ON THE EXISTENCE OF A MINIMAL GENERATING SET FOR σ-ALGEBRAS

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Abstract. Does there exist for any σ-algebra a minimal (with respect to inclusion) generating set? We formulate this problem and answer it in the very special instance of partition generated and standard measurable spaces, the general case remaining open.

1. Introduction

Beyond serving as domains of measures, σ-fields are also interpreted as representing aggregates of information. It is then natural to consider the existence of a smallest ensemble of pieces of information, which generates the same body of knowledge, in the following precise sense;

Definition 1.1. Let $\mathcal{F}$ be a σ-algebra (on Ω), $\mathcal{C} \subset 2^\Omega$.[1] Then $\mathcal{C}$ is called a, respectively minimal, generating set for $\mathcal{F}$, if $\sigma(\mathcal{C}) = \mathcal{F}$ and, respectively, for all $\mathcal{D} \subset 2^\Omega$, the conjunction $\mathcal{D} \subset \mathcal{C}$ and $\sigma(\mathcal{D}) = \mathcal{F}$ implies $\mathcal{D} = \mathcal{C}$.

Our question: Does every σ-algebra posses a minimal generating set? Note that this is similar to asking for a minimal generating set in vector spaces (which yields a basis); cf. also [2] for the case of groups, rings and fields.

Remark 1.2. Since the property of being a (respectively minimal) generating set remains invariant under σ-isomorphisms (one-to-one and onto mappings between σ-fields, which preserve countable set operations), so the property of possessing a minimal generating set is invariant under σ-isomorphisms. Remark also that every Borel isomorphism (i.e. measurable bijection with a measurable inverse) induces a natural σ-isomorphism.

2. Special cases

The following notion will prove useful:

Definition 2.1. A collection $\mathcal{D}$ of subsets of Ω discerns points $a \neq b$ from Ω, if one can find $A \in \mathcal{D}$ such that neither $\{a, b\} \subset A$ nor $\{a, b\} \subset \Omega \backslash A$. $\mathcal{D}$ is separating, if it discerns any two distinct elements of Ω. We also say $D \subset \Omega$ discerns $a \neq b$ from Ω, if $\{D\}$ does so.

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[1]Throughout 2^Ω will be used to denote the power set of Ω.

Remark 2.2. The property of not discerning a given pair of points is preserved under the σ-operation; i.e. if $\mathcal{D} \subset 2^\Omega$ does not discern a and b, then $\sigma(\mathcal{D})$ does not discern a and b either (trivial; $\mathcal{D} \subset \{B \in 2^\Omega : B$ does not discern a and $b\}$, which is a σ-field on Ω).

Proposition 2.3. *If a σ-field $\mathcal{F}$ on Ω is either (a) generated by a partition $\mathcal{P}$ of Ω or else (b) is standard (i.e. Borel isomorphic to some (equivalently, Borel subset of a) Polish space), then $\mathcal{F}$ admits a minimal generating set.*

Proof. As regards (a), it is assumed without loss of generality that $\mathcal{P}$ is the collection of all the singletons of Ω (from Remark 1.2). Then, if (i) $\mathcal{P} = \emptyset$, then $\mathcal{P}$ itself is minimal generating; and if (ii) $\mathcal{P}$ is non-empty but countable, pick $P \in \mathcal{P}$, in which case $\mathcal{P}\backslash\{P\}$ becomes a minimal generating set (from Remark 2.2, say); whilst, finally, if (iii) $\mathcal{P}$ is uncountable, then $\mathcal{P}$ is again itself a minimal generating set (for if $\mathcal{Q} \subset \mathcal{P}$, but $\mathcal{Q} \neq \mathcal{P}$, then there is a $P \in \mathcal{P}\backslash\mathcal{Q}$, while $\sigma(\mathcal{Q})$ is included in the σ-field of all the countable unions of elements of $\mathcal{Q}$ and their complements, which does not have P for an element, so $\sigma(\mathcal{Q}) \neq \mathcal{F}$).

As regards (b), since by the Isomorphism theorem all uncountable standard measurable space are Borel isomorphic, from what we have just shown in (a), and from Remark 1.2, it remains to verify the claim in the case of $\mathcal{F}$ being the Borel σ-field on $\mathbb{R}$. Furthermore, the latter is certainly Souslin, and hence we know that any separating denumerable family of Borel sets is automatically generating for its Borel σ-field [1, p. 32, Theorem 6.8.9]. The idea is thus to take the "nice" denumerable generating family $\{(r, +\infty) : r \in \mathbb{Q}\}$, not itself minimal generating, and to "tweak" it, so as to make it minimal separating (hence minimal generating).

Specifically, let $\mathcal{C} := \{A_r : r \in \mathbb{Q}\backslash\{0\}\}$, where $A_r := [(r, +\infty)\backslash\mathbb{Q}] \cup \{r\}$ for $r \in \mathbb{Q}\backslash\{0\}$. Clearly $\mathcal{C} \subset \mathcal{B}(\mathbb{R})$ is denumerable. To show that it is separating, let a and b be any two distinct real numbers, assume $a < b$, without loss of generality. If b is irrational, let r be any non-zero rational number with $a < r < b$. Then A_r discerns between a and b, since $a \notin A_r$, but $b \in A_r$. Further, if both a and b are rational, b non-zero, then $b \in A_b$, but $a \notin A_b$; likewise if a is rational and $b = 0$, $a \in A_a$, but $0 \notin A_a$. Finally, suppose a is irrational and b is rational. Let r be any non-zero rational number strictly less than a. Then $a \in A_r$, but $b \notin A_r$.

Thus $\mathcal{C}$ is separating and denumerable, hence generating. It is also minimal in doing so, since if $q \in \mathbb{Q}\backslash\{0\}$, then any subset of $\mathcal{C}\backslash\{A_q\}$ does not discern between 0 and q, and hence cannot generate the Borel σ-field. □

3. General case

The general case remains an open question and indeed the *ad hoc* methods of Section 2 do not really seem to offer any firm guidance as to whether the answer is to the affirmative or the negative. However, the conjecture that a minimal generating set always obtains, seems at least to be motivated by the findings thereof.

References

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