

Exercise Sheet 3, ST213

- 1) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a measure space and $\tilde{\Omega} \subseteq \Omega$ an arbitrary subset. Define

$$\tilde{\mathcal{F}} = \{\tilde{\Omega} \cap A : A \in \mathcal{F}\}.$$

Show that $\tilde{\mathcal{F}}$ is a σ -algebra on $\tilde{\Omega}$. Assume now $\tilde{\Omega} \in \mathcal{F}$. Show that then $\mathcal{P}(\tilde{\Omega}) \cap \mathcal{F} = \tilde{\mathcal{F}} \subseteq \mathcal{F}$ and $\tilde{P}|_{\tilde{\mathcal{F}}}$ is a measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$.

- 2) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}$ be a random variable. Show that the distribution function F_X is right-continuous, non-decreasing and satisfies $F_X(+\infty) = 1$, where $F_X(+\infty) = \sup\{F_X(x) : x \in \mathbb{R}\}$.
- 3) Let (Ω, \mathcal{F}) be a measurable space, $X : \Omega \rightarrow \tilde{\Omega}$ a function. Show that $\tilde{\mathcal{F}} = \{\tilde{A} \in \mathcal{P}(\tilde{\Omega}) : X^{-1}(\tilde{A}) \in \mathcal{F}\}$ is a σ -algebra on $\tilde{\Omega}$, the largest σ -algebra \mathcal{G} on $\tilde{\Omega}$ for which X is \mathcal{F}/\mathcal{G} -measurable.
- 4) Let (Ω, \mathcal{F}) be a measurable space and $X : \Omega \rightarrow \mathbb{R}$ a function. Show that X is a Borel function if and only if $\{X < q\} \in \mathcal{F}$ for all $q \in \mathbb{Q}$, where \mathbb{Q} denotes the set of all rational numbers.

Complements.

- (i) Let (A, \mathcal{A}) and (B, \mathcal{B}) be two measurable spaces, $f : A \rightarrow B$. Suppose furthermore $\mathcal{C} \subset \mathcal{P}(B)$ is such that $\sigma(\mathcal{C}) = \mathcal{B}$. Show that f is \mathcal{A}/\mathcal{B} -measurable if and only if $f^{-1}(C) \in \mathcal{A}$ for every $C \in \mathcal{C}$.
- (ii) Show that a nondecreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ admits at most denumerably many points of discontinuity.
- (iii) *Find an everywhere dense open subset of \mathbb{R} of finite Lebesgue measure.
- (iv) **Fix $(\Omega, \mathcal{F}, \mathbf{P})$, a probability space. Following Meyer¹, let us call a subset $A \subseteq \Omega$, internally \mathbf{P} -negligible, if every measurable subset of A has zero probability. Let \mathcal{N} be a collection of internally \mathbf{P} -negligible sets, and suppose \mathcal{N} is closed under countable unions. Define $\mathcal{F}' := \sigma_{\Omega}(\mathcal{F} \cup \mathcal{N})$ and $\mathcal{M} := \cup_{N \in \mathcal{N}} \mathcal{P}(N)$.
- Show that $\mathcal{G} := \{F \triangle M : (F, M) \in \mathcal{F} \times \mathcal{M}\}$ is a σ -field containing \mathcal{F}' .
 - Show that \mathbf{P} can be extended in a unique manner to a probability measure \mathbf{P}' on \mathcal{F}' , such that every member of \mathcal{N} is \mathbf{P}' -negligible. **Hint:** Extend \mathbf{P} to a probability measure \mathbf{Q} on \mathcal{G} , by insisting that every member of \mathcal{M} becomes \mathbf{Q} -negligible.

¹Meyer, P. A.: Probability and potentials, p22.