FLUCTUATION THEORY FOR UPWARDS SKIP-FREE LÉVY CHAINS

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Abstract. A fluctuation theory and, in particular, a theory of scale functions is developed for upwards skip-free Lévy chains, i.e. for right-continuous random walks embedded into continuous time as compound Poisson processes. This is done by analogy to the spectrally negative class of Lévy processes.

1. Introduction

It was shown in [22] that precisely two types of Lévy processes exhibit the property of non-random overshoots: those with no positive jumps a.s., and compound Poisson processes, whose jump chain is (for some $h > 0$) a random walk on $\mathbb{Z}_h := \{hk\colon k \in \mathbb{Z}\}$, skip-free to the right. The latter class was then referred to as "upwards skip-free Lévy chains". Also in the same paper it was remarked that this common property which the two classes share results in a more explicit fluctuation theory (including the Wiener-Hopf factorization) than for a general Lévy process, this being rarely the case (cf. [14, p. 172, Subsection 6.5.4]).

Now, with reference to existing literature on fluctuation theory, the spectrally negative case (when there are no positive jumps, a.s.) is dealt with in detail in [3, Chapter VII] [20, Section 9.46] and especially [14, Chapter 8]. On the other hand no equally exhaustive treatment of the right-continuous random walk seems to have been presented thus far, but see e.g. [17, 7] [23, Section 4] [9, Section 7] [10, Section 9.3] [21, *passim*]. In particular, no such exposition appears forthcoming for the continuous-time analogue of such random walks, wherein the connection and analogy to the spectrally negative class of Lévy processes becomes most transparent and direct.

In the present paper we proceed to do just that, i.e. we develop, by analogy to the spectrally negative case, a complete fluctuation theory (including theory of scale functions) for upwards skip-free Lévy chains. Indeed, the transposition of the results from the spectrally negative to the skip-free setting is essentially straightforward. Over and above this, however, and beyond what is purely analogous to the exposition of the spectrally negative case, further specifics of the reflected process (see Theorem 3.1(i)) and of the excursions from the supremum (see Theorem 3.1(iii)) are

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2010 *Mathematics Subject Classification.* Primary: 60G51; Secondary: 60G50.

Key words and phrases. Lévy processes, non-random overshoots, skip-free random walks, fluctuation theory, scale functions.

The support of the Slovene Human Resources Development and Scholarship Fund under contract number 11010-543/2011 is acknowledged.

identified, and a linear recursion is presented which allows us to directly compute the families of scale functions (see (4.10), (4.11), Proposition 4.14 and Corollary 4.15).

Application-wise, note e.g. that the classical continuous-time Bienaymé-Galton-Watson branching process is associated to upwards skip-free Lévy chains via a suitable time change [14, Section 1.3.4].

The organisation of the rest of this paper is as follows. Section 2 introduces the setting and notation. Then Section 3 develops the relevant fluctuation theory, in particular details of the Wiener-Hopf factorization. Finally, Section 4 deals with the two-sided exit problem and the accompanying families of scale functions.

2. Setting and notation

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space supporting a Lévy process [14, p. 2, Definition 1.1] X (X is assumed to be $\mathbb{F}$-adapted and to have independent increments relative to $\mathbb{F}$). The Lévy measure [20, p. 38, Definition 8.2] of X is denoted by λ. Next, recall from [22] (with $\mathrm{supp}(\nu)$ denoting the support [12, p. 9] of a measure ν defined on the Borel σ-field of some topological space):

Definition 2.1 (Upwards skip-free Lévy chain)**.** X is an *upwards skip-free Lévy chain*, if it is a compound Poisson process [20, p. 18, Definition 4.2], and for some $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$, whereas $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.

In the sequel, X will be assumed throughout an upwards skip-free Lévy chain, with $\lambda(\{h\}) > 0$ ($h > 0$) and characteristic exponent $\Psi(p) = \int (e^{ipx} - 1)\lambda(dx)$ ($p \in \mathbb{R}$). In general, we insist on (i) every sample path of X being càdlàg (i.e. right-continuous, admitting left limits) and (ii) $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ satisfying the standard assumptions (i.e. the σ-field $\mathcal{F}$ is P-complete, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all P-null sets). Nevertheless, we shall, sometimes and then only provisionally, relax assumption (ii), by transferring X as the coordinate process onto the canonical space $\mathbb{D}_h := \{\omega \in \mathbb{Z}_h^{[0,\infty)} : \omega \text{ is càdlàg}\}$ of càdlàg paths, mapping $[0, \infty) \to \mathbb{Z}_h$, equipping $\mathbb{D}_h$ with the σ-algebra and natural filtration of evaluation maps; this, however, will always be made explicit. We allow e_1 to be exponentially distributed, mean one, and independent of X; then define $e_p := e_1/p$ ($p \in (0, \infty)\backslash\{1\}$).

Further, for $x \in \mathbb{R}$, introduce $T_x := \inf\{t \geq 0 : X_t \geq x\}$, the first entrance time of X into $[x, \infty)$. Note that T_x is an $\mathbb{F}$-stopping time [12, p. 101, Theorem 6.7]. The supremum or maximum (respectively infimum or minimum) process of X is denoted $\overline{X}_t := \sup\{X_s : s \in [0, t]\}$ (respectively $\underline{X}_t := \inf\{X_s : s \in [0, t]\}$) ($t \geq 0$). $\underline{X}_\infty := \inf\{X_s : s \in [0, \infty)\}$ is the overall infimum.

With regard to miscellaneous general notation we have:

(1) The nonnegative, nonpositive, positive and negative real numbers are denoted by $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_- := \{x \in \mathbb{R} : x \leq 0\}$, $\mathbb{R}^+ := \mathbb{R}_+\backslash\{0\}$ and $\mathbb{R}^- := \mathbb{R}_-\backslash\{0\}$, respectively.

Then $\mathbb{Z}_+ := \mathbb{R}_+ \cap \mathbb{Z}$, $\mathbb{Z}_- := \mathbb{R}_- \cap \mathbb{Z}$, $\mathbb{Z}^+ := \mathbb{R}^+ \cap \mathbb{Z}$ and $\mathbb{Z}^- := \mathbb{R}^- \cap \mathbb{Z}$ are the nonnegative, nonpositive, positive and negative integers, respectively.

(2) Similarly, for $h > 0$: $\mathbb{Z}_h^+ := \mathbb{Z}_h \cap \mathbb{R}_+$, $\mathbb{Z}_h^{++} := \mathbb{Z}_h \cap \mathbb{R}^+$, $\mathbb{Z}_h^- := \mathbb{Z}_h \cap \mathbb{R}_-$ and $\mathbb{Z}_h^{--} := \mathbb{Z}_h \cap \mathbb{R}^-$ are the apposite elements of $\mathbb{Z}_h$.

(3) The following introduces notation for the relevant half-planes of $\mathbb{C}$; the arrow notation is meant to be suggestive of which half-plane is being considered: $\mathbb{C}^{\rightarrow} := \{z \in \mathbb{C} : \Re z > 0\}$, $\mathbb{C}^{\leftarrow} := \{z \in \mathbb{C} : \Re z < 0\}$, $\mathbb{C}^{\downarrow} := \{z \in \mathbb{C} : \Im z < 0\}$ and $\mathbb{C}^{\uparrow} := \{z \in \mathbb{C} : \Im z > 0\}$. $\overline{\mathbb{C}^{\rightarrow}}$, $\overline{\mathbb{C}^{\leftarrow}}$, $\overline{\mathbb{C}^{\downarrow}}$ and $\overline{\mathbb{C}^{\uparrow}}$ are then the respective closures of these sets.

(4) $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the positive and nonnegative integers, respectively. $\lceil x \rceil := \inf\{k \in \mathbb{Z} : k \geq x\}$ ($x \in \mathbb{R}$) is the ceiling function. For $\{a, b\} \subset [-\infty, +\infty]$: $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

(5) The Laplace transform of a measure μ on $\mathbb{R}$, concentrated on $[0, \infty)$, is denoted $\hat{\mu}$: $\hat{\mu}(\beta) = \int_{[0,\infty)} e^{-\beta x} \mu(dx)$ (for all $\beta \geq 0$ such that this integral is finite). To a nondecreasing right-continuous function $F : \mathbb{R} \to \mathbb{R}$, a measure dF may be associated in the Lebesgue-Stieltjes sense.

The geometric law $\mathrm{geom}(p)$ with success parameter $p \in (0, 1]$ has $\mathrm{geom}(p)(\{k\}) = p(1-p)^k$ ($k \in \mathbb{N}_0$), $1 - p$ is then the failure parameter. The exponential law $\mathrm{Exp}(\beta)$ with parameter $\beta > 0$ is specified by the density $\mathrm{Exp}(\beta)(dt) = \beta e^{-\beta t} \mathbb{1}_{(0,\infty)}(t) dt$. A function $f : [0, \infty) \to [0, \infty)$ is said to be of exponential order, if there are $\{\alpha, A\} \subset \mathbb{R}_+$, such that $f(x) \leq A e^{\alpha x}$ ($x \geq 0$); $f(+\infty) := \lim_{x \to \infty} f(x)$, when this limit exists. DCT (respectively MCT) stands for the dominated (respectively monotone) convergence theorem. Finally, increasing (respectively decreasing) will mean strictly increasing (respectively strictly decreasing), nondecreasing (respectively nonincreasing) being used for the weaker alternative; we will understand $a/0 = \pm\infty$ for $a \in \pm(0, \infty)$.

3. Fluctuation theory

In the following, to fully appreciate the similarity (and eventual differences) with the spectrally negative case, the reader is invited to directly compare the exposition of this subsection with that of [3, Section VII.1] and [14, Section 8.1].

3.1. **Laplace exponent, the reflected process, local times and excursions from the supremum, supremum process and long-term behaviour, exponential change of measure.** Since the Poisson process admits exponential moments of all orders, it follows that $\mathsf{E}[e^{\beta \overline{X}_t}] < \infty$ and, in particular, $\mathsf{E}[e^{\beta X_t}] < \infty$ for all $\{\beta, t\} \subset [0, \infty)$. Indeed, it may be seen by a direct computation that for $\beta \in \overline{\mathbb{C}^{\rightarrow}}$, $t \geq 0$, $\mathsf{E}[e^{\beta X_t}] = \exp\{t\psi(\beta)\}$, where $\psi(\beta) := \int_{\mathbb{R}} (e^{\beta x} - 1)\lambda(dx)$ is the Laplace exponent of X. Moreover, ψ is continuous (by the DCT) on $\overline{\mathbb{C}^{\rightarrow}}$ and analytic in $\mathbb{C}^{\rightarrow}$ (use the theorems of Cauchy [19, p. 206, 10.13 Cauchy's theorem for triangle], Morera [19, p. 209, 10.17 Morera's theorem] and Fubini).

Next, note that $\psi(\beta)$ tends to $+\infty$ as $\beta \to \infty$ over the reals, due to the presence of the atom of λ at h. Upon restriction to $[0,\infty)$, ψ is strictly convex, as follows first on $(0,\infty)$ by using differentiation under the integral sign and noting that the second derivative is strictly positive, and then extends to $[0,\infty)$ by continuity.

Denote then by $\Phi(0)$ the largest root of $\psi|_{[0,\infty)}$. Indeed, 0 is always a root, and due to strict convexity, if $\Phi(0) > 0$, then 0 and $\Phi(0)$ are the only two roots. The two cases occur, according as to whether $\psi'(0+) \geq 0$ or $\psi'(0+) < 0$, which is clear. It is less obvious, but nevertheless true, that this right derivative at 0 actually exists, indeed $\psi'(0+) = \int_{\mathbb{R}} x\lambda(dx) \in [-\infty,\infty)$. This follows from the fact that $(e^{\beta x}-1)/\beta$ is nonincreasing as $\beta \downarrow 0$ for $x \in \mathbb{R}_-$ and hence monotone convergence applies. Continuing from this, and with a similar justification, one also gets the equality $\psi''(0+) = \int x^2\lambda(dx) \in (0,+\infty]$ (where we agree $\psi''(0+) = +\infty$ if $\psi'(0+) = -\infty$). In any case, $\psi : [\Phi(0),\infty) \to [0,\infty)$ is continuous and increasing, it is a bijection and we let $\Phi : [0,\infty) \to [\Phi(0),\infty)$ be the inverse bijection, so that $\psi \circ \Phi = \mathrm{id}_{\mathbb{R}_+}$.

With these preliminaries having been established, our first theorem identifies characteristics of the reflected process, the local time of X at the maximum (for a definition of which see e.g. [14, p. 140, Definition 6.1]), its inverse, as well as the expected length of excursions and the probability of an infinite excursion therefrom (for definitions of these terms see e.g. [14, pp. 140-147]; we agree that an excursion (from the maximum) starts immediately X leaves its running maximum and ends immediately it returns to it; by its length we mean the amount of time between these two time points).

Theorem 3.1 (Reflected process; (inverse) local time; excursions). *Let $q_n := \lambda(\{-nh\})/\lambda(\mathbb{R})$ for $n \in \mathbb{N}$ and $p := \lambda(\{h\})/\lambda(\mathbb{R})$.*

(i) The generator matrix $\tilde{Q}$ of the Markov process $Y := \overline{X} - X$ on $\mathbb{Z}_h^+$ is given by (with $\{s,s'\} \subset \mathbb{Z}_h^+$): $\tilde{Q}_{ss'} = \lambda(\{s-s'\}) - \delta_{ss'}\lambda(\mathbb{R})$, unless $s = s' = 0$, in which case we have $\tilde{Q}_{ss'} = -\lambda((-\infty,0))$.

(ii) For the reflected process Y, 0 is a holding point. The actual time spent at 0 by Y, which we shall denote L, is a local time at the maximum. Its right-continuous inverse L^{-1}, given by $L_t^{-1} := \inf\{s \geq 0 : L_s > t\}$ (for $0 \leq t < L_\infty$; $L_t^{-1} := \infty$ otherwise), is then a (possibly killed) compound Poisson subordinator with unit positive drift.

(iii) Assuming that $\lambda((-\infty,0)) > 0$ to avoid the trivial case, the expected length of an excursion away from the supremum is equal to $\frac{\lambda(\{h\})h - \psi'(0+)}{(\psi'(0+)\vee 0)\lambda((-\infty,0))}$; whereas the probability of such an excursion being infinite is $\frac{\lambda(\{h\})}{\lambda((-\infty,0))}(e^{\Phi(0)h}-1) =: p^*$.

(iv) Assume again $\lambda((-\infty,0)) > 0$ to avoid the trivial case. Let N, taking values in $\mathbb{N} \cup \{+\infty\}$, be the number of jumps the chain makes before returning to its running maximum, after it has first left it (it does so with probability 1). Then the law of L^{-1} is given by (for

$\theta\in[0,+\infty)$):

$$-\log \mathsf{E}\left[\exp(-\theta L_1^{-1})\mathbb{1}_{\{L_1^{-1}<+\infty\}}\right]=\theta+\lambda((-\infty,0))\left(1-\sum_{k=1}^\infty \mathsf{P}(N=k)\left(\frac{\lambda(\mathbb{R})}{\lambda(\mathbb{R})+\theta}\right)^k\right).$$

In particular, L^{-1} has a killing rate of $\lambda((-\infty,0))p^*$, Lévy mass $\lambda((-\infty,0))(1-p^*)$ and its jumps have the probability law on $(0,+\infty)$ given by the length of a generic excursion from the supremum, conditional on it being finite, i.e. that of an independent N-fold sum of independent $\mathrm{Exp}(\lambda(\mathbb{R}))$-distributed random variables, conditional on N being finite. Moreover, one has, for $k\in\mathbb{N}$, $\mathsf{P}(N=k)=\sum_{l=1}^k q_l p_{l,k}$, where the coefficients $(p_{l,k})_{l,k=1}^\infty$ satisfy the initial conditions:

$$p_{l,1}=p\delta_{l1},\quad l\in\mathbb{N};$$

the recursions:

$$p_{l,k+1}=\begin{cases}0 & \text{if } l=k \text{ or } l>k+1\\ \sum_{m=1}^{k-1}q_m p_{m+1,k} & \text{if } l=1\\ p^{k+1} & \text{if } l=k+1\\ pp_{l-1,k}+\sum_{m=1}^{k-l}q_m p_{m+l,k} & \text{if } 1<l<k\end{cases},\qquad \{l,k\}\subset\mathbb{N};$$

and $p_{l,k}$ may be interpreted as the probability of X reaching level 0 starting from level $-lh$ for the first time on precisely the k-th jump ($\{l,k\}\subset\mathbb{N}$).

Proof. (i) is clear, since, e.g. Y transitions away from 0 at the rate at which X makes a negative jump; and from $s\in\mathbb{Z}_h^+\backslash\{0\}$ to 0 at the rate at which X jumps up by s or more etc.

(ii) is standard [14, p. 141, Example 6.3 & p. 149, Theorem 6.10].

We next establish (iii). Denote, provisionally, by β the expected excursion length. Further, let the discrete-time Markov chain W (on the state space $\mathbb{N}_0$) be endowed with the initial distribution $w_j:=\frac{1}{\lambda((-\infty,0))}q_j$ for $j\in\mathbb{N}$, $w_0:=0$; and transition matrix P, given by $P_{0i}=\delta_{0i}$, whereas for $i\geq 1$: $P_{ij}=p$, if $j=i-1$; $P_{ij}=q_{j-i}$, if $j>i$; and $P_{ij}=0$ otherwise (W jumps down with probability p, up i steps with probability q_i, $i\geq 1$, until it reaches 0, where it gets stuck). Let further N be the first hitting time for W of $\{0\}$, so that a typical excursion length of X is equal in distribution to an independent sum of N (possibly infinite) $\mathrm{Exp}(\lambda(\mathbb{R}))$-random variables. It is Wald's identity that $\beta=(1/\lambda(\mathbb{R}))\mathsf{E}[N]$. Then (in the obvious notation, where $\underline{\infty}$ indicates the sum is inclusive of ∞), by Fubini: $\mathsf{E}[N]=\sum_{n=1}^{\underline{\infty}} n\sum_{l=1}^\infty w_l\mathsf{P}_l(N=n)=\sum_{l=1}^\infty w_l k_l$, where k_l is the mean hitting time of $\{0\}$ for W, if it starts from $l\in\mathbb{N}_0$, as in [15, p. 12]. From the skip-free property of the chain W it is moreover transparent that $k_i=\alpha i$, $i\in\mathbb{N}_0$, for some $0<\alpha\leq\infty$ (with the usual convention $0\cdot\infty=0$). Moreover we know [15, p. 17, Theorem 1.3.5] that $(k_i:i\in\mathbb{N}_0)$ is the minimal solution to $k_0=0$ and $k_i=1+\sum_{j=1}^\infty P_{ij}k_j$ ($i\in\mathbb{N}$). Plugging in $k_i=\alpha i$, the last system of linear equations is equivalent to (provided $\alpha<\infty$) $0=1-p\alpha+\alpha\zeta$, where $\zeta:=\sum_{j=1}^\infty jq_j$. Thus, if $\zeta<p$, the

minimal solution to the system is $k_i = i/(p-\zeta)$, $i \in \mathbb{N}_0$, from which $\beta = \zeta/(\lambda((-\infty,0))(p-\zeta))$ follows at once. If $\zeta \geq p$, clearly we must have $\alpha = +\infty$, hence $\mathsf{E}[N] = +\infty$ and thus $\beta = +\infty$.

To establish the probability of an excursion being infinite, i.e. $\sum_{i=1}^{\infty} q_i(1-\alpha_i)/\sum_{i=1}^{\infty} q_i$, where $\alpha_i := \mathsf{P}_i(N < \infty) > 0$, we see that (by the skip-free property) $\alpha_i = \alpha_1^i$, $i \in \mathbb{N}_0$, and by the strong Markov property, for $i \in \mathbb{N}$, $\alpha_i = p\alpha_{i-1} + \sum_{j=1}^{\infty} q_j \alpha_{i+j}$. It follows that $1 = p\alpha_1^{-1} + \sum_{j=1}^{\infty} q_j \alpha_1^j$, i.e. $0 = \psi(\log(\alpha_1^{-1})/h)$. Hence, by Theorem 3.2(ii), whose proof will be independent of this one, $\alpha_1 = e^{-\Phi(0)h}$ (since $\alpha_1 < 1$, if and only if X drifts to $-\infty$).

Finally, (iv) is straightforward. □

We turn our attention now to the supremum process $\overline{X}$. First, using the lack of memory property of the exponential law and the skip-free nature of X, we deduce from the strong Markov property applied at the time T_a, that for every $a, b \in \mathbb{Z}_h^+$, $p > 0$: $\mathsf{P}(T_{a+b} < e_p) = \mathsf{P}(T_a < e_p)\mathsf{P}(T_b < e_p)$. In particular, for any $n \in \mathbb{N}_0$: $\mathsf{P}(T_{nh} < e_p) = \mathsf{P}(T_h < e_p)^n$. And since for $s \in \mathbb{Z}_h^+$, $\{T_s < e_p\} = \{\overline{X}_{e_p} \geq s\}$ (P-a.s.) one has (for $n \in \mathbb{N}_0$): $\mathsf{P}(\overline{X}_{e_p} \geq nh) = \mathsf{P}(\overline{X}_{e_p} \geq h)^n$. Therefore $\overline{X}_{e_p}/h \sim \mathrm{geom}(1 - \mathsf{P}(\overline{X}_{e_p} \geq h))$.

Next, to identify $\mathsf{P}(\overline{X}_{e_p} \geq h)$, $p > 0$, observe that (for $\beta \geq 0$, $t \geq 0$): $\mathsf{E}[\exp\{\Phi(\beta)X_t\}] = e^{t\beta}$ and hence $(\exp\{\Phi(\beta)X_t - \beta t\})_{t\geq 0}$ is an $(\mathbb{F}, \mathsf{P})$-martingale by stationary independent increments of X, for each $\beta \geq 0$. Then apply the Optional Sampling Theorem at the bounded stopping time $T_x \wedge t$ ($t, x \geq 0$) to get:

$$\mathsf{E}[\exp\{\Phi(\beta)X(T_x \wedge t) - \beta(T_x \wedge t)\}] = 1.$$

Note that $X(T_x \wedge t) \leq h\lceil x/h \rceil$ and $\Phi(\beta)X(T_x \wedge t) - \beta(T_x \wedge t)$ converges to $\Phi(\beta)h\lceil x/h\rceil - \beta T_x$ (P-a.s.) as $t \to \infty$ on $\{T_x < \infty\}$. It converges to $-\infty$ on the complement of this event, P-a.s., provided $\beta + \Phi(\beta) > 0$. Therefore we deduce by dominated convergence, first for $\beta > 0$ and then also for $\beta = 0$, by taking limits:

$$\mathsf{E}[\exp\{-\beta T_x\}\mathbb{1}_{\{T_x<\infty\}}] = \exp\{-\Phi(\beta)h\lceil x/h\rceil\}. \tag{3.1}$$

Before we formulate out next theorem, recall also that any non-zero Lévy process either drifts to $+\infty$, oscillates or drifts to $-\infty$ [20, pp. 255-256, Proposition 37.10 and Definition 37.11].

Theorem 3.2 (Supremum process and long-term behaviour)**.**

(i) The failure probability for the geometrically distributed $\overline{X}_{e_p}/h$ *is* $\exp\{-\Phi(p)h\}$ *(*$p > 0$*).*

(ii) X *drifts to* $+\infty$*, oscillates or drifts to* $-\infty$ *according as to whether* $\psi'(0+)$ *is positive, zero, or negative. In the latter case* $\overline{X}_\infty/h$ *has a geometric distribution with failure probability* $\exp\{-\Phi(0)h\}$*.*

(iii) $(T_{nh})_{n\in\mathbb{N}_0}$ *is a discrete-time increasing stochastic process, vanishing at* 0 *and having stationary independent increments up to the explosion time, which is an independent geometric random variable; it is a killed random walk.*

Remark 3.3. Unlike in the spectrally negative case [3, p. 189], the supremum process cannot be obtained from the reflected process, since the latter does not discern a point of increase in X when the latter is at its running maximum.

Proof. We have for every $s \in \mathbb{Z}_h^+$:

$$\mathsf{P}(\overline{X}_{e_p} \geq s) = \mathsf{P}(T_s < e_p) = \mathsf{E}[\exp\{-pT_s\}\mathbb{1}_{\{T_s<\infty\}}] = \exp\{-\Phi(p)s\}. \tag{3.2}$$

Thus (i) obtains.

For (ii) note that letting $p \downarrow 0$ in (3.2), we obtain $\overline{X}_\infty < \infty$ (P-a.s.), if and only if $\Phi(0) > 0$, which is equivalent to $\psi'(0+) < 0$. If so, $\overline{X}_\infty/h$ is geometrically distributed with failure probability $\exp\{-\Phi(0)h\}$ and then (and only then) does X drift to $-\infty$.

It remains to consider drifting to $+\infty$ (the cases being mutually exclusive and exhaustive). Indeed, X drifts to $+\infty$, if and only if $\mathsf{E}[T_s]$ is finite for each $s \in \mathbb{Z}_h^+$ [3, p. 172, Proposition VI.17]. Using again the nondecreasingness of $(e^{-\beta T_s} - 1)/\beta$ in $\beta \in [0, \infty)$, we deduce from (3.1), by monotone convergence, that one may differentiate under the integral sign, to get $\mathsf{E}[T_s\mathbb{1}_{\{T_s<\infty\}}] = (\beta \mapsto -\exp\{-\Phi(\beta)s\})'(0+)$. So the $\mathsf{E}[T_s]$ are finite, if and only if $\Phi(0) = 0$ (so that $T_s < \infty$ P-a.s.) and $\Phi'(0+) < \infty$. Since Φ is the inverse of $\psi|_{[\Phi(0),\infty)}$, this is equivalent to saying $\psi'(0+) > 0$.

Finally, (iii) is clear. $\square$

TABLE 1. Connections between the quantities $\psi'(0+)$, $\Phi(0)$, $\Phi'(0+)$. Behaviour of X at large times and of its excursions away from the running supremum (the latter if $\lambda((-\infty, 0)) > 0$).

$\psi'(0+)$	$\Phi(0)$	$\Phi'(0+)$	Long-term behaviour	Excursion length
$\in (0, \infty)$	0	$\in (0, \infty)$	drifts to $+\infty$	finite expectation
0	0	$+\infty$	oscillates	a.s. finite with infinite expectation
$\in [-\infty, 0)$	$\in (0, \infty)$	$\in (0, \infty)$	drifts to $-\infty$	infinite with a positive probability

We conclude this section by offering a way to reduce the general case of an upwards skip-free Lévy chain to one which necessarily drifts to $+\infty$. This will prove useful in the sequel. First, there is a pathwise approximation of an oscillating X, by (what is again) an upwards skip-free Lévy chain, but drifting to infinity.

Remark 3.4. Suppose X oscillates. Let (possibly by enlarging the probability space to accommodate for it) N be an independent Poisson process with intensity 1 and $N_t^\epsilon := N_{t\epsilon}$ ($t \geq 0$) so that N^ϵ is a Poisson process with intensity ϵ, independent of X. Define $X^\epsilon := X + hN^\epsilon$. Then, as $\epsilon \downarrow 0$, X^ϵ converges to X, uniformly on bounded time sets, almost surely, and is clearly an upwards skip-free Lévy chain drifting to $+\infty$.

The reduction of the case when X drifts to $-\infty$ is somewhat more involved and is done by a change of measure. For this purpose assume until the end of this subsection, that X is already the coordinate process on the canonical space $\Omega = \mathbb{D}_h$, equipped with the σ-algebra $\mathcal{F}$ and filtration $\mathbb{F}$ of evaluation maps (so that P *coincides* with the law of X on $\mathbb{D}_h$ and $\mathcal{F} = \sigma(\mathrm{pr}_s : s \in [0, +\infty))$, whilst for $t \geq 0$, $\mathcal{F}_t = \sigma(\mathrm{pr}_s : s \in [0, t])$, where $\mathrm{pr}_s(\omega) = \omega(s)$, for $(s, \omega) \in [0, +\infty) \times \mathbb{D}_h$). We make this transition in order to be able to apply the Kolmogorov extension theorem in the proposition, which follows. Note, however, that we are no longer able to assume standard conditions on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$. Notwithstanding this, $(T_x)_{x\in\mathbb{R}}$ remain $\mathbb{F}$-stopping times, since by the nature of the space $\mathbb{D}_h$, for $x \in \mathbb{R}$, $t \geq 0$, $\{T_x \leq t\} = \{\overline{X}_t \geq x\} \in \mathcal{F}_t$.

Proposition 3.5 (Exponential change of measure)**.** *Let $c \geq 0$. Then, demanding:*

$$\mathsf{P}_c(\Lambda) = \mathsf{E}[\exp\{cX_t - \psi(c)t\}\mathbb{1}_\Lambda] \quad (\Lambda \in \mathcal{F}_t, t \geq 0) \tag{3.3}$$

this introduces a unique measure P_c on $\mathcal{F}$. Under the new measure, X remains an upwards skip-free Lévy chain with Laplace exponent $\psi_c = \psi(\cdot + c) - \psi(c)$, drifting to $+\infty$, if $c \geq \Phi(0)$, unless $c = \psi'(0+) = 0$. Moreover, if λ_c is the new Lévy measure of X under P_c, then $\lambda_c \ll \lambda$ and $\frac{d\lambda_c}{d\lambda}(x) = e^{cx}$ λ-a.e. in $x \in \mathbb{R}$. Finally, for every $\mathbb{F}$-stopping time T, $\mathsf{P}_c \ll \mathsf{P}$ on restriction to $\mathcal{F}'_T := \{A \cap \{T < \infty\} : A \in \mathcal{F}_T\}$, and:

$$\frac{d\mathsf{P}_c|_{\mathcal{F}'_T}}{d\mathsf{P}|_{\mathcal{F}'_T}} = \exp\{cX_T - \psi(c)T\}.$$

Proof. That P_c is introduced consistently as a probability measure on $\mathcal{F}$ follows from the Kolmogorov extension theorem [16, p. 143, Theorem 4.2]. Indeed, $M := (\exp\{cX_t - \psi(c)t\})_{t\geq 0}$ is a nonnegative martingale (use independence and stationarity of increments of X and the definition of the Laplace exponent), equal identically to 1 at time 0.

Further, for all $\beta \in \overline{\mathbb{C}^{\rightarrow}}$, $\{t, s\} \subset \mathbb{R}_+$ and $\Lambda \in \mathcal{F}_t$:

$$\begin{aligned} \mathsf{E}_c[\exp\{\beta(X_{t+s} - X_t)\}\mathbb{1}_\Lambda] &= \mathsf{E}[\exp\{cX_{t+s} - \psi(c)(t+s)\}\exp\{\beta(X_{t+s} - X_t)\}\mathbb{1}_\Lambda] \\ &= \mathsf{E}[\exp\{(c+\beta)(X_{t+s} - X_t) - \psi(c)s\}]\mathsf{E}[\exp\{cX_t - \psi(c)t\}\mathbb{1}_\Lambda] \\ &= \exp\{s(\psi(c+\beta) - \psi(c))\}\mathsf{P}_c(\Lambda). \end{aligned}$$

An application of the Functional Monotone Class Theorem then shows that X is indeed a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P}_c)$ and its Laplace exponent under P_c is as stipulated (that $X_0 = 0$ P_c-a.s. follows from the absolute continuity of P_c with respect to P on restriction to $\mathcal{F}_0$).

Next, from the expression for ψ_c, the claim regarding λ_c follows at once. Then clearly X remains an upwards skip-free Lévy chain under P_c, drifting to $+\infty$, if $\psi'(c+) > 0$.

Finally, let $A \in \mathcal{F}_T$ and $t \geq 0$. Then $A \cap \{T \leq t\} \in \mathcal{F}_{T\wedge t}$, and by the Optional Sampling Theorem:

$$\mathsf{P}_c(A\cap\{T \leq t\}) = \mathsf{E}[M_t \mathbb{1}_{A\cap\{T\leq t\}}] = \mathsf{E}[\mathsf{E}[M_t\mathbb{1}_{A\cap\{T\leq t\}}|\mathcal{F}_{T\wedge t}]] = \mathsf{E}[M_{T\wedge t}\mathbb{1}_{A\cap\{T\leq t\}}] = \mathsf{E}[M_T\mathbb{1}_{A\cap\{T\leq t\}}].$$

Using the MCT, letting $t \to \infty$, we obtain the equality $\mathsf{P}_c(A \cap \{T < \infty\}) = \mathsf{E}[M_T \mathbb{1}_{A\cap\{T<\infty\}}]$. □

Proposition 3.6 (Conditioning to drift to $+\infty$)**.** *Assume $\Phi(0) > 0$ and denote $\mathsf{P}^\natural := \mathsf{P}_{\Phi(0)}$ (see (3.3)). We then have as follows.*

1. *For every $\Lambda \in \mathcal{A} := \cup_{t\geq 0}\mathcal{F}_t$, $\lim_{n\to\infty} \mathsf{P}(\Lambda|\overline{X}_\infty \geq nh) = \mathsf{P}^\natural(\Lambda)$.*
2. *For every $x \geq 0$, the stopped process $X^{T_x} = (X_{t\wedge T_x})_{t\geq 0}$ is identical in law under the measures $\mathsf{P}^\natural$ and $\mathsf{P}(\cdot|T_x < \infty)$ on the canonical space $\mathbb{D}_h$.*

Proof. With regard to (1), we have as follows. Let $t \geq 0$. By the Markov property of X at time t, the process $\overset{\triangle}{X} := (X_{t+s} - X_t)_{s\geq 0}$ is identical in law with X on $\mathbb{D}_h$ and independent of $\mathcal{F}_t$ under P. Thus, letting $\overset{\triangle}{T}_y := \inf\{t \geq 0 : \overset{\triangle}{X}_t \geq y\}$ ($y \in \mathbb{R}$), one has for $\Lambda \in \mathcal{F}_t$ and $n \in \mathbb{N}_0$, by conditioning:

$$\mathsf{P}(\Lambda \cap \{t < T_{nh} < \infty\}) = \mathsf{E}[\mathsf{E}[\mathbb{1}_\Lambda \mathbb{1}_{\{t<T_{nh}\}} \mathbb{1}_{\{\overset{\triangle}{T}_{nh-X_t}<\infty\}}|\mathcal{F}_t]] = \mathsf{E}[e^{\Phi(0)(X_t-nh)}\mathbb{1}_{\Lambda\cap\{t<T_{nh}\}}],$$

since $\{\Lambda, \{t < T_{nh}\}\} \cup \sigma(X_t) \subset \mathcal{F}_t$. Next, noting that $\{\overline{X}_\infty \geq nh\} = \{T_{nh} < \infty\}$:

$$\begin{aligned}
\mathsf{P}(\Lambda|\overline{X}_\infty > nh) &= e^{\Phi(0)nh}\left(\mathsf{P}(\Lambda \cap \{T_{nh} \leq t\}) + \mathsf{P}(\Lambda \cap \{t < T_{nh} < \infty\})\right) \\
&= e^{\Phi(0)nh}\left(\mathsf{P}(\Lambda \cap \{T_{nh} \leq t\}) + \mathsf{E}[e^{\Phi(0)(X_t-nh)}\mathbb{1}_{\Lambda\cap\{t<T_{nh}\}}]\right) \\
&= e^{\Phi(0)nh}\mathsf{P}(\Lambda \cap \{T_{nh} \leq t\}) + \mathsf{P}^\natural(\Lambda \cap \{t < T_{nh}\}).
\end{aligned}$$

The second term clearly converges to $\mathsf{P}^\natural(\Lambda)$ as $n \to \infty$. The first converges to 0, because by (3.2) $\mathsf{P}(\overline{X}_{e_1} \geq nh) = e^{-nh\Phi(1)} = o(e^{-nh\Phi(0)})$, as $n \to \infty$, and we have the estimate $\mathsf{P}(T_{nh} \leq t) = \mathsf{P}(\overline{X}_t \geq nh) = \mathsf{P}(\overline{X}_t \geq nh|e_1 \geq t) \leq \mathsf{P}(\overline{X}_{e_1} \geq nh|e_1 \geq t) \leq e^t\mathsf{P}(\overline{X}_{e_1} \geq nh)$.

We next show (2). Note first that X is $\mathbb{F}$-progressively measurable (in particular, measurable), hence the stopped process X^{T_x} is measurable as a mapping into $\mathbb{D}_h$ [13, p. 5, Problem 1.16].

Further, by the strong Markov property, conditionally on $\{T_x < \infty\}$, $\mathcal{F}_{T_x}$ is independent of the future increments of X after T_x, hence also of $\{T_{x'} < \infty\}$ for any $x' > x$. We deduce that the law of X^{T_x} is the same under $\mathsf{P}(\cdot|T_x < \infty)$ as it is under $\mathsf{P}(\cdot|T_{x'} < \infty)$ for any $x' > x$. (2) then follows from (1) by letting x' tend to $+\infty$, the algebra $\mathcal{A}$ being sufficient to determine equality in law by a π/λ-argument. □

3.2. Wiener-Hopf factorization.

Definition 3.7. We define, for $t \geq 0$, $\overline{G}_t^* := \inf\{s \in [0, t] : X_s = \overline{X}_t\}$, i.e., P-a.s., $\overline{G}_t^*$ is the last time in the interval $[0, t]$ that X attains a new maximum. Similarly we let $\underline{G}_t := \sup\{s \in [0, t] : X_s = \underline{X}_s\}$ be, P-a.s., the last time on $[0, t]$ of attaining the running infimum ($t \geq 0$).

While the statements of the next proposition are given for the upwards skip-free Lévy chain X, they in fact hold true for the Wiener-Hopf factorization of *any* compound Poisson process. Moreover, they are (essentially) known [14]. Nevertheless, we begin with these general observations, in order to (a) introduce further relevant notation and (b) provide the reader with the prerequisites needed to understand the remainder of this subsection. Immediately following Proposition 3.8, however, we particularize to our the skip-free setting.

Proposition 3.8. *Let $p > 0$. Then:*

(i) The pairs $(\overline{G}^*_{e_p}, \overline{X}_{e_p})$ and $(e_p - \overline{G}^*_{e_p}, \overline{X}_{e_p} - X_{e_p})$ are independent and infinitely divisible, yielding the factorisation:

$$\frac{p}{p - i\eta - \Psi(\theta)} = \Psi^+_p(\eta,\theta)\Psi^-_p(\eta,\theta),$$

where for $\{\theta,\eta\} \subset \mathbb{R}$,

$$\Psi^+_p(\eta,\theta) := \mathsf{E}[\exp\{i\eta\overline{G}^*_{e_p} + i\theta\overline{X}_{e_p}\}] \text{ and } \Psi^-_p(\eta,\theta) := \mathsf{E}[\exp\{i\eta\underline{G}_{e_p} + i\theta\underline{X}_{e_p}\}].$$

Duality: $(e_p - \overline{G}^*_{e_p}, \overline{X}_{e_p} - X_{e_p})$ is equal in distribution to $(\underline{G}_{e_p}, -\underline{X}_{e_p})$. Ψ^+_p and Ψ^-_p are the Wiener-Hopf factors.

(ii) The Wiener-Hopf factors may be identified as follows:

$$\mathsf{E}[\exp\{-\alpha\overline{G}^*_{e_p} - \beta\overline{X}_{e_p}\}] = \frac{\kappa^*(p,0)}{\kappa^*(p+\alpha,\beta)}$$

and

$$\mathsf{E}[\exp\{-\alpha\underline{G}_{e_p} + \beta\underline{X}_{e_p}\}] = \frac{\hat{\kappa}(p,0)}{\hat{\kappa}(p+\alpha,\beta)}$$

for $\{\alpha,\beta\} \subset \overline{\mathbb{C}^{\rightarrow}}$.

(iii) Here, in terms of the law of X,

$$\kappa^*(\alpha,\beta) := k^* \exp\left(\int_0^\infty \int_{(0,\infty)} (e^{-t} - e^{-\alpha t - \beta x})\frac{1}{t}\mathsf{P}(X_t \in dx)dt\right)$$

and

$$\hat{\kappa}(\alpha,\beta) = \hat{k} \exp\left(\int_0^\infty \int_{(-\infty,0]} (e^{-t} - e^{-\alpha t + \beta x})\frac{1}{t}\mathsf{P}(X_t \in dx)dt\right)$$

for $\alpha \in \mathbb{C}^{\rightarrow}$, $\beta \in \overline{\mathbb{C}^{\rightarrow}}$ and some constants $\{k^*, \hat{k}\} \subset \mathbb{R}^+$.

Proof. These claims are contained in the remarks regarding compound Poisson processes in [14, p. 167] pursuant to the proof of Theorem 6.16 therein. Analytic continuations have been effected in part (iii) using properties of zeros of holomorphic functions [19, p. 209, Theorem 10.18], the theorems of Cauchy, Morera and Fubini, and finally the finiteness/integrability properties of q-potential measures [20, p. 203, Theorem 30.10(ii)]. □

Now, thanks to the skip-free nature of X, we can expand on the contents of Proposition 3.8, by offering further details of the Wiener-Hopf factorization. Indeed, if we let $N_t := \overline{X}_t/h$ and $T_k := T_{kh}$ ($t \geq 0$, $k \in \mathbb{N}_0$) then clearly $T := (T_k)_{k\geq 0}$ are the arrival times of a renewal process (with a possibly defective inter-arrival time distribution) and $N := (N_t)_{t\geq 0}$ is the 'number of arrivals' process. One also has the relation: $\overline{G}^*_t = T_{N_t}$, $t \geq 0$ (P-a.s.). Thus the random variables entering the Wiener-Hopf factorization are determined in terms of the renewal process (T, N).

Moreover, we can proceed to calculate explicitly the Wiener-Hopf factors as well as $\hat{\kappa}$ and κ^*. Let $p > 0$. First, since $\overline{X}_{e_p}/h$ is a geometrically distributed random variable, we have, for any $\beta \in \overline{\mathbb{C}^{\rightarrow}}$:

$$\mathsf{E}[e^{-\beta \overline{X}_{e_p}}] = \sum_{k=0}^{\infty} e^{-\beta hk}(1 - e^{-\Phi(p)h})e^{-\Phi(p)hk} = \frac{1 - e^{-\Phi(p)h}}{1 - e^{-\beta h - \Phi(p)h}}. \tag{3.4}$$

Note here that $\Phi(p) > 0$ for all $p > 0$. On the other hand, using conditioning (for any $\alpha \geq 0$):

$$\begin{aligned}
\mathsf{E}\left[e^{-\alpha \overline{G}^*_{e_p}}\right] &= \mathsf{E}\left[\left((u,t) \mapsto \sum_{k=0}^{\infty} \mathbb{1}_{[0,\infty)}(t_k)e^{-\alpha t_k}\mathbb{1}_{[t_k,t_{k+1})}(u)\right) \circ (e_p, T)\right] \\
&= \mathsf{E}\left[\left(t \mapsto \sum_{k=0}^{\infty} \mathbb{1}_{[0,\infty)}(t_k)e^{-\alpha t_k}(e^{-pt_k} - e^{-pt_{k+1}})\right) \circ T\right], \text{ since } e_p \perp T \\
&= \mathsf{E}\left[\sum_{k=0}^{\infty} \mathbb{1}_{\{T_k<\infty\}}\left(e^{-(p+\alpha)T_k} - e^{-(p+\alpha)T_k}e^{-p(T_{k+1}-T_k)}\right)\right] \\
&= \mathsf{E}\left[\sum_{k=0}^{\infty} e^{-(p+\alpha)T_k}\mathbb{1}_{\{T_k<\infty\}}\left(1 - e^{-p(T_{k+1}-T_k)}\right)\right].
\end{aligned}$$

Now, conditionally on $T_k < \infty$, $T_{k+1} - T_k$ is independent of T_k and has the same distribution as T_1. Therefore, by (3.1) and the theorem of Fubini:

$$\mathsf{E}[e^{-\alpha \overline{G}^*_{e_p}}] = \sum_{k=0}^{\infty} e^{-\Phi(p+\alpha)hk}(1 - e^{-\Phi(p)h}) = \frac{1 - e^{-\Phi(p)h}}{1 - e^{-\Phi(p+\alpha)h}}. \tag{3.5}$$

We identify from (3.4) for any $\beta \in \overline{\mathbb{C}^{\rightarrow}}$: $\frac{\kappa^*(p,0)}{\kappa^*(p,\beta)} = \frac{1-e^{-\Phi(p)h}}{1-e^{-\beta h-\Phi(p)h}}$ and therefore for any $\alpha \geq 0$: $\frac{\kappa^*(p+\alpha,0)}{\kappa^*(p+\alpha,\beta)} = \frac{1-e^{-\Phi(p+\alpha)h}}{1-e^{-\beta h-\Phi(p+\alpha)h}}$. We identify from (3.5) for any $\alpha \geq 0$: $\frac{\kappa^*(p,0)}{\kappa^*(p+\alpha,0)} = \frac{1-e^{-h\Phi(p)}}{1-e^{-\Phi(p+\alpha)h}}$. Therefore, multiplying the last two equalities, for $\alpha \geq 0$ and $\beta \in \overline{\mathbb{C}^{\rightarrow}}$, the equality:

$$\frac{\kappa^*(p,0)}{\kappa^*(p+\alpha,\beta)} = \frac{1 - e^{-\Phi(p)h}}{1 - e^{-\beta h - \Phi(p+\alpha)h}} \tag{3.6}$$

obtains. In particular, for $\alpha > 0$ and $\beta \in \overline{\mathbb{C}^{\rightarrow}}$, we recognize for some constant $k^* \in (0,\infty)$: $\kappa^*(\alpha,\beta) = k^*(1 - e^{-(\beta+\Phi(\alpha))h})$. Next, observe that by independence and duality (for $\alpha \geq 0$ and $\theta \in \mathbb{R}$):

$$\mathsf{E}[\exp\{-\alpha \overline{G}^*_{e_p} + i\theta \overline{X}_{e_p}\}]\mathsf{E}[\exp\{-\alpha \underline{G}_{e_p} + i\theta \underline{X}_{e_p}\}] = \int_0^{\infty} dtpe^{-pt}\mathsf{E}[\exp\{-\alpha t + i\theta X_t\}] = \int_0^{\infty} dtpe^{-pt-\alpha t+\Psi(\theta)t} = \frac{p}{p + \alpha - \Psi(\theta)}.$$

Therefore:

$$(p + \alpha - \psi(i\theta))\frac{\hat{\kappa}(p,0)}{\hat{\kappa}(p+\alpha,i\theta)} = p\frac{1 - e^{i\theta h - \Phi(p+\alpha)h}}{1 - e^{-\Phi(p)h}}.$$

Both sides of this equality are continuous in $\theta \in \overline{\mathbb{C}^{\downarrow}}$ and analytic in $\theta \in \mathbb{C}^{\downarrow}$. They agree on $\mathbb{R}$, hence agree on $\overline{\mathbb{C}^{\downarrow}}$ by analytic continuation. Therefore (for all $\alpha \geq 0$, $\beta \in \overline{\mathbb{C}^{\rightarrow}}$):

$$(p+\alpha-\psi(\beta))\frac{\hat{\kappa}(p,0)}{\hat{\kappa}(p+\alpha,\beta)} = p\frac{1-e^{\beta h-\Phi(p+\alpha)h}}{1-e^{-\Phi(p)h}}, \tag{3.7}$$

i.e. for all $\beta \in \overline{\mathbb{C}^{\rightarrow}}$ and $\alpha \geq 0$ for which $p+\alpha \neq \psi(\beta)$ one has:

$$\mathsf{E}[\exp\{-\alpha \underline{G}_{e_p} + \beta \underline{X}_{e_p}\}] = \frac{p}{p+\alpha-\psi(\beta)}\frac{1-e^{(\beta-\Phi(p+\alpha))h}}{1-e^{-\Phi(p)h}}.$$

Moreover, for the unique $\beta_0 > 0$, for which $\psi(\beta_0) = p+\alpha$, one can take the limit $\beta \to \beta_0$ in the above to obtain: $\mathsf{E}[\exp\{-\alpha \underline{G}_{e_p} + \beta_0 \underline{X}_{e_p}\}] = \frac{ph}{\psi'(\beta_0)(1-e^{-\Phi(p)h})} = \frac{ph\Phi'(p+\alpha)}{1-e^{-\Phi(p)h}}$. We also recognize from (3.7) for $\alpha > 0$ and $\beta \in \overline{\mathbb{C}^{\rightarrow}}$ with $\alpha \neq \psi(\beta)$, and some constant $\hat{k} \in (0,\infty)$: $\hat{\kappa}(\alpha,\beta) = \hat{k}\frac{\alpha-\psi(\beta)}{1-e^{(\beta-\Phi(\alpha))h}}$. With $\beta_0 = \Phi(\alpha)$ one can take the limit in the latter as $\beta \to \beta_0$ to obtain: $\hat{\kappa}(\alpha,\beta_0) = \hat{k}\psi'(\beta_0)/h = \frac{\hat{k}}{h\Phi'(\alpha)}$.

In summary:

Theorem 3.9 (Wiener-Hopf factorization for upwards skip-free Lévy chains)**.** *We have the following identities in terms of ψ and Φ:*

(i) For every $\alpha \geq 0$ and $\beta \in \overline{\mathbb{C}^{\rightarrow}}$:

$$\mathsf{E}[\exp\{-\alpha \overline{G}^*_{e_p} - \beta \overline{X}_{e_p}\}] = \frac{1-e^{-\Phi(p)h}}{1-e^{-(\beta+\Phi(p+\alpha))h}}$$

and

$$\mathsf{E}[\exp\{-\alpha \underline{G}_{e_p} + \beta \underline{X}_{e_p}\}] = \frac{p}{p+\alpha-\psi(\beta)}\frac{1-e^{(\beta-\Phi(p+\alpha))h}}{1-e^{-\Phi(p)h}}$$

(the latter whenever $p+\alpha \neq \psi(\beta)$; for the unique $\beta_0 > 0$ such that $\psi(\beta_0) = p+\alpha$, i.e. for $\beta_0 = \Phi(p+\alpha)$, one has the right-hand side given by $\frac{ph}{\psi'(\beta_0)(1-e^{-\Phi(p)h})} = \frac{ph\Phi'(p+\alpha)}{1-e^{-\Phi(p)h}}$).

(ii) For some $\{k^*, \hat{k}\} \subset \mathbb{R}^+$ and then for every $\alpha > 0$ and $\beta \in \overline{\mathbb{C}^{\rightarrow}}$:

$$\kappa^*(\alpha,\beta) = k^*(1-e^{-(\beta+\Phi(\alpha))h})$$

and

$$\hat{\kappa}(\alpha,\beta) = \hat{k}\frac{\alpha-\psi(\beta)}{1-e^{(\beta-\Phi(\alpha))h}}$$

(the latter whenever $\alpha \neq \psi(\beta)$; for the unique $\beta_0 > 0$ such that $\psi(\beta_0) = \alpha$, i.e. for $\beta_0 = \Phi(\alpha)$, one has the right-hand side given by $\hat{k}\psi'(\beta_0)/h = \frac{\hat{k}}{h\Phi'(\alpha)}$).

As a consequence of Theorem 3.9(i), we obtain the formula for the Laplace transform of the running infimum evaluated at an independent exponentially distributed random time:

$$\mathsf{E}[e^{\beta \underline{X}_{e_p}}] = \frac{p}{p-\psi(\beta)}\frac{1-e^{(\beta-\Phi(p))h}}{1-e^{-\Phi(p)h}} \quad (\beta \in \mathbb{R}_+\backslash\{\Phi(p)\}) \tag{3.8}$$

(and $\mathsf{E}[e^{\Phi(p)\underline{X}_{e_p}}] = \frac{p\Phi'(p)h}{1-e^{-\Phi(p)h}}$). In particular, if $\psi'(0+) > 0$, then letting $p \downarrow 0$ in (3.8), one obtains by the DCT:

$$\mathsf{E}[e^{\beta \underline{X}_{\infty}}] = \frac{e^{\beta h}-1}{\Phi'(0+)h\psi(\beta)} \quad (\beta > 0). \tag{3.9}$$

4. Theory of scale functions

Again the reader is invited to compare the exposition of the following section with that of [3, Section VII.2] and [14, Section 8.2], which deal with the spectrally negative case.

4.1. **The scale function W.** It will be convenient to consider in this subsection the times at which X attains a new maximum. We let D_1, D_2 and so on, denote the depths (possibly zero, or infinity) of the excursions below these new maxima. For $k \in \mathbb{N}$, it is agreed that $D_k = +\infty$ if the process X never reaches the level $(k-1)h$. Then it is clear that for $y \in \mathbb{Z}_h^+$, $x \geq 0$ (cf. [8, p. 137, Paragraph 6.2.4(a)] [10, Section 9.3]):

$$\mathsf{P}(\underline{X}_{T_y} \geq -x) = \mathsf{P}(D_1 \leq x, D_2 \leq x+h, \ldots, D_{y/h} \leq x+y-h) =$$

$$\mathsf{P}(D_1 \leq x) \cdot \mathsf{P}(D_1 \leq x+h) \cdots \mathsf{P}(D_1 \leq x+y-h) = \frac{\prod_{r=1}^{\lfloor (y+x)/h \rfloor} \mathsf{P}(D_1 \leq (r-1)h)}{\prod_{r=1}^{\lfloor x/h \rfloor h} \mathsf{P}(D_1 \leq (r-1)h)} = \frac{W(x)}{W(x+y)},$$

where we have introduced (up to a multiplicative constant) the *scale function*:

$$W(x) := 1/\prod_{r=1}^{\lfloor x/h \rfloor} \mathsf{P}(D_1 \leq (r-1)h) \quad (x \geq 0). \tag{4.1}$$

(When convenient, we extend W by 0 on $(-\infty, 0)$.)

Remark 4.1. If needed, we can of course express $\mathsf{P}(D_1 \leq hk)$, $k \in \mathbb{N}_0$, in terms of the usual excursions away from the maximum. Thus, let $\tilde{D}_1$ be the depth of the first excursion away from the current maximum. By the time the process attains a new maximum (that is to say h), conditionally on this event, it will make a total of N departures away from the maximum, where (with J_1 the first jump time of X, $p := \lambda(\{h\})/\lambda(\mathbb{R})$, $\tilde{p} := \mathsf{P}(X_{J_1} = h | T_h < \infty) = p/\mathsf{P}(T_h < \infty)$) $N \sim \mathrm{geom}(\tilde{p})$. So, denoting $\tilde{\theta}_k := \mathsf{P}(\tilde{D}_1 \leq hk)$, one has $\mathsf{P}(D_1 \leq hk) = \mathsf{P}(T_h < \infty) \sum_{l=0}^{\infty} \tilde{p}(1-\tilde{p})^l \tilde{\theta}_k^l = \frac{p}{1-(1-e^{\Phi(0)h}p)\tilde{\theta}_k}$, $k \in \mathbb{N}_0$.

The following theorem characterizes the scale function in terms of its Laplace transform.

Theorem 4.2 (The scale function)**.** *For every $y \in \mathbb{Z}_h^+$ and $x \geq 0$ one has:*

$$\mathsf{P}(\underline{X}_{T_y} \geq -x) = \frac{W(x)}{W(x+y)} \tag{4.2}$$

and $W : [0, \infty) \to [0, \infty)$ is (up to a multiplicative constant) the unique right-continuous and piecewise continuous function of exponential order with Laplace transform:

$$\hat{W}(\beta) = \int_0^\infty e^{-\beta x} W(x) dx = \frac{e^{\beta h} - 1}{\beta h \psi(\beta)} \quad (\beta > \Phi(0)). \tag{4.3}$$

Proof. (For uniqueness see e.g. [11, p. 14, Theorem 10]. It is clear that W is of exponential order, simply from the definition (4.1).)

Suppose first X tends to $+\infty$. Then, letting $y\to\infty$ in (4.2) above, we obtain $\mathsf{P}(-\underline{X}_\infty\leq x)=W(x)/W(+\infty)$. Here, since the left-hand side limit exists by the DCT, is finite and non-zero at least for all large enough x, so does the right-hand side, and $W(+\infty)\in(0,\infty)$.

Therefore $W(x)=W(+\infty)\mathsf{P}(-\underline{X}_\infty\leq x)$ and hence the Laplace-Stieltjes transform of W is given by (3.9) — here we consider W as being extended by 0 on $(-\infty,0)$:

$$\int_{[0,\infty)}e^{-\beta x}dW(x)=W(+\infty)\frac{e^{\beta h}-1}{\Phi'(0+)h\psi(\beta)}\quad(\beta>0).$$

Since (integration by parts [18, Chapter 0, Proposition 4.5]) $\int_{[0,\infty)}e^{-\beta x}dW(x)=\beta\int_{(0,\infty)}e^{-\beta x}W(x)dx$,

$$\int_0^\infty e^{-\beta x}W(x)dx=\frac{W(+\infty)}{\Phi'(0+)}\frac{e^{\beta h}-1}{\beta h\psi(\beta)}\quad(\beta>0).\tag{4.4}$$

Suppose now that X oscillates. Via Remark 3.4, approximate X by the processes X^ϵ, $\epsilon>0$. In (4.4), fix β, carry over everything except for $\frac{W(+\infty)}{\Phi'(0+)}$, divide both sides by $W(0)$, and then apply this equality to X^ϵ. Then on the left-hand side, the quantities pertaining to X^ϵ will converge to the ones for the process X as $\epsilon\downarrow 0$ by the MCT. Indeed, for $y\in\mathbb{Z}_h^+$, $\mathsf{P}(\underline{X}_{T_y}=0)=W(0)/W(y)$ and (in the obvious notation): $1/\mathsf{P}(\underline{X^\epsilon}_{T_y^\epsilon}=0)\uparrow 1/\mathsf{P}(\underline{X}_{T_y}=0)=W(y)/W(0)$, since $X^\epsilon\downarrow X$, uniformly on bounded time sets, almost surely as $\epsilon\downarrow 0$. (It is enough to have convergence for $y\in\mathbb{Z}_h^+$, as this implies convergence for all $y\geq 0$, W being the right-continuous piecewise constant extension of $W|_{\mathbb{Z}_h^+}$.) Thus we obtain in the oscillating case, for some $\alpha\in(0,\infty)$ which is the limit of the right-hand side as $\epsilon\downarrow 0$:

$$\int_0^\infty e^{-\beta x}W(x)dx=\alpha\frac{e^{\beta h}-1}{\beta h\psi(\beta)}\quad(\beta>0).\tag{4.5}$$

Finally, we are left with the case when X drifts to $-\infty$. We treat this case by a change of measure (see Proposition 3.5 and the paragraph immediately preceding it). To this end assume, provisionally, that X is already the coordinate process on the canonical filtered space $\mathbb{D}_h$. Then we calculate by Proposition 3.6(2) (for $y\in\mathbb{Z}_h^+$, $x\geq 0$):

$$\mathsf{P}(\underline{X}_{T_y}\geq -x)=\mathsf{P}(T_y<\infty)\mathsf{P}(\underline{X}_{T_y}\geq -x|T_y<\infty)=e^{-\Phi(0)y}\mathsf{P}(\underline{X^{T_y}}_\infty\geq -x|T_y<\infty)=$$
$$e^{-\Phi(0)y}\mathsf{P}^\natural(\underline{X^{T_y}}_\infty\geq -x)=e^{-\Phi(0)y}\mathsf{P}^\natural(\underline{X}_{T(y)}\geq -x)=e^{-\Phi(0)y}W^\natural(x)/W^\natural(x+y),$$

where the third equality uses the fact that $(\omega\mapsto\inf\{\omega(s):s\in[0,\infty)\}):(\mathbb{D}_h,\mathcal{F})\to([-\infty,\infty),\mathcal{B}([-\infty,\infty))$ is a measurable transformation. Here $W^\natural$ is the scale function corresponding to X under the measure $\mathsf{P}^\natural$, with Laplace transform:

$$\int_0^\infty e^{-\beta x}W^\natural(x)dx=\frac{e^{\beta h}-1}{\beta h\psi(\Phi(0)+\beta)}\quad(\beta>0).$$

Note that the equality $\mathsf{P}(\underline{X}_{T_y}\geq -x)=e^{-\Phi(0)y}W^\natural(x)/W^\natural(x+y)$ remains true if we revert back to our original X (no longer assumed to be in its canonical guise). This is so because we can always go from X to its canonical counter-part by taking an image measure. Then the law of the process, hence the Laplace exponent and the probability $\mathsf{P}(\underline{X}_{T_y} \geq -x)$ do not change in this transformation.

Now define $\tilde{W}(x) := e^{\Phi(0)\lfloor 1+x/h \rfloor h} W^\natural(x)$ ($x \geq 0$). Then $\tilde{W}$ is the right-continuous piecewise-constant extension of $\tilde{W}|_{\mathbb{Z}_h^+}$. Moreover, for all $y \in \mathbb{Z}_h^+$ and $x \geq 0$, (4.2) obtains with W replaced by $\tilde{W}$. Plugging in $x = 0$ into (4.2), $\tilde{W}|_{\mathbb{Z}_h}$ and $W|_{\mathbb{Z}_h}$ coincide up to a multiplicative constant, hence $\tilde{W}$ and W do as well. Moreover, for all $\beta > \Phi(0)$, by the MCT:

$$
\begin{aligned}
\int_0^\infty e^{-\beta x}\tilde{W}(x)dx &= e^{\Phi(0)h}\sum_{k=0}^\infty \int_{kh}^{(k+1)h} e^{-\beta x}e^{\Phi(0)kh}W^\natural(kh)dx \\
&= e^{\Phi(0)h}\sum_{k=0}^\infty \frac{1}{\beta}e^{-\beta kh}(1-e^{-\beta h})e^{\Phi(0)kh}W^\natural(kh) \\
&= e^{\Phi(0)h}\frac{\beta-\Phi(0)}{\beta}\frac{1-e^{-\beta h}}{1-e^{-(\beta-\Phi(0))h}}\int_0^\infty e^{-(\beta-\Phi(0))x}W^\natural(x)dx \\
&= e^{\Phi(0)h}\frac{\beta-\Phi(0)}{\beta}\frac{1-e^{-\beta h}}{1-e^{-(\beta-\Phi(0))h}}\frac{e^{(\beta-\Phi(0))h}-1}{(\beta-\Phi(0))h\psi(\beta)} = \frac{(e^{\beta h}-1)}{\beta h\psi(\beta)}.
\end{aligned}
$$

□

Remark 4.3. Henceforth the normalization of the scale function W will be understood so as to enforce the validity of (4.3).

Proposition 4.4. *$W(0) = 1/(h\lambda(\{h\}))$, and $W(+\infty) = 1/\psi'(0+)$ if $\Phi(0) = 0$. If $\Phi(0) > 0$, then $W(+\infty) = +\infty$.*

Proof. Integration by parts and the DCT yield $W(0) = \lim_{\beta\to\infty}\beta\hat{W}(\beta)$. (4.3) and another application of the DCT then show that $W(0) = 1/(h\lambda(\{h\}))$. Similarly, integration by parts and the MCT give the identity $W(+\infty) = \lim_{\beta\downarrow 0}\beta\hat{W}(\beta)$. The conclusion $W(+\infty) = 1/\psi'(0+)$ is then immediate from (4.3) when $\Phi(0) = 0$. If $\Phi(0) > 0$, then the right-hand side of (4.3) tends to infinity as $\beta \downarrow \Phi(0)$ and thus, by the MCT, necessarily $W(+\infty) = +\infty$. □

4.2. **The scale functions $W^{(q)}$, $q \geq 0$.**

Definition 4.5. For $q \geq 0$, let $W^{(q)}(x) := e^{\Phi(q)\lfloor 1+x/h \rfloor h}W_{\Phi(q)}(x)$ ($x \geq 0$), where W_c plays the role of W but for the process (X, P_c) ($c \geq 0$; see Proposition 3.5). Note that $W^{(0)} = W$. When convenient we extend $W^{(q)}$ by 0 on $(-\infty, 0)$.

Theorem 4.6. *For each $q \geq 0$, $W^{(q)} : [0,\infty) \to [0,\infty)$ is the unique right-continuous and piecewise continuous function of exponential order with Laplace transform:*

$$\widehat{W^{(q)}}(\beta) = \int_0^\infty e^{-\beta x}W^{(q)}(x)dx = \frac{e^{\beta h}-1}{\beta h(\psi(\beta)-q)} \quad (\beta > \Phi(q)). \tag{4.6}$$

Moreover, for all $y \in \mathbb{Z}_h^+$ and $x \geq 0$:

$$\mathsf{E}[e^{-qT_y}\mathbb{1}_{\{\underline{X}_{T_y}\geq -x\}}] = \frac{W^{(q)}(x)}{W^{(q)}(x+y)}. \tag{4.7}$$

Proof. The claim regarding the Laplace transform follows from Proposition 3.5, Theorem 4.2 and Definition 4.5 as it did in the case of the scale function W (cf. final paragraph of the proof of Theorem 4.2). For the second assertion, let us calculate (moving onto the canonical space $\mathbb{D}_h$ as usual, using Proposition 3.5 and noting that $X_{T_y} = y$ on $\{T_y < \infty\}$):

$$\mathsf{E}[e^{-qT_y}\mathbb{1}_{\{\underline{X}_{T_y}\geq -x\}}] = \mathsf{E}[e^{\Phi(q)X_{T_y}-qT_y}\mathbb{1}_{\{\underline{X}_{T_y}\geq -x\}}]e^{-\Phi(q)y} =$$

$$e^{-\Phi(q)y}\mathsf{P}_{\Phi(q)}(\underline{X}_{T_y} \geq -x) = e^{-\Phi(q)y}\frac{W_{\Phi(q)}(x)}{W_{\Phi(q)}(x+y)} = \frac{W^{(q)}(x)}{W^{(q)}(x+y)}.$$

□

Proposition 4.7. *For all $q > 0$: $W^{(q)}(0) = 1/(h\lambda(\{h\}))$ and $W^{(q)}(+\infty) = +\infty$.*

Proof. As in Proposition 4.4, $W^{(q)}(0) = \lim_{\beta\to\infty}\beta\widehat{W^{(q)}}(\beta) = 1/(h\lambda(\{h\}))$. Since $\Phi(q) > 0$, $W^{(q)}(+\infty) = +\infty$ also follows at once from the expression for $\widehat{W^{(q)}}$. □

Moreover:

Proposition 4.8. *For $q \geq 0$:*

(i) If $\Phi(q) > 0$ or $\psi'(0+) > 0$, then $\lim_{x\to\infty} W^{(q)}(x)e^{-\Phi(q)\lfloor 1+x/h\rfloor h} = 1/\psi'(\Phi(q))$.
(ii) If $\Phi(q) = \psi'(0+) = 0$ (hence $q = 0$), then $W^{(q)}(+\infty) = +\infty$, but $\limsup_{x\to\infty} W^{(q)}(x)/x < \infty$. Indeed, $\lim_{x\to\infty} W^{(q)}(x)/x = 2/m_2$, if $m_2 := \int y^2\lambda(dy) < \infty$ and $\lim_{x\to\infty} W^{(q)}(x)/x = 0$, if $m_2 = \infty$.

Proof. The first claim is immediate from Proposition 4.4, Definition 4.5 and Proposition 3.5. To handle the second claim, let us calculate, for the Laplace transform $\widehat{dW}$ of the measure dW, the quantity (using integration by parts, Theorem 4.2 and the fact that (since $\psi'(0+) = 0$) $\int y\lambda(dy) = 0$):

$$\lim_{\beta\downarrow 0}\beta\widehat{dW}(\beta) = \lim_{\beta\downarrow 0}\frac{\beta^2}{\psi(\beta)} = \frac{2}{m_2} \in [0, +\infty).$$

For:

$$\lim_{\beta\downarrow 0}\int(e^{\beta y} - 1)\lambda(dy)/\beta^2 = \lim_{\beta\downarrow 0}\int\frac{e^{\beta y} - \beta y - 1}{\beta^2 y^2}y^2\lambda(dy) = \frac{m_2}{2},$$

by the MCT, since $(u \mapsto \frac{e^{-u}+u-1}{u^2})$ is nonincreasing on $(0, \infty)$ (the latter can be checked by comparing derivatives). The claim then follows by the Karamata Tauberian Theorem [6, p. 37, Theorem 1.7.1 with $\rho = 1$]. □

4.3. **The functions $Z^{(q)}$, $q \geq 0$.**

Definition 4.9. For each $q \geq 0$, let $Z^{(q)}(x) := 1 + q\int_0^{\lfloor x/h\rfloor h} W^{(q)}(z)dz$ $(x \geq 0)$. When convenient we extend these functions by 1 on $(-\infty, 0)$.

Definition 4.10. For $x \geq 0$, let $T_x^- := \inf\{t \geq 0 : X_t < -x\}$.

Proposition 4.11. *In the sense of measures on the real line, for every $q > 0$:*

$$\mathsf{P}_{-\underline{X}_{e_q}} = \frac{qh}{e^{\Phi(q)h}-1}dW^{(q)} - qW^{(q)}(\cdot - h)\cdot\Delta,$$

where $\Delta := h\sum_{k=1}^{\infty}\delta_{kh}$ is the normalized counting measure on $\mathbb{Z}_h^{++}\subset\mathbb{R}$, $\mathsf{P}_{-\underline{X}_{e_q}}$ is the law of $-\underline{X}_{e_q}$ under P, and $(W^{(q)}(\cdot - h)\cdot\Delta)(A) = \int_A W^{(q)}(y-h)\Delta(dy)$ for Borel subsets A of $\mathbb{R}$.

Theorem 4.12. *For each $x \geq 0$,*

$$\mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_x^-<\infty\}}] = Z^{(q)}(x) - \frac{qh}{e^{\Phi(q)h}-1}W^{(q)}(x) \tag{4.8}$$

when $q > 0$, and $\mathsf{P}(T_x^- < \infty) = 1 - W(x)/W(+\infty)$. The Laplace transform of $Z^{(q)}$, $q \geq 0$, is given by:

$$\widehat{Z^{(q)}}(\beta) = \int_0^\infty Z^{(q)}(x)e^{-\beta x}dx = \frac{1}{\beta}\left(1+\frac{q}{\psi(\beta)-q}\right), \quad (\beta > \Phi(q)). \tag{4.9}$$

Proof of Proposition 4.11 and Theorem 4.12. First, with regard to the Laplace transform of $Z^{(q)}$, we have the following derivation (using integration by parts, for every $\beta > \Phi(q)$):

$$\begin{aligned}
\int_0^\infty Z^{(q)}(x)e^{-\beta x}dx &= \int_0^\infty \frac{e^{-\beta x}}{\beta}dZ^{(q)}(x) = \frac{1}{\beta}\left(1+q\sum_{k=1}^{\infty}e^{-\beta kh}W^{(q)}((k-1)h)h\right)\\
&= \frac{1}{\beta}\left(1+\frac{qe^{-\beta h}\beta h}{1-e^{-\beta h}}\sum_{k=1}^{\infty}\frac{(1-e^{-\beta h})}{\beta}e^{-\beta(k-1)h}W^{(q)}((k-1)h)\right)\\
&= \frac{1}{\beta}\left(1+q\frac{\beta h}{e^{\beta h}-1}\widehat{W^{(q)}}(\beta)\right) = \frac{1}{\beta}\left(1+\frac{q}{\psi(\beta)-q}\right).
\end{aligned}$$

Next, to prove Proposition 4.11, note that it will be sufficient to check the equality of the Laplace transforms [4, p. 109, Theorem 8.4]. By what we have just shown, (3.8), integration by parts, and Theorem 4.6, we need then only establish, for $\beta > \Phi(q)$:

$$\frac{q}{\psi(\beta)-q}\frac{e^{(\beta-\Phi(q))h}-1}{1-e^{-\Phi(q)h}} = \frac{qh}{e^{\Phi(q)h}-1}\frac{\beta(e^{\beta h}-1)}{(\psi(\beta)-q)\beta h} - \frac{q}{\psi(\beta)-q},$$

which is clear.

Finally, let $x \in \mathbb{Z}_h^+$. For $q > 0$, evaluate the measures in Proposition 4.11 at $[0, x]$, to obtain:

$$\begin{aligned}
\mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_x^-<\infty\}}] &= \mathsf{P}(e_q \geq T_x^-) = \mathsf{P}(\underline{X}_{e_q} < -x) = 1 - \mathsf{P}(\underline{X}_{e_q} \geq -x)\\
&= 1 + q\int_0^x W^{(q)}(z)dz - \frac{qh}{e^{\Phi(q)h}-1}W^{(q)}(x),
\end{aligned}$$

whence the claim follows. On the other hand, when $q = 0$, the following calculation is straightforward: $\mathsf{P}(T_x^- < \infty) = \mathsf{P}(\underline{X}_\infty < -x) = 1 - \mathsf{P}(\underline{X}_\infty \geq -x) = 1 - W(x)/W(+\infty)$ (we have passed to the limit $y \to \infty$ in (4.2) and used the DCT on the left-hand side of this equality). $\square$

Proposition 4.13. *Let $q \geq 0$, $x \geq 0$, $y \in \mathbb{Z}_h^+$. Then:*

$$\mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_x^-<T_y\}}] = Z^{(q)}(x) - Z^{(q)}(x+y)\frac{W^{(q)}(x)}{W^{(q)}(x+y)}.$$

Proof. Observe that $\{T_x^- = T_y\} = \emptyset$, P-a.s. The case when $q = 0$ is immediate and indeed contained in Theorem 4.2, since, P-a.s., $\Omega\backslash\{T_x^- < T_y\} = \{T_x^- \geq T_y\} = \{\underline{X}_{T_y} \geq -x\}$. For $q > 0$ we observe that by the strong Markov property, Theorem 4.6 and Theorem 4.12:

$$\begin{aligned}
&\mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_x^-<T_y\}}] = \mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_x^-<\infty\}}] - \mathsf{E}[e^{-qT_x^-}\mathbb{1}_{\{T_y<T_x^-<\infty\}}] \\
= \;& Z^{(q)}(x) - \frac{qh}{e^{\Phi(q)h}-1}W^{(q)}(x) - \mathsf{E}[e^{-qT_y}\mathbb{1}_{\{T_y<T_x^-\}}]\mathsf{E}[e^{-qT_{x+y}^-}\mathbb{1}_{\{T_{x+y}^-<\infty\}}] \\
= \;& Z^{(q)}(x) - \frac{qh}{e^{\Phi(q)h}-1}W^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(x+y)}\left(Z^{(q)}(x+y) - \frac{qh}{e^{\Phi(q)h}-1}W^{(q)}(x+y)\right) \\
= \;& Z^{(q)}(x) - Z^{(q)}(x+y)\frac{W^{(q)}(x)}{W^{(q)}(x+y)}.
\end{aligned}$$

□

4.4. **Calculating scale functions.** In this subsection it will be assumed for notational convenience, but without loss of generality, that $h = 1$ and that X is the canonical process on $\Omega = \mathbb{D}_h$ equipped with the usual σ-algebra and filtration. We define:

$$\gamma := \lambda(\mathbb{R}), \quad p := \lambda(\{1\})/\gamma, \quad q_k := \lambda(\{-k\})/\gamma,\ k \geq 1.$$

Fix $q \geq 0$. Then denote, provisionally, $e_{m,k} := \mathsf{E}[e^{-qT_k}\mathbb{1}_{\{\underline{X}_{T_k}\geq -m\}}]$, and $e_k := e_{0,k}$, where $\{m,k\} \subset \mathbb{N}_0$ and note that, thanks to Theorem 4.6, $e_{m,k} = \frac{e_{m+k}}{e_m}$ for all $\{m,k\} \subset \mathbb{N}_0$. Now, $e_0 = 1$. Moreover, by the strong Markov property, for each $k \in \mathbb{N}_0$, by conditioning on $\mathcal{F}_{T_k}$ and then on $\mathcal{F}_J$, where J is the time of the first jump after T_k (so that, conditionally on $T_k < \infty$, $J - T_k \sim \mathrm{Exp}(\gamma)$):

$$\begin{aligned}
e_{k+1} = \;& \mathsf{E}\Big[e^{-qT_k}\mathbb{1}_{\{\underline{X}_{T_k}\geq 0\}}e^{-q(J-T_k)}\big(\mathbb{1}(\text{next jump after } T_k \text{ up}) + \\
& \mathbb{1}(\text{next jump after } T_k \text{ 1 down, then up 2 before down more than } k-1) + \cdots + \\
& \mathbb{1}(\text{next jump after } T_k\ k \text{ down \& then up } k+1 \text{ before down more than } 0)\big)e^{-q(T_{k+1}-J)}\Big] \\
= \;& e_k\frac{\gamma}{\gamma+q}[p + q_1e_{k-1,2} + \cdots + q_ke_{0,k+1}] = e_k\frac{\gamma}{\gamma+q}[p + q_1\frac{e_{k+1}}{e_{k-1}} + \cdots + q_k\frac{e_{k+1}}{e_0}].
\end{aligned}$$

Upon division by e_ke_{k+1}, we obtain:

$$W^{(q)}(k) = \frac{\gamma}{\gamma+q}[pW^{(q)}(k+1) + q_1W^{(q)}(k-1) + \cdots + q_kW^{(q)}(0)].$$

Put another way, for all $k \in \mathbb{Z}_+$:

$$pW^{(q)}(k+1) = \left(1 + \frac{q}{\gamma}\right)W^{(q)}(k) - \sum_{l=1}^{k} q_lW^{(q)}(k-l). \tag{4.10}$$

Coupled with the initial condition $W^{(q)}(0) = 1/(\gamma p)$ (from Proposition 4.7 and Proposition 4.4), this is an explicit recursion scheme by which the values of $W^{(q)}$ obtain (cf. [23, Section 4, Equations (6) & (7)] [9, Section 7, Equations (7.1) & (7.5)]). We can also see the vector $W^{(q)} = (W^{(q)}(k))_{k\in\mathbb{Z}}$ as a suitable eigenvector of the transition matrix P associated to the jump chain of X. Namely, we have for all $k \in \mathbb{Z}_+$: $\left(1 + \frac{q}{\gamma}\right)W^{(q)}(k) = \sum_{l\in\mathbb{Z}} P_{kl}W^{(q)}(l)$.

Now, with regard to the function $Z^{(q)}$, its values can be computed directly from the values of $W^{(q)}$ by a straightforward summation, $Z^{(q)}(n) = 1+q\sum_{k=0}^{n-1} W^{(q)}(k)$ $(n \in \mathbb{N}_0)$. Alternatively, (4.10) yields immediately its analogue, valid for each $n \in \mathbb{Z}^+$ (make a summation $\sum_{k=0}^{n-1}$ and multiply by q, using Fubini's theorem for the last sum):

$$pZ^{(q)}(n+1) - p - pqW^{(q)}(0) = \left(1+\frac{q}{\gamma}\right)(Z^{(q)}(n)-1) - \sum_{l=1}^{n-1} q_l(Z^{(q)}(n-l)-1),$$

i.e. for all $k \in \mathbb{Z}_+$:

$$pZ^{(q)}(k+1) + \left(1-p-\sum_{l=1}^{k-1} q_l\right) = \left(1+\frac{q}{\gamma}\right)Z^{(q)}(k) - \sum_{l=1}^{k-1} q_l Z^{(q)}(k-l). \tag{4.11}$$

Again this can be seen as an eigenvalue problem. Namely, for all $k \in \mathbb{Z}_+$: $\left(1+\frac{q}{\gamma}\right)Z^{(q)}(k) = \sum_{l\in\mathbb{Z}} P_{kl} Z^{(q)}(l)$. In summary:

Proposition 4.14 (Calculation of $W^{(q)}$ and $Z^{(q)}$)**.** *Let $h = 1$ and $q \geq 0$. Seen as vectors, $W^{(q)} := (W^{(q)}(k))_{k\in\mathbb{Z}}$ and $Z^{(q)} := (Z^{(q)}(k))_{k\in\mathbb{Z}}$ satisfy, entry-by-entry (P being the transition matrix associated to the jump chain of X; $\lambda_q := 1 + q/\lambda(\mathbb{R})$):*

$$(PW^{(q)})|_{\mathbb{Z}_+} = \lambda_q W^{(q)}|_{\mathbb{Z}_+} \text{ and } (PZ^{(q)})|_{\mathbb{Z}_+} = \lambda_q Z^{(q)}|_{\mathbb{Z}_+}, \tag{4.12}$$

i.e. (4.10) and (4.11) hold true for $k \in \mathbb{Z}_+$. Additionally, $W^{(q)}|_{\mathbb{Z}^-} = 0$ with $W^{(q)}(0) = 1/\lambda(\{1\})$, whereas $Z^{(q)}|_{\mathbb{Z}_-} = 1$.

An alternative form of recursions (4.10) and (4.11) is as follows:

Corollary 4.15. *We have for all $n \in \mathbb{N} \cup \{0\}$:*

$$W^{(q)}(n+1) = W^{(q)}(0) + \sum_{k=1}^{n+1} W^{(q)}(n+1-k)\frac{q+\lambda(-\infty,-k]}{\lambda(\{1\})}, \quad W^{(q)}(0) = 1/\lambda(\{1\}), \tag{4.13}$$

and for $\widetilde{Z^{(q)}} := Z^{(q)} - 1$,

$$\widetilde{Z^{(q)}}(n+1) = (n+1)\frac{q}{\lambda\{1\}} + \sum_{k=1}^{n} \widetilde{Z^{(q)}}(n+1-k)\frac{q+\lambda(-\infty,-k]}{\lambda(\{1\})}, \quad \widetilde{Z^{(q)}}(0) = 0. \tag{4.14}$$

Proof. Recursion (4.13) obtains from (4.10) as follows (cf. also [1, (proof of) Proposition XVI.1.2]):

$$pW^{(q)}(n+1)+\sum_{k=1}^{n}q_kW^{(q)}(n-k)=\nu_qW^{(q)}(n),\forall n\in\mathbb{N}_0\Rightarrow$$

$$pW^{(q)}(k+1)+\sum_{m=0}^{k-1}q_{k-m}W^{(q)}(m)=\nu_qW^{(q)}(k),\forall k\in\mathbb{N}_0\Rightarrow\text{ (making a summation }\sum_{k=0}^{n})$$

$$p\sum_{k=0}^{n}W^{(q)}(k+1)+\sum_{k=0}^{n}\sum_{m=0}^{k-1}q_{k-m}W^{(q)}(m)=\nu_q\sum_{k=0}^{n}W^{(q)}(k),\forall n\in\mathbb{N}_0\Rightarrow\text{ (Fubini)}$$

$$pW^{(q)}(n+1)+p\sum_{k=0}^{n}W^{(q)}(k)+\sum_{m=0}^{n-1}W^{(q)}(m)\sum_{k=m+1}^{n}q_{k-m}=pW^{(q)}(0)+\nu_q\sum_{k=0}^{n}W^{(q)}(k),\forall n\in\mathbb{N}_0\Rightarrow\text{ (relabeling)}$$

$$pW^{(q)}(n+1)+p\sum_{k=0}^{n}W^{(q)}(k)+\sum_{k=0}^{n-1}W^{(q)}(k)\sum_{l=1}^{n-k}q_l=pW^{(q)}(0)+(1+q/\gamma)\sum_{k=0}^{n}W^{(q)}(k),\forall n\in\mathbb{N}_0\Rightarrow\text{ (rearranging)}$$

$$W^{(q)}(n+1)=W^{(q)}(0)+\sum_{k=0}^{n}W^{(q)}(k)\frac{q+\gamma\sum_{l=n-k+1}^{\infty}q_l}{p\gamma},\forall n\in\mathbb{N}_0\Rightarrow\text{ (relabeling)}$$

$$W^{(q)}(n+1)=W^{(q)}(0)+\sum_{k=1}^{n+1}W^{(q)}(n+1-k)\frac{q+\gamma\sum_{l=k}^{\infty}q_l}{p\gamma},\forall n\in\mathbb{N}_0.$$

Then (4.14) follows from (4.13) by another summation from $n=0$ to $n=w-1$, $w\in\mathbb{N}_0$, say, and an interchange in the order of summation for the final sum. □

For the purposes of the following remark and corollary (cf. [5, Equation (12)] and [2, Remark 5], respectively, for their spectrally negative analogues), it is no longer assumed that $h=1$ or, indeed, that the underlying filtered probability space is the canonical one, i.e. we revert back to our original setting.

Remark 4.16. Let L be the infinitesimal generator [20, p. 208, Theorem 31.5] of X. It is seen from (4.12), that for each $q\geq 0$, $((L-q)W^{(q)})|_{\mathbb{R}_+}=((L-q)Z^{(q)})|_{\mathbb{R}_+}=0$.

Corollary 4.17. *For each $q\geq 0$, the stopped processes Y and Z, defined by $Y_t:=e^{-q(t\wedge T_0^-)}W^{(q)}\circ X_{t\wedge T_0^-}$ and $Z_t:=e^{-q(t\wedge T_0^-)}W^{(q)}\circ X_{t\wedge T_0^-}$, $t\geq 0$, are nonnegative* P*-martingales with respect to the natural filtration $\mathbb{F}^X=(\mathcal{F}^X_s)_{s\geq 0}$ of X.*

Proof. We argue for the case of the process Y, the justification for Z being similar. Let $(H_k)_{k\geq 1}$, $H_0:=0$, be the sequence of jump times of X (where, possibly by discarding a P-negligible set, we may insist on all of the T_k, $k\in\mathbb{N}_0$, being finite and increasing to $+\infty$ as $k\to\infty$). Let $0\leq s<t$, $A\in\mathcal{F}^X_s$. By the MCT it will be sufficient to establish for $\{l,k\}\subset\mathbb{N}_0$, $l\leq k$, that:

$$\mathsf{E}[\mathbb{1}(H_l\leq s<H_{l+1})\mathbb{1}_AY_t\mathbb{1}(H_k\leq t<H_{k+1})]=\mathsf{E}[\mathbb{1}(H_l\leq s<H_{l+1})\mathbb{1}_AY_s\mathbb{1}(H_k\leq t<H_{k+1})].\quad(4.15)$$

On the left-hand (respectively right-hand) side of (4.15) we may now replace Y_t (respectively Y_s) by Y_{H_k} (respectively Y_{H_l}) and then harmlessly insist on $l<k$. Moreover, up to a completion, $\mathcal{F}^X_s\subset\sigma((H_m\wedge s,X(H_m\wedge s))_{m\geq 0})$. Therefore, by a π/λ-argument, we need only verify (4.15) for sets A of the form: $A=\bigcap_{m=1}^M\{H_m\wedge s\in A_m\}\cap\{X(H_m\wedge s)\in B_m\}$, A_m, B_m Borel subsets of $\mathbb{R}$, $1\leq m\leq M$, $M\in\mathbb{N}$. Due to the presence of the indicator $\mathbb{1}(H_l\leq s<H_{l+1})$, we may also take, without loss of generality, $M=l$ and hence $A\in\mathcal{F}^X_{H_l}$. Further, $\mathcal{H}:=\sigma(H_{l+1}-H_l,H_k-H_l,H_{k+1}-H_l)$

is independent of $\mathcal{F}^X_{H_l} \vee \sigma(Y_{H_k})$ and then $\mathsf{E}[Y_{H_k}|\mathcal{F}^X_{H_l} \vee \mathcal{H}] = \mathsf{E}[Y_{H_k}|\mathcal{F}^X_{H_l}] = Y_{H_l}$, P-a.s. (as follows at once from (4.12) of Proposition 4.14), whence (4.15) obtains. $\square$

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