Markov chain approximations to scale functions of Lévy processes

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Computational Methods for Jump Processes, University of Warwick

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Outline of talk

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A continuous-time $\mathbb{F}$-adapted stochastic process $X = (X_t)_{t\geq 0}$, with state space $\mathbb{R}$, is a *Lévy process* (relative to $(\mathbb{F}, \mathsf{P})$), if it starts at 0, a.s., $X_{t-s} \sim X_t - X_s \perp \mathcal{F}_s$ $(0 \leq s \leq t)$ and it is càdlàg off a null set.

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For this class of Lévy processes, fluctuation theory in terms of the two families of scale functions, $(W^{(q)})_{q\in[0,\infty)}$ and (their integrals) $(Z^{(q)})_{q\in[0,\infty)}$, has been developed.

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Note: ψ is given by [$(\sigma^2, \lambda, \mu)_{\tilde{c}}$ – characteristic triplet; $\tilde{c} := \mathrm{id}_{\mathbb{R}} \mathbb{1}_{[-V,0)}$, $V \in \{0, 1\}$]:

$$\psi(\beta) = \frac{1}{2}\sigma^2\beta^2 + \mu\beta + \int_{(-\infty,0)} \left(e^{\beta y} - \beta\tilde{c}(y) - 1\right) \lambda(dy), \quad \beta \in \overline{\mathbb{C}^{\rightarrow}}.$$

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Analytic characterization of scale functions: is in terms of their Laplace transforms;

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q} \quad (\beta > \Phi(q)).$$

$(Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy,\ x \geq 0.)$

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(i) **Two-sided exit problem**. For $a \geq 0$ let T_a (respectively T_a^-) be the first entrance time of X to $[a, \infty)$ (respectively $(-\infty, -a)$). Then:

$$\mathsf{P}(T_x^- > T_a) = \frac{W(x)}{W(a+x)},$$

whenever $\{a, x\} \subset (0, \infty) =: \mathbb{R}^+$.

(ii) **Ruin probabilities**. In the case that X drifts to $+\infty$ we have for $x \in \mathbb{R}^+$, the generalised Cramér-Lundberg identity:

$$\mathsf{P}(T_x^- = \infty) = W(x)\psi'(0+).$$

(cont'd)

(iii) **Continuous-state branching processes**. Under mild conditions, the law of the supremum of a continuous state branching process Y is given by the identity (for $x \in \mathbb{R}^+$, $y \in \mathbb{R}$):

$$\mathsf{P}_y(\sup_{s \geq 0} Y_s \leq x) = \frac{W(x-y)}{W(x)},$$

where W is the scale function of the associated Lévy process.

(iv) **Population biology**. The typical branch length H between two consecutive individuals alive at time $t \in \mathbb{R}^+$, conditionally on there being at least two extant individuals at said time, satisfies the identity:

$$\mathsf{P}(H < s) = \frac{1 - W(s)^{-1}}{1 - W(t)^{-1}},$$

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Miscellaneous other areas featuring scale functions $W^{(q)}$ (together with their derivatives and the integrals $Z^{(q)}$), include queuing theory, optimal stopping and control problems, fragmentation processes etc.

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- Typically not possible to perform the inversion explicitly; user is faced with a Laplace inversion (e.g. Filon's (with FFT), Gaver-Stehfest, Euler's, fixed Talbot's) algorithm, which:
  (a) requires evaluation of the Laplace exponent of X (at complex values of its argument);
  (b) says little about the dependence of the scale function on the Lévy triplet of X;
  (c) fails to *a priori* ensure that the computed values of the scale function are probabilistically meaningful.

Kuznetsov, Kyprianou, Rivero: The Theory of Scale Functions for Spectrally Negative Lévy Processes (in S. Cohen et al., Lévy Matters II, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2012).

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(v) Main result establishes **sharp rates of convergence** of this algorithm providing an **explicit link** between the semimartingale characteristics of X and its scale functions.

The algorithm

To compute $W(x)$ for some $x > 0$, choose small $h > 0$ such that x/h is an integer and define the approximation $W_h(x)$ by the formula:

$$W_h(y+h) = W_h(0) + \sum_{k=1}^{y/h+1} W_h(y+h-kh)\frac{\gamma_{-kh}}{\gamma_h}, \qquad W_h(0) = (\gamma_h h)^{-1}$$

for $y = 0, h, 2h, \dots, x-h$, where the coefficients γ_h and $(\gamma_{-kh})_{k\geq 1}$ are given *explicitly* in terms of $(\sigma^2, \lambda, \mu)_{\tilde{c}}$ (next slide).

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Two main results:

1. approximating scale functions converge pointwise to those of the original Lévy process (as $h \downarrow 0$);
2. sharp rates on $\mathbb{Z}_h := h\mathbb{Z}$ (under a weak condition on the Lévy measure).

(cont'd)

Recall: $(\sigma^2, \lambda, \mu)_{\tilde{c}}$ – characteristic triplet; $\tilde{c} := \mathrm{id}_{\mathbb{R}} \mathbb{1}_{[-V,0)}$.

$V = 0$ or $V = 1$, according as to whether $\lambda(\mathbb{R}) < +\infty$ or $\lambda(\mathbb{R}) = +\infty$.

$c_0^h := \int_{[-h/2,0)} y^2 \mathbb{1}_{[-V,0)}(y)\lambda(dy)$ and $\mu^h := \sum_{y \in \mathbb{Z}_h^{--}} y \int_{[-y-h/2,-y+h/2)} \mathbb{1}_{[-V,0)}(z)\lambda(dz)$.

$$\tilde{\sigma}_h^2 := \frac{1}{2h^2}\left(\sigma^2 + c_0^h\right) \text{ and } \tilde{\mu}^h := \frac{1}{2h}\left(\mu - \mu^h\right).$$

$\gamma_h := \tilde{\sigma}_h^2 + \mathbb{1}_{(0,\infty)}(\sigma^2)\tilde{\mu}^h + \mathbb{1}_{\{0\}}(\sigma^2)2\tilde{\mu}^h$, $\quad \gamma_{-h} := \tilde{\sigma}_h^2 - \mathbb{1}_{(0,\infty)}(\sigma^2)\tilde{\mu}^h + \lambda(-\infty, -h/2]$

$\gamma_{-kh} := \lambda(-\infty, -kh + h/2]$, $\quad$ where $k \geq 2$.

(cont'd)

Key attractions:

(a) *consistency*: for each fixed $h > 0$, algorithm calculates precisely the values of the scale function W_h for the process X^h;

(b) *conceptual simplicity*: a weak approximation of X by a skip-free-to-the-right-CTMC X^h provides a natural way of encoding the underlying probabilistic structure of the problem in the design of the algorithm;

(c) *robustness*: method valid for all spectrally negative Lévy processes;

(d) *straightforwardness of the algorithm*: implementation requires only to effect a linear recursion;

(e) *no evaluations of Laplace exponent* necessary;

(f) *convergence rates* known;

(g) *stability*: algorithm is numerically stable.

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A two step programme, which yields these formulae, as follows:

(i) Approximate the spectrally negative Lévy process X by a CTMC X^h with state space $\mathbb{Z}_h := \{hk\colon k \in \mathbb{Z}\}$ $(h \in (0, h_\star)$, $h_\star \in (0, +\infty])$:
   - (a) Brownian motion with drift $\to$ asymmetric random walk;
   - (b) Lévy measure becomes concentrated on $\mathbb{Z}_h$;
   - (c) $+$ details for the part of the Lévy measure around the origin (when the latter infinite).

   Mijatović, V., Jacka: Markov chain approximations for transition densities of Lévy processes, (EJP 19(7), 2014).

(ii) Find an algorithm for computing the scale functions $W_h^{(q)}$ and $Z_h^{(q)}$ of the chain X^h.

   V.: Fluctuation theory for upwards skip-free Lévy chains.

Rates of convergence

Fix $q \geq 0$; let K, G be bounded subset of $(0, \infty)$; K bounded away from zero when $\sigma^2 = 0$; and define:

$$\Delta_W^K(h) := \sup_{x \in \mathbb{Z}_h \cap K} \left| W_h^{(q)}(x - \delta^0 h) - W^{(q)}(x) \right| \text{ and } \Delta_Z^G(h) := \sup_{x \in \mathbb{Z}_h \cap G} \left| Z_h^{(q)}(x) - Z^{(q)}(x) \right|,$$

where δ^0 equals 0 if X has sample paths of finite variation and 1 otherwise. Further introduce:

$$\kappa(\delta) := \int_{[-1,-\delta)} |y| \lambda(dy), \qquad \text{for any } \delta \geq 0.$$

If the jump part of X has paths of infinite variation, assume:

Assumption

There exists $\epsilon \in (1, 2)$ *with:*

(1) $\limsup_{\delta \downarrow 0} \delta^\epsilon \lambda(-1, -\delta) < \infty$ *and*

(2) $\liminf_{\delta \downarrow 0} \int_{[-\delta, 0)} x^2 \lambda(dx) / \delta^{2-\epsilon} > 0.$

(cont'd)

Theorem

Then the rates of convergence of the scale functions are summarized by the following table:

$\lambda(\mathbb{R}) = 0$	$\Delta_W^K(h) = O(h^2)$ *and* $\Delta_Z^G(h) = O(h)$
$0 < \lambda(\mathbb{R})$ & $\kappa(0) < \infty$	$\Delta_W^K(h) + \Delta_Z^G(h) = O(h)$
$\kappa(0) = \infty$	$\Delta_W^K(h) + \Delta_Z^G(h) = O(h^{2-\epsilon})$

Moreover, the rates so established are sharp in the sense that for each of the three entries in the table above, examples of spectrally negative Lévy processes are constructed for which the rate of convergence is no better than stipulated.

A numerical illustration

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and (with $V = 1$) $\mu = 10$.

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(i) Lévy measure has fat tails at 0 and $-\infty$, a discontinuity (indeed, a pole) in the density of its absolutely continuous part (which, in particular, is not completely monotone), two atoms. No Gaussian component. Sample paths of the process have infinite variation.

(ii) No explicit expression in terms of elementary/special functions for the Laplace/Furrier transform of the Lévy measure, and thus not the Laplace exponent.

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Example: $\sigma^2 = 0$; the Lévy measure $\lambda(dy)$ given by

$$\frac{3}{2(-y)^{5/2}}\mathbb{1}_{[-1,0)}(y)dy + \frac{1}{2}\left(\delta_{-1}+\delta_{-2}\right)(dy)+$$

$$\frac{1}{2(-y-1)^{1/2}}\mathbb{1}_{[-2,-1)}(y)dy + \left(\frac{e^{\cos(y)}(2+y\sin(y))}{(-y)^3}+\frac{e}{y^2}\right)\mathbb{1}_{(-\infty,-1)}(y)dy;$$

and (with $V=1$) $\mu = 10$.

(i) Lévy measure has fat tails at 0 and $-\infty$, a discontinuity (indeed, a pole) in the density of its absolutely continuous part (which, in particular, is not completely monotone), two atoms. No Gaussian component. Sample paths of the process have infinite variation.

(ii) No explicit expression in terms of elementary/special functions for the Laplace/Furrier transform of the Lévy measure, and thus not the Laplace exponent.

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(cont'd)

n	5	7	9	11	13
max rel$_n$	0.4058	0.2152	0.1040	0.0467	0.0182

Table : Convergence of W_{h_n} on the decreasing sequence $(h_n = 1/2^n)_{n\in\{5,7,9,11,13\}}$, on the interval $[0,3]$. Number of computed values of W_{h_n} is $N_n = 3\cdot 2^n$ in each case. Relative error: max rel$_n := \max_{i\in[96]} \frac{|W_{h_n}(x_i-h_n)-W_{h_{16}}(x_i-h_{16})|}{W_{h_{16}}(x_i-h_{16})}$, where $x_i = i/32$, $i\in[96]$.

(cont'd)

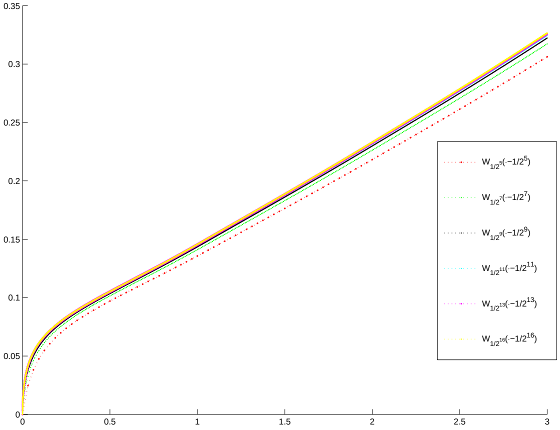

Figure : Convergence of $W_h(\cdot - h)$ to W (as $h \downarrow 0$).

(cont'd)

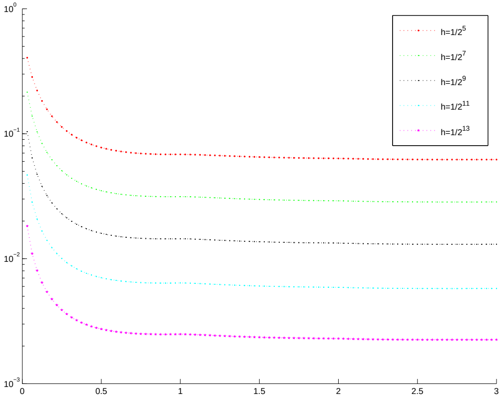

Figure : Relative error $\frac{|W_h(\cdot - h) - W_{h_{16}}(\cdot - h_{16})|}{W_{h_{16}}(\cdot - h_{16})}$.

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- Drawback: # of necessary operations to achieve given precision.
- Efficiency of algorithm sensitive to tail of Lévy measure at 0, rather than smoothness of the target scale function.
  Abate, Whitt: A Unified Framework for Numerically Inverting Laplace Transforms (INFORMS Journal on Computing 18(4), 2006).
- Relates scale functions directly to $(\sigma^2, \lambda, \mu)_{\tilde{c}}$ in a probabilistically meaningful way; they become one limit process, rather than a Laplace exponent, and then an inversion, away from $(\sigma^2, \lambda, \mu)_{\tilde{c}}$.
- Only one parameter to vary (h), whilst e.g. in the (o/w very robust and fast, albeit quite complicated) Filon's method (with FFT) one has also to decide on the cut-off in the Bromwich integral.
- No evaluations of the Laplace exponent ψ (at complex values of its argument) — so complements existing methods, when ψ is not easily computable in terms of elementary/special functions!

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