}(c, \mathcal{S})$ *there exists a* $d'' \in \mathcal{D}(c, \mathcal{S})$ *such that*

$$J(d'', \mathcal{S}) \geq (M \wedge J(d, \mathcal{S})) \vee (M \wedge J(d', \mathcal{S})) - \epsilon$$

P^c*-a.s.*

A 'technical' condition (cont'd)

A 'technical' condition (cont'd)

Assumption (Upwards lattice property)

For all $c \in \mathbf{C}$, $\mathcal{S} \in \mathbf{G}$ *and* $\{\epsilon, M\} \subset (0, \infty)$, $(J(d, \mathcal{S}))_{d \in \mathcal{D}(c,\mathcal{S})}$ *enjoys the* (ϵ, M)*-upwards lattice property.*

A 'technical' condition (cont'd)

Assumption (Upwards lattice property)

For all $c \in \mathbf{C}$, $\mathcal{S} \in \mathbf{G}$ *and* $\{\epsilon, M\} \subset (0, \infty)$, $(J(d, \mathcal{S}))_{d \in \mathcal{D}(c, \mathcal{S})}$ *enjoys the* (ϵ, M)*-upwards lattice property.*

This is only seemingly of merely a 'technical' nature ....

A 'technical' condition (cont'd)

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For all $c \in \mathbf{C}$, $\mathcal{S} \in \mathbf{G}$ *and* $\{\epsilon, M\} \subset (0, \infty)$, $(J(d, \mathcal{S}))_{d \in \mathcal{D}(c, \mathcal{S})}$ *enjoys the* (ϵ, M)*-upwards lattice property.*

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A 'technical' condition (cont'd)

Assumption (Upwards lattice property)

For all $c \in \mathbf{C}$, $\mathcal{S} \in \mathbf{G}$ *and* $\{\epsilon, M\} \subset (0, \infty)$, $(J(d, \mathcal{S}))_{d \in \mathcal{D}(c,\mathcal{S})}$ *enjoys the* (ϵ, M)*-upwards lattice property.*

This is only seemingly of merely a 'technical' nature .... In fact, it represents a direct linking between $\mathbf{C}$, $\mathbf{G}$ and the collection $(\mathcal{G}^c)_{c \in \mathbf{C}}$. In particular, it may fail at deterministic times!!

Super/-/sub-martingale systems

Super/-/sub-martingale systems

Definition ($(\mathbf{C}, \mathbf{G})$-super/-/sub-martingale systems)

A collection $X = (X(c, \mathcal{S}))_{(c,\mathcal{S}) \in (\mathbf{C},\mathbf{G})}$ of functions from $[-\infty, +\infty]^\Omega$ is a $(\mathbf{C}, \mathbf{G})$- (resp. **super-**, **sub-**) **martingale system**, if for each $(c, \mathcal{S}) \in \mathbf{C} \times \mathbf{G}$ (i) $X(c, \mathcal{S})$ is $\mathcal{G}^c_{\mathcal{S}^c}$-measurable, (ii) $X(c, \mathcal{S}) = X(d, \mathcal{S})$ P^c-a.s. and P^d-a.s., whenever $c \sim_{\mathcal{S}} d$, (iii) (resp. the negative, positive part of) $X(c, \mathcal{S})$ is integrable and (iv) for all $\{\mathcal{S}, \mathcal{T}\} \subset \mathbf{G}$ and $c \in \mathbf{C}$ with $\mathcal{S}^d \leq \mathcal{T}^d$ $\{\mathsf{P}^d$-a.s.$\}$ for $d \in \mathcal{D}(c, \mathcal{T})$,

$$\mathsf{E}^{\mathsf{P}^c}[X(c, \mathcal{T}) | \mathcal{G}^c_{\mathcal{S}^c}] = X(c, \mathcal{S})$$

(resp. $\mathsf{E}^{\mathsf{P}^c}[X(c, \mathcal{T}) | \mathcal{G}^c_{\mathcal{S}^c}] \leq X(c, \mathcal{S})$, $\mathsf{E}^{\mathsf{P}^c}[X(c, \mathcal{T}) | \mathcal{G}^c_{\mathcal{S}^c}] \geq X(c, \mathcal{S})$) P^c-a.s.

Bellman's principle

Bellman's principle

Theorem (Bellman's principle)

Bellman's principle

Theorem (Bellman's principle)

$(V(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ *is a* $(\mathbf{C},\mathbf{G})$*-supermartingale system. Moreover, if* $c^*\in\mathbf{C}$ *is optimal, then* $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ *has a constant* P^{c^*}*-expectation (equal to the optimal value* $v=\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)$*). If further* $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$*, then* $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ *is a* $\mathbf{G}$*-martingale in the sense that (i) for each* $\mathcal{T}\in\mathbf{G}$*,* $V(c^*,\mathcal{T})$ *is* $\mathcal{G}^{c^*}_{\mathcal{T}^{c^*}}$*-measurable and* P^{c^*}*-integrable and (ii) for any* $\{\mathcal{S},\mathcal{T}\}\subset\mathbf{G}$ *with* $\mathcal{S}^d\leq\mathcal{T}^d$ $\{\mathsf{P}^d$*-a.s.*$\}$ *for* $d\in\mathcal{D}(c^*,\mathcal{T})$*,* P^{c^*}*-a.s.,*

$$\mathsf{E}^{\mathsf{P}^{c^*}}[V(c^*,\mathcal{T})|\mathcal{G}^{c^*}_{\mathcal{S}^{c^*}}]=V(c^*,\mathcal{S}).$$

Bellman's principle

Theorem (Bellman's principle)

$(V(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ is a $(\mathbf{C},\mathbf{G})$-supermartingale system. Moreover, if $c^*\in\mathbf{C}$ is optimal, then $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ has a constant P^{c^*}-expectation (equal to the optimal value $v=\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)$). If further $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ is a $\mathbf{G}$-martingale in the sense that (i) for each $\mathcal{T}\in\mathbf{G}$, $V(c^*,\mathcal{T})$ is $\mathcal{G}^{c^*}_{\mathcal{T}^{c^*}}$-measurable and P^{c^*}-integrable and (ii) for any $\{\mathcal{S},\mathcal{T}\}\subset\mathbf{G}$ with $\mathcal{S}^d\leq\mathcal{T}^d$ $\{\mathsf{P}^d$-a.s.$\}$ for $d\in\mathcal{D}(c^*,\mathcal{T})$, P^{c^*}-a.s.,

$$\mathsf{E}^{\mathsf{P}^{c^*}}[V(c^*,\mathcal{T})|\mathcal{G}^{c^*}_{\mathcal{S}^{c^*}}]=V(c^*,\mathcal{S}).$$

Furthermore, if $c^*\in\mathbf{C}$ is conditionally optimal at $\mathcal{S}\in\mathbf{G}$ and $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then c^* is conditionally optimal at $\mathcal{T}$ for any $\mathcal{T}\in\mathbf{G}$ satisfying $\mathcal{T}^d\geq\mathcal{S}^d$ $\{\mathsf{P}^d$-a.s.$\}$ for $d\in\mathcal{D}(c^*,\mathcal{T})$. In particular, if c^* is optimal, then it is conditionally optimal at 0, so that if further $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then c^* must be conditionally optimal at any $\mathcal{S}\in\mathbf{G}$.

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Theorem (Bellman's principle)

$(V(c,\mathcal{S}))_{(c,\mathcal{S})\in\mathbf{C}\times\mathbf{G}}$ is a $(\mathbf{C},\mathbf{G})$-supermartingale system. Moreover, if $c^*\in\mathbf{C}$ is optimal, then $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ has a constant P^{c^*}-expectation (equal to the optimal value $v=\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)$). If further $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then $(V(c^*,\mathcal{T}))_{\mathcal{T}\in\mathbf{G}}$ is a $\mathbf{G}$-martingale in the sense that (i) for each $\mathcal{T}\in\mathbf{G}$, $V(c^*,\mathcal{T})$ is $\mathcal{G}^{c^*}_{\mathcal{T}^{c^*}}$-measurable and P^{c^*}-integrable and (ii) for any $\{\mathcal{S},\mathcal{T}\}\subset\mathbf{G}$ with $\mathcal{S}^d\leq\mathcal{T}^d$ $\{\mathsf{P}^d$-a.s.$\}$ for $d\in\mathcal{D}(c^*,\mathcal{T})$, P^{c^*}-a.s.,

$$\mathsf{E}^{\mathsf{P}^{c^*}}[V(c^*,\mathcal{T})|\mathcal{G}^{c^*}_{\mathcal{S}^{c^*}}]=V(c^*,\mathcal{S}).$$

Furthermore, if $c^*\in\mathbf{C}$ is conditionally optimal at $\mathcal{S}\in\mathbf{G}$ and $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then c^* is conditionally optimal at $\mathcal{T}$ for any $\mathcal{T}\in\mathbf{G}$ satisfying $\mathcal{T}^d\geq\mathcal{S}^d$ $\{\mathsf{P}^d$-a.s.$\}$ for $d\in\mathcal{D}(c^*,\mathcal{T})$. In particular, if c^* is optimal, then it is conditionally optimal at 0, so that if further $\mathsf{E}^{\mathsf{P}^{c^*}}J(c^*)<\infty$, then c^* must be conditionally optimal at any $\mathcal{S}\in\mathbf{G}$. Conversely, and regardless of whether the "upwards lattice assumption" holds true, if $\mathbf{G}$ includes a sequence $(\mathcal{S}_n)_{n\in\mathbb{N}_0}$ for which (i) $\mathcal{S}_0=0$, (ii) the family $(V(c^*,\mathcal{S}_n))_{n\geq 0}$ has a constant P^{c^*}-expectation and is uniformly integrable, and (iii) $V(c^*,\mathcal{S}_n)\to V(c^*,\infty)$, P^{c^*}-a.s. (or even just in P^{c^*}-probability), as $n\to\infty$, then c^* is optimal.

A bird's eye view

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Start "measure-theoretically", no completions, no probability.

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One's naïve expectation/ "it must be true":

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> *If X is a process, and S a time, then S is a stopping time of $\mathcal{F}^X$, if and only if it is a stopping time of $\mathcal{F}^{X^S}$.*

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> *If X is a process, and S a time, then S is a stopping time of $\mathcal{F}^X$, if and only if it is a stopping time of $\mathcal{F}^{X^S}$. When so, then $\mathcal{F}^X_S = \sigma(X^S)$.*

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Is it true always/"can it be proved"?

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In parts, maybe (so basically, I don't know). Can be proved if (imprecisely) "the underlying space is Blackwell, and the space in which the processes live is also 'nice enough' ".

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What about if one "completes everything"?

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In parts, maybe (so basically, I don't know). Can be proved if (imprecisely) "the underlying space is Blackwell, and the space in which the processes live is also 'nice enough' ".

What about if one "completes everything"? Then it's trickier ...

Tools

Tools

Blackwell's Theorem. *Let $(\Omega, \mathcal{F})$ be a Blackwell space, $\mathcal{G}$ a sub-σ-field of $\mathcal{F}$ and $\mathcal{S}$ a separable sub-σ-field of $\mathcal{F}$. Then $\mathcal{G} \subset \mathcal{S}$, if and only if every atom of $\mathcal{G}$ is a union of atoms of $\mathcal{S}$. In particular, a $\mathcal{F}$-measurable real function g is $\mathcal{S}$-measurable, if and only if g is constant on every atom of $\mathcal{S}$.*

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Lemma (Key lemma)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0, \infty)\}$ and values in $(E, \mathcal{E})$), S an $\mathcal{F}^X$-stopping time, $A \in \mathcal{F}^X_S$. If $X_t(\omega) = X_t(\omega')$ for all $t \in T$ with $t \leq S(\omega) \wedge S(\omega')$, then $S(\omega) = S(\omega')$, $X^S(\omega) = X^S(\omega')$ and $\mathbb{1}_A(\omega) = \mathbb{1}_A(\omega')$.

Key results – stopping times

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Theorem (Stopping times)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), $S : \Omega \to T \cup \{\infty\}$ a time. If $T = \mathbb{N}_0$, or else if the conditions:

(1) $\sigma(X|_{[0,t]})$ *and* $\sigma(X^{S\wedge t})$ *are separable,* $(\mathrm{Im}X|_{[0,t]}, \mathcal{E}^{\otimes[0,t]})$ *and* $(\mathrm{Im}X^{S\wedge t}, \mathcal{E}^{\otimes T}|_{\mathrm{Im}X^{S\wedge t}})$ *Hausdorff for each* $t \in [0,\infty)$.

(2) X^S *and* X *are both measurable with respect to a Blackwell* σ*-field* $\mathcal{G}$ *on* Ω.

are met, then the following statements are equivalent:

(i) S *is an* $\mathcal{F}^X$*-stopping time.*

(ii) S *is an* $\mathcal{F}^{X^S}$*-stopping time.*

Key results – Galmarino's test

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Theorem (Generalized Galmarino's test)

Let X be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), S an $\mathcal{F}^X$-stopping time. If $T = \mathbb{N}_0$, then $\sigma(X^S) = \mathcal{F}^X_S$. Moreover, if X^S is $\mathcal{F}^X_S/\mathcal{E}^{\otimes T}$-measurable (in particular, if it is adapted to the stopped filtration $(\mathcal{F}^X_{t\wedge S})_{t\in T}$) and either one of the conditions:

(1) $\mathrm{Im}X^S \subset \mathrm{Im}X$.

(2) (a) $(\Omega,\mathcal{G})$ *is Blackwell for some σ-field* $\mathcal{G} \supset \mathcal{F}^X_\infty$.
(b) $\sigma(X^S)$ *is separable.*
(c) $(\mathrm{Im}X^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im}X^S})$ *is Hausdorff.*

is met, then the following statements are equivalent:

(i) $A \in \mathcal{F}^X_S$.

(ii) $\mathbb{1}_A$ *is constant on every set on which X^S is constant and* $A \in \mathcal{F}^X_\infty$.

(iii) $A \in \sigma(X^S)$.

Key results – informational consistency

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Corollary (Observational consistency)

Let X and Y be two processes (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E, \mathcal{E})$), S an $\mathcal{F}^X$ and an $\mathcal{F}^Y$-stopping time. Suppose furthermore $X^S = Y^S$. If any one of the conditions

(1) $T = \mathbb{N}_0$.

(2) $\mathrm{Im}X = \mathrm{Im}Y$.

(3) (a) $(\Omega, \mathcal{G})$ *(resp.* $(\Omega, \mathcal{H})$*) is Blackwell for some* σ*-field* $\mathcal{G} \supset \mathcal{F}^X_\infty$ *(resp.* $\mathcal{H} \supset \mathcal{F}^Y_\infty$*).*
(b) $\sigma(X^S)$ *(resp.* $\sigma(Y^S)$*) is separable and contained in* $\mathcal{F}^X_S$ *(resp.* $\mathcal{F}^Y_S$*).*
(c) $(\mathrm{Im}X^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im}X^S})$ *(resp.* $(\mathrm{Im}Y^S, \mathcal{E}^{\otimes T}|_{\mathrm{Im}Y^S})$*) is Hausdorff.*

is met, then $\mathcal{F}^X_S = \mathcal{F}^Y_S$.

Key results – monotonicity of information

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Proposition (Monotonicity of information)

Let Z be a process (on Ω, with time domain $T \in \{\mathbb{N}_0, [0,\infty)\}$ and values in $(E,\mathcal{E})$), $U \leq V$ two stopping times of $\mathcal{F}^Z$. If either $T = \mathbb{N}_0$ or else the conditions:

1. *$(\Omega,\mathcal{G})$ is Blackwell for some σ-field $\mathcal{G} \supset \sigma(Z^V) \vee \sigma(Z^U)$.*
2. *$(\mathrm{Im} Z^V, \mathcal{E}^{\otimes T}|_{\mathrm{Im} Z^V})$ is Hausdorff.*
3. *$\sigma(Z^V)$ is separable.*

are met, then $\sigma(Z^U) \subset \sigma(Z^V)$.

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4. Are true, if the stopping times are predictable . . .

A counter-example

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Example

Let $\Omega = (0,\infty) \times \{-2,-1,0\}$ be endowed with the law $\mathsf{P} = \mathrm{Exp}(1) \times \mathrm{Unif}(\{-2,-1,0\})$, defined on the tensor product of the Lebesgue σ-field on $(0,\infty)$ and the power set of $\{-2,-1,0\}$. Denote by e, respectively I, the projection onto the first, respectively second, coordinate. Define the process $X_t := I(t-e)\mathbb{1}_{[0,t]}(e)$, $t \in [0,\infty)$, and the process $Y_t := (-1)(t-e)\mathbb{1}_{[0,t]}(e)\mathbb{1}_{\{-1,-2\}} \circ I$, $t \in [0,\infty)$. The completed natural filtrations of X and Y are already right-continuous. The first entrance time S of X into $(-\infty,0)$ is equal to the first entrance time of Y into $(-\infty,0)$, and this is a stopping time of $\overline{\mathcal{F}^X}^{\mathsf{P}}$ as it is of $\overline{\mathcal{F}^Y}^{\mathsf{P}}$ (but not of $\mathcal{F}^X$ and not of $\mathcal{F}^Y$). Moreover, $X^S = 0 = Y^S$. Finally, consider the event $A := \{I = -1\}$. Then $A \in \overline{\mathcal{F}^X}^{\mathsf{P}}_S$, however, $A \notin \overline{\mathcal{F}^Y}^{\mathsf{P}}_S$. ◇

Handling the predictable case (in continuous time)

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Proposition

Let $T = [0, \infty)$, $\mathcal{G}$ be a filtration on Ω. Let furthermore P be a complete probability measure on Ω, whose domain includes $\mathcal{G}_\infty$; S a predictable stopping time relative to $\overline{\mathcal{G}}^{\mathsf{P}}$. Then S is P-a.s. equal to a predictable stopping time P of $\mathcal{G}$. Moreover, if U is any $\mathcal{G}$-stopping time, P-a.s. equal to S, then $\overline{\mathcal{G}}^{\mathsf{P}}_S = \overline{\mathcal{G}_U}^{\mathsf{P}}$. Finally, if S' is another random time, P-a.s equal to S, then it is a $\overline{\mathcal{G}}^{\mathsf{P}}$-stopping time, and $\overline{\mathcal{G}}^{\mathsf{P}}_S = \overline{\mathcal{G}}^{\mathsf{P}}_{S'}$.

Further work?

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- Regarding the 'technical' condition: A more precise investigation into the relationship between the validity of Bellman's principle, and the linking between $\mathbf{C}$, $\mathbf{G}$ and the collection $(\mathcal{G}^c)_{c\in\mathbf{C}}$.
- Regarding the theme of informational consistency: Try and relax/drop the Blackwell-ian assumption. Alternatively (or in addition) find relevant counter-examples!

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