## Exercise Sheet 2, ST213

In the following, $a(\mathcal{C})$ will denote the algebra generated by $\mathcal{C}$ (on the sample space $\Omega$ ), while $\sigma(\mathcal{C})$ will denote the $\sigma$-algebra generated by $\mathcal{C}$ (on the sample space $\Omega$ ).

1) Assume $A_{1}, \ldots, A_{n} \in \mathcal{A}$ where $\mathcal{A}$ is an algebra of subsets of a sample space $\Omega$. Set

$$
B_{i}=A_{i} \backslash\left(A_{1} \cup \ldots \cup A_{i-1}\right), \quad i=1, \ldots, n
$$

a) Only using the properties stated in our definition of an algebra, show that $B_{i} \in \mathcal{A}$ for $i=1, \ldots, n$.
b) Show that $\bigcup_{i=1}^{n} A_{i}=\bigcup_{i=1}^{n} B_{i}$.
2) Let $\mathbf{P}_{n}, n=1,2, \ldots$, be a sequence of contents on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is an algebra of subsets of a sample space $\Omega$. Define $\mathbf{P}: \mathcal{A} \rightarrow[0,+\infty]$ by setting for $A \in \mathcal{A}$ :

$$
\mathbf{P}(A)=\left\{\begin{array}{cl}
\sum_{n=1}^{\infty} \mathbf{P}_{n}(A), & \text { if this series converges; } \\
+\infty, & \text { otherwise }
\end{array}\right.
$$

Show that $\mathbf{P}$ is a content on $(\Omega, \mathcal{A})$. For short, we write $\sum_{n=1}^{\infty} \mathbf{P}_{n}$ for this content.
3) Let $\mathcal{C}$ be a set of subsets of a sample space $\Omega$. Show that $a(\mathcal{C}) \subseteq \sigma(\mathcal{C})$, while $\sigma(a(\mathcal{C}))=\sigma(\mathcal{C})$.
4) Establish that any finite algebra is automatically a $\sigma$-algebra (on a given space $\Omega$ ).
5) Let $(\Omega, \mathcal{F})$ be a measurable space.
a) Show that if $\left\{A_{i}\right\}_{i=1}^{\infty} \subseteq \mathcal{F}$ then $\bigcap_{i=1}^{\infty} A_{i} \in \mathcal{F}$.
b) Show that a measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ is sub-additive, that is $\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_{i}\right) \leq \sum_{i=1}^{\infty} \mathbf{P}\left(A_{i}\right)$ for every $\left\{A_{i}\right\}_{i=1}^{\infty} \subseteq \mathcal{F}$.

## Complements.

(i) Let $\Omega$ be any set. Find an explicit description for $\rho:=\sigma(\{\{\omega\}: \omega \in \Omega\})$. In particular, decide whether $\rho=\mathcal{P}(\Omega)$ depending on whether or not $\Omega$ is denumerable.
(ii) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Show that:

$$
\sigma(\mathcal{A})=\bigcup_{\mathcal{I} \subseteq \mathcal{A}, \mathcal{I} \text { denumerable }} \sigma(\mathcal{I})
$$

(iii) Let $\mathbf{P}$ be a finite content on an algebra $\mathcal{A}$ of subsets of $\Omega$. Prove the inclusion-exclusion (Bonferroni in-) equalities, i.e. show that ${ }^{1}$ :
(a) For any $n \in \mathbb{N}$, any sequence $A_{1}, \ldots, A_{n}$ of members of $\mathcal{A}$, and any $0 \leq k \leq n$ :

$$
\left(\mathbf{P}\left(\bigcup_{i=1}^{n} A_{i}\right)-\sum_{i=1}^{k}(-1)^{i+1} \sum_{1 \leq l_{1}<\cdots<l_{i} \leq n} \mathbf{P}\left(A_{l_{1}} \cap \cdots \cap A_{l_{i}}\right)\right)(-1)^{k} \geq 0
$$

with equality for $k=n$.

[^0](b) Conclude that, if $\Omega$ is finite, then for any $n \in \mathbb{N}, 0 \leq k \leq n$ and subsets $A_{1}, \ldots, A_{n}$ of $\Omega$ :
$$
\left(\left|\bigcup_{i=1}^{n} A_{i}\right|-\sum_{i=1}^{k}(-1)^{i+1} \sum_{1 \leq l_{1}<\cdots<l_{i} \leq n}\left|A_{l_{1}} \cap \cdots \cap A_{l_{i}}\right|\right)(-1)^{k} \geq 0
$$
with equality for $k=n$.


[^0]:    ${ }^{1}$ We always understand $\sum_{\emptyset}:=0$, while $\cup_{\emptyset}:=\emptyset$. In particular, $\sum_{i=1}^{0}:=0$ and $\cup_{i=1}^{0}:=\emptyset$.

